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Epha Diana Supandi
*State Islamic University, Sunan Kalijaga, Yogyakarta, Indonesia*, epha.supandi@uin-suka.ac.id

Dedi Rosadi
*Gadjah Mada University, Yogyakarta, Indonesia*, dedirosadi@ugm.ac.id

Abdurakhman
*Gadjah Mada University, Yogyakarta, Indonesia*, rachmanstat@ugm.ac.id



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# An Empirical Comparison between Robust Estimation and Robust Optimization to Mean-Variance Portfolio

**Epha Diana Supandi**
State Islamic University, Sunan Kalijaga
Yogyakarta, Indonesia

**Dedi Rosadi**
Gadjah Mada University
Yogyakarta, Indonesia

**Abdurakhman**
Gadjah Mada University
Yogyakarta, Indonesia

Mean-variance portfolios constructed using the sample mean and covariance matrix of asset returns perform poorly out-of-sample due to estimation error. Recently, there are two approaches designed to reduce the effect of estimation error: robust statistics and robust optimization. Two different robust portfolios were examined by assessing the out-of-sample performance and the stability of optimal portfolio compositions. The performance of the proposed robust portfolios was compared to classical portfolios via expected return, risk, and Sharpe Ratio. The aim is to shed light on the debate concerning the importance of the estimation error and weights stability in the portfolio allocation problem, and the potential benefits coming from robust strategies in comparison to classical portfolios.

*Keywords:* Mean-variance portfolio, robust statistics, robust optimization

## Introduction

The portfolio optimization approach proposed by Markowitz (1952) undoubtedly is one of the most important models in financial portfolio selection. This model is based upon the fundamental trade-off between expected return and risk, measured by the mean and standard deviation of return respectively. Therefore, Markowitz's model is called the mean-variance portfolio since this technique is highly reliant upon the value of a set of inputs, i.e. the mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. The goal of the portfolio allocation problem is to find weights $\mathbf{w}$ which represent the percentage of capital to be invested in each asset.

*Epha Diana Supandi is a Lecturer in the Department of Mathematics. Email her at: epha.supandi@uin-suka.ac.id.*

To compute the mean-variance portfolios, the mean vector $\hat{\boldsymbol{\mu}}$ and covariance matrix $\hat{\boldsymbol{\Sigma}}$ need to be estimated and both inputs are obtained from historical data. These estimators plug into an analytical or numerical solution to the investor's optimization problem. This leads to an important drawback in the mean-variance approach: the estimation error.

The fact that mean-variance "optimal" portfolios are sensitive to small changes in input data is well documented in the literature. Chopra and Ziemba (1993) showed that even slight changes to the estimates of expected return or risk can produce vastly different mean-variance optimized portfolios. Best and Grauer (1991) analyzed the sensitivity of optimal portfolios to changes in expected return estimates. Broadie (1993), meanwhile, showed how the estimated efficient frontier overestimates the expected returns of portfolios for various levels of estimation errors. Because of the ill effects of estimation errors on optimal portfolios, portfolio optimization has been called "error maximization" (see Michaud, 1989).

There are two standard methods extensively adopted in the literature to combat the impact of estimation error on portfolio selection. The first method is robust estimation, which can be quite robust to distributional assumptions. The introduction of robust estimation to portfolio optimization is relatively recent compared to the Markowitz foundational paper. Nevertheless, the subject has become very active in the last decade, as seen in the works of Lauprête (2001), Lauprete, Samarov, and Welsch (2002), Mendes and Leal (2003), Perret-Gentil and Victoria-Feser (2004), Welsch and Zhou (2007), and DeMiguel and Nogales (2009). The main difference among these approaches is in the term of the type of robust estimator used. Lauprête (2001) and Lauprete et al. (2002) used the least absolute deviation Huber estimator and trimean estimator, Mendes and Leal (2003) used the *M*-estimator, Perret-Gentil and Victoria-Feser (2004) used the *S*-estimator, Welsch and Zhou (2007) used the minimum covariance determinant estimator and Winsorization, and DeMiguel and Nogales (2009) used the *M*-estimator and the *S*-estimator. In their investigations, the portfolios constructed using a robust estimator outperformed those created using traditional mean-variance portfolio in the majority of cases.

The second method to deal with the estimation error is robust optimization. Robust portfolio optimization is a fundamentally different way of handling estimation error in the portfolio construction process. Unlike the previously-mentioned approaches, robust optimization considers the estimation error directly in the optimization problem itself. Introduced by Ben-Tal and Nemirovski (2002) for robust truss topology design, robust optimization is an emerging branch in the field of optimization in which the solutions for optimization problems are obtained from uncertain parameters. The uncertainty is described using an uncertainty set which includes all, or most, possible realizations of the uncertain input parameters (see Pachamanova, Kolm, Fabozzi, & Focardi, 2007). The true mean and covariance matrix of asset returns lie in a fixed range. A robust portfolio, the one that optimizes the worst-case performance concerning with all possible values the mean vector and covariance matrix. The worst-case for robust optimization probably happened in the uncertainty sets (see, for example, Goldfarb & Iyengar, 2003; Tütüncü & Koenig, 2004; Engels, 2004; Garlappi, Uppal, & Wang, 2007; Lu, 2011).

The aim of this study is to shed light on the recent debate regarding the importance of the estimation error and weights' stability in the portfolio allocation problem and the potential benefits coming from robust portfolios in comparison to classical techniques. Here, two different robust portfolios have been investigated. The first portfolio was obtained by robust estimator to the mean-variance portfolio towards the *S*-estimators, constrained *M*-estimators, Minimum Covariance Determinant (MCD), and Minimum Volume Ellipsoid (MVE). The second one was obtained by robust optimization to the sample mean-variance portfolio where the formulation and the algorithm used in this paper were based on those developed by Tütüncü and Koenig (2004). We empirically compared two versions of robust asset allocation through the out-of-sample performance of those portfolio allocation approaches corresponding to the methodology of rolling horizon as proposed in DeMiguel and Nogales (2009).

## The Mean-Variance Portfolio (Classical Portfolio)

It is assumed that the random vector $\mathbf{r} = (r_1, r_2, \ldots, r_N)'$ denotes random returns of the $N$ risky assets with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. A portfolio is defined to be a list of weights $w_i$ for the assets $i = 1, \ldots, N$ that represent the amount of capital to be invested in each asset. We assumed that

$$\sum_{i=1}^{N} w_i = 1$$

meaning that capital is fully invested.

For a given portfolio $\mathbf{w}$, the expected return and variance were respectively given by: $E(\mathbf{w}'\mathbf{r}) = \mathbf{w}'\boldsymbol{\mu}$ and $Var(\mathbf{w}'\mathbf{r}) = \mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w}$. Then, the classical mean-variance portfolio models of Markowitz were formulated mathematically as the optimization problem:

$$\max_{\mathbf{w}} \mathbf{w}'\boldsymbol{\mu} - \frac{\gamma}{2}\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}, \quad \text{s.t. } \mathbf{e}'\mathbf{w} = 1, \mathbf{w} \geq 0 \tag{1}$$

where $\boldsymbol{\mu} \in \Re^N$ is the vector of expected return, $\boldsymbol{\Sigma} \in \Re^{N \times N}$ is the covariance matrix of return, where $\Re^{N \times N}$ denotes the set of all $N \times N$ positive definite symmetric matrices, and $\mathbf{w} \in \Re^N$ is the vector of portfolio weight. The restriction $\mathbf{w} \geq 0$ means that short-selling is not allowed. The parameter $\gamma$ can be interpreted as a risk aversion, since it takes into account the trade-off between risk and return of the portfolios.

The main criticisms against the Markowitz models centers on the observation that the optimal portfolios generated by this approach are often quite sensitive to the input parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$. To make matters worse, these parameters can never be observed, and one has to settle for estimates found using some particular techniques.

## Robust Portfolio Estimation

In this section, the class of portfolio policies based on the robust estimators is proposed where portfolio optimization and robust estimation are performed in two steps. It began by computing the robust estimators of the mean vector and covariance matrix of asset returns and followed by computing the portfolio policies by solving the classical minimum-variance problem (1), but replacing the sample mean and covariance matrix by their robust counterparts.

One of the most popular classes of robust estimators is affine equivariant robust estimators (see Maronna, Martin, & Yohai, 2007). Let $\left(\hat{\boldsymbol{\mu}}(\mathbf{r}), \hat{\boldsymbol{\Sigma}}(\mathbf{r})\right)$ be location and dispersion estimates corresponding to a sample = $(r_1, r_2, \ldots, r_N)'$. Then the estimates are affine equivariant if

$$\hat{\boldsymbol{\mu}}(\mathbf{A}\mathbf{r} + \mathbf{b}) = \mathbf{A}\hat{\boldsymbol{\mu}}(\mathbf{r}) + \mathbf{b} \quad \text{and} \quad \hat{\boldsymbol{\Sigma}}(\mathbf{A}\mathbf{r} + \mathbf{b}) = \mathbf{A}\hat{\boldsymbol{\Sigma}}(\mathbf{r})\mathbf{A}'$$

for any constant $N$-dimensional vector $\mathbf{b}$ and any non-singular $N \times N$ matrix $\mathbf{A}$. There are many different robust estimators for the mean and covariance in this class, such as $S$-estimators (Rousseeuw & Yohai, 1984), MVE and MCD proposed by Rousseeuw (1984), as well as $CM$-estimators (Kent & Tyler, 1996).

### *S*-Estimators

*S*-estimators were first introduced (in the context of regression) by Rousseeuw and Yohai (1984). Later, they were applied to the multivariate scale and location estimation problem (Davies, 1992).

Let $\mathbf{r}$ be a data set in $\Re^N$. The *S*-estimators of the multivariate location $\hat{\boldsymbol{\mu}}(\mathbf{r}) \in \Re^N$ and scatter $\hat{\boldsymbol{\Sigma}}(\mathbf{r}) \in \Re^{N\times N}$ are defined as the solution to the problem of minimizing $|\boldsymbol{\Sigma}|$ subject to

$$\frac{1}{n}\sum_{i=1}^{n}\rho\left[\left\{(\mathbf{r}_i-\boldsymbol{\mu})'\,\boldsymbol{\Sigma}^{-1}(\mathbf{r}_i-\boldsymbol{\mu})\right\}^{\frac{1}{2}}\right]=b_0 \tag{2}$$

where $\rho$ denotes the loss function and $b_0$ satisfies $0 < b_0 < a_0 = \sup\{\rho\}$. As stated by Alqallaf (2003), it is natural to choose $b_0 = \mathrm{E}\left(\rho\left(\|\mathbf{r}\|\right)\right)$.

Let $\mathbf{r}$ be a data set in $\Re^N$ and $c_0 = b_0/\sup\rho$. If $c_0 \le (n-N)/2n$, where $n \ge N+1$, then the breakdown point $\varepsilon^* = \lceil nc_0\rceil/n$, where $\lceil k\rceil$ denotes the nearest integer greater than or equal to $k$. The breakdown point for *S*-estimators is

$$\varepsilon^* = \frac{n-N+1}{2n}$$

when

$$c_0 = \frac{(n-N)}{2n}$$

Portfolios based on *S*-estimators with biweight function were examined by Perret-Gentil and Victoria-Feser (2004) and, in a one-step approach, by DeMiguel and Nogales (2009).

### *CM*-Estimators

As stated by Kent and Tyler (1996), the *CM*-estimator is defined via the minimization of an objective function subject to some constraints. For the data set $\mathbf{r}$ we defined the *CM*-estimators of the multivariate location $\hat{\boldsymbol{\mu}}(\mathbf{r}) \in \Re^N$ and scatter $\hat{\boldsymbol{\Sigma}}(\mathbf{r}) \in \Re^{N\times N}$ to be any pair which minimized the objective function

$$\mathrm{L}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{r}) = \frac{1}{n}\sum_{i=1}^{n}\rho(\mathbf{d}_i) + \frac{1}{2}\log|\boldsymbol{\Sigma}| \tag{3}$$

subject to the constraint

$$\frac{1}{n}\sum_{i=1}^{n}\rho(\mathbf{d}_i) \le \varepsilon\rho(\infty) \tag{4}$$

where $\mathbf{d}_i = (\mathbf{r}_i - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{r}_i - \boldsymbol{\mu})$, ρ denotes the loss function, and $\varepsilon \in (0, 1)$ refers to the breakdown point. Kent and Tyler (1996) showed that the breakdown point of the *CM*-estimate for data **r** in general is

$$\varepsilon^* = \min\left(\left\lceil \frac{n\varepsilon}{n} \right\rceil, \left\lceil \frac{n(1-\varepsilon)-N}{n} \right\rceil\right)$$

## Minimum Volume Ellipsoid (MVE) Estimators

Rousseeuw (1984) introduced a highly robust estimator, the MVE estimator, $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ where $\boldsymbol{\mu}$ was taken to be the center of the minimum volume ellipsoid covering at least half of the observations, and $\boldsymbol{\Sigma}$ was an $N$ by $N$ matrix representing the shape of the ellipsoid.

This approach attempted to seek the ellipsoid with the smallest volume covering $h$ data points where $n/2 \le h \le n$. Formally, the estimate is defined as these $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$ that minimized $|\boldsymbol{\Sigma}|$ subject to

$$\#\left\{i; (\mathbf{r}_i - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\mathbf{r}_i - \boldsymbol{\mu}) \le c^2\right\} \ge \left[\frac{n+N+1}{2}\right] \tag{5}$$

The constant $c$ is chosen as $\chi^2_{N,0.5}$ and # denotes the cardinality. Portfolios based on MVE estimators were used by Kaszuba (2013). Let **r** be a data set in $\Re^N$ with $N \ge 2$, and let $n \ge N + 1$; then the breakdown point of MVE is

$$\varepsilon^* = \frac{\lfloor (n-N+1)/2 \rfloor}{n}$$

## Minimum Covariance Determinant (MCD) Estimators

The MCD estimators are highly robust estimators of multivariate location and scatter introduced by Rousseeuw (1984). Given an $n \times N$ data matrix $\mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n)'$ with $\mathbf{r}_i = (\mathbf{r}_{i1}, \mathbf{r}_{i2}, \ldots, \mathbf{r}_{iN})'$, it is focused on finding $h$ (with $\lfloor (n + N + 1)/2 \rfloor \leq h \leq n$) observations whose classical covariance matrix has the lowest possible determinant. Then, the MCD estimator of location is the average of these $h$ points, whereas the MCD estimator of scatter is their covariance matrix.

In 1999, Rousseeuw and Van Diressen constructed a very fast algorithm to calculate the MCD estimator. The new algorithm was called Fast-MCD based on the $C$-step. The Fast-MCD algorithm is defined as follows:

***Algorithm 1.*** The Fast-MCD (Rousseeuw & Van Diressen, 1999)

1. Set an initial $h$-subset $\mathrm{H}_1$, that is, beginning with a random $(N + 1)$-subset J.
2. Compute

$$\hat{\boldsymbol{\mu}}_0 = \frac{1}{N+1}\sum_{i\in \mathrm{J}} \mathbf{r}_i \quad \text{and} \quad \hat{\Sigma}_0 = \frac{1}{N+1}\sum_{i\in \mathrm{J}} \left(\mathbf{r}_i - \hat{\boldsymbol{\mu}}_0\right)\left(\mathbf{r}_i - \hat{\boldsymbol{\mu}}_0\right)'$$

   If $\left|\hat{\boldsymbol{\Sigma}}_0\right| = 0$, random observations are added to J until $\left|\hat{\boldsymbol{\Sigma}}_0\right| > 0$.

3. Apply the $C$-step to the initial $h$-subset $\mathrm{H}_1$, and obtain the $\left(\hat{\boldsymbol{\mu}}_1, \hat{\boldsymbol{\Sigma}}_1\right)$. If $\left|\hat{\boldsymbol{\Sigma}}_0\right| = 0$ or $\left|\hat{\boldsymbol{\Sigma}}_0\right| = \left|\hat{\boldsymbol{\Sigma}}_1\right|$, stop; otherwise, running another C-step produces $\left|\hat{\boldsymbol{\Sigma}}_2\right|$, and so on, until convergence is reached.

If the data are sampled from a continuous distribution, then these estimators have the breakdown point

$$\varepsilon^* = \min\left(\frac{n-h+1}{n}, \frac{h-p}{n}\right)$$

Portfolios based on MCD estimators were investigated by Zhou (2006), Welsch and Zhou (2007), and, in a modified version, by Mendes and Leal (2005).

*S*-estimators, *CM*-estimators, MVE, and MCD are used to construct robust portfolio mean-variance. A two-step approach to robust portfolio estimation is proposed. First, compute a robust estimate of the mean vector and covariance matrix of asset returns. Second, solve the classical mean-variance problem (1), but replacing the sample mean and covariance matrix by their robust counterparts. Thus, given the robust estimators, the robust portfolio estimation can be found by solving the following optimization problem:

$$\max_{\mathbf{w}} \mathbf{w}'\hat{\boldsymbol{\mu}}_{\text{rob}} - \frac{\gamma}{2}\mathbf{w}'\hat{\boldsymbol{\Sigma}}_{\text{rob}}\mathbf{w}, \quad \text{s.t. } \mathbf{e}'\mathbf{w} = 1, \mathbf{w} \geq 0 \tag{6}$$

## Robust Portfolio Optimization

Robust optimization has been developed to solve any problems related to the uncertainty in the decision environment and, therefore, sometimes it is referred to uncertain optimization (Ben-Tal & Nemirovski, 2002). Robust models have been adapted in portfolio optimization to resolve the sensitivity issue of the mean-variance portfolio to its inputs.

Robust portfolio optimization is to represent all available information about the unknown input parameters in the form of an uncertainty set that contains most of the possible values for these parameters.

Tütüncü and Koenig (2004) proposed a bootstrap method to determine the uncertainty sets. This method attempted to capture the uncertainty regarding the parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ in their uncertainty sets $\mathbb{U}_{\boldsymbol{\mu}}$ and $\mathbb{U}_{\boldsymbol{\Sigma}}$ by carrying out the following algorithm:

***Algorithm 2.*** The construction of $\mathbb{U}_{\boldsymbol{\mu}}$ and $\mathbb{U}_{\boldsymbol{\Sigma}}$ using a block bootstrap method

1. Choose the block length ($l$). In our experiment, we used the non-overlapping block. Divide the data into $n/l$ blocks in which block 1 became $\{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_l\}$ and block 2 became $\{\mathbf{r}_{l+1}, \mathbf{r}_{l+2}, \ldots, \mathbf{r}_{2l}\}$, ..., etc.
2. Resample the blocks and generate the bootstrap sample.
3. Compute the classical estimators of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ from bootstrap data.
4. Construct the empirical distribution of estimators by repeating step 2 and step 3 $B$ times and sorting the bootstrap estimators from the smallest to largest ones.
5. Determine the $(1-\alpha)100\%$ percent quintile of distribution of estimators

From algorithm 2, the uncertainty sets are defined as

$$\mathbb{U}_{\boldsymbol{\mu}} = \left\{ \boldsymbol{\mu} : \hat{\boldsymbol{\mu}}^{\mathrm{L}} \leq \boldsymbol{\mu} \leq \hat{\boldsymbol{\mu}}^{\mathrm{U}} \right\} \tag{7}$$

$$\mathbb{U}_{\boldsymbol{\Sigma}} = \left\{ \boldsymbol{\Sigma} : \hat{\boldsymbol{\Sigma}}^{\mathrm{L}} \leq \boldsymbol{\Sigma} \leq \hat{\boldsymbol{\Sigma}}^{\mathrm{U}}, \boldsymbol{\Sigma} \succeq 0 \right\} \tag{8}$$

Given the uncertainty sets of mean vector (7) and covariance matrix (8), then robust optimization (Rob.Opt) can be defined as follows:

$$\max_{\mathbf{w}} \mathbf{w}^{\mathrm{T}} \hat{\boldsymbol{\mu}}^{\mathrm{L}} - \frac{\gamma}{2} \mathbf{w}' \hat{\boldsymbol{\Sigma}}^{\mathrm{U}} \mathbf{w}, \quad \text{s.t. } \mathbf{e}^{\mathrm{T}} \mathbf{w} = 1, \mathbf{w} \geq 0 \tag{9}$$

## Empirical Study

Data used in this study were collected from the Jakarta Stocks Exchange (JSE) consisting of 20 companies categorized as the blue chip. A blue chip is a stock in “a nationally recognized, well-established and financially sound company.” (“Blue Chip”, n.d.). Table 1 presents the list of companies.

The time series data span was from 04/02/2008 to 29/12/2014 with a total of 360 weekly returns. The first 260 observations (02/01/2008 to 07/01/2013) were used as the first window to perform the estimation and the uncertainty set. The last 100 observations (14/01/2013 to 29/12/2014) referred to the out-of-sample period and were used for the ex-post effectiveness analysis.

**Table 1.** Asset name for empirical analysis

| No | Asset Name | No | Asset name |
|---|---|---|---|
| 1 | AALI = Astra Argo Lestari, Tbk | 11 | JSMR = Jasa Marga (Persero) Tbk |
| 2 | AKRA = Akr Corporindo Tbk | 12 | KLBF = Kalbe Farma Tbk |
| 3 | BBCA = Bank Centra Asia Tbk | 13 | LPKR = Lippo Karawaci Tbk |
| 4 | BBNI = Bank Negara Indonesia (Persero) Tbk | 14 | MNCN = Media Nusantara Citra Tbk |
| 5 | BBRI = Bank Rakyat Indonesia (Persero) Tbk | 15 | PGAS = Perusahaan Gas Neagara (Persero) Tbk |
| 6 | BMRI = Bank Mandiri (Persero) Tbk | 16 | PTBA = Tambang Batu Bara Asam (Persero) Tbk |
| 7 | CPIN = Charoen Pokphand Indonesia Tbk | 17 | SMGR = Semen Indonesia (Persero) Tbk |
| 8 | INDF = Indofood Sukses Makmur Tbk | 18 | TLKM = Telekomunikasi Indonesia (Persero) Tbk |
| 9 | INTP = Indocement Tunggal Prakarsa Tbk | 19 | UNTR = United Tractors Tbk |
| 10 | ITMG = Indo Tambangraya Megah Tbk | 20 | UNVR = Unilever Indonesia Tbk |

## Research Methodology

For an empirical analysis, several parameters have to be set. Firstly, for robust portfolio estimation, a translated biweight function is used as the loss function and the breakdown point is set at 45%. Meanwhile, in robust portfolio optimization, an important question is how to determine the uncertainty sets. The value $\alpha$ determines the most extreme parameter values that are still included in the uncertainty sets. The smaller $\alpha$ is, the larger an uncertainty set will be, and thus the greater the worst-case estimation errors will be. Hence, $\alpha$ can be interpreted as a parameter that captures the investor's tolerance for estimation errors (Fastrich & Winker, 2009). Therefore, to measure the level of sensitivity of the Rob.Opt model, set $\alpha = 0.05$, 0.10, and 0.20.

Use the rolling-horizon procedure to compute the out-of-sample performance measures. This procedure has been implemented similarly as in DeMiguel and Nogales (2009). First, chose the window $T = 260$ to perform the estimation and the uncertainty sets. Second, using the return data in the estimation window, compute some optimal portfolio policies according to each strategy (classical portfolio, robust portfolio estimation, and robust portfolio optimization). Third, repeat the rolling-window procedure for the next month by including the four data points for the new date and dropping the four data points for the earliest period of the estimation window (we assumed that investors would rebalance their portfolios every one month). Continue this until the end of the dataset is reached. Therefore, at the end there is a time series of 25 portfolio weight vectors for each of the portfolios considered in the analysis.

The out-of-sample performance of each strategy was evaluated according to the following statistics: mean return, risk, Sharpe ratio, and portfolio turnover. Holding the portfolio $\mathbf{w}_t^s$ for one trading period gave the following out-of-sample excess return at time $t + 1$, that is $\hat{\mathbf{r}}_{t+1} = \mathbf{w}_t'^s \mathbf{r}_{t+1}^s$. After collecting the time series of 25 excess returns $\hat{\mathbf{r}}_{t+1}$, the out-of-sample mean return, standard deviation (risk), Sharpe ratio, and portfolio turnover are:

$$\hat{\mu}^s = \frac{1}{25}\sum_{t=1}^{25} \mathbf{w}_t'^s \mathbf{r}_{t+1}$$

$$\hat{\sigma}^s = \sqrt{\frac{1}{24}\sum_{t=1}^{25}\left(\mathbf{w}_t'^s \mathbf{r}_{t+1} - \hat{\boldsymbol{\mu}}^s\right)^2}$$

$$\mathrm{SR}^s = \frac{\hat{\mu}^s}{\hat{\sigma}^s}$$

$$\text{Turnover} = \frac{1}{24}\sum_{t=1}^{25}\sum_{j=1}^{5}\left(\left|w_{j,t+1} - w_{j,t}\right|\right)$$

where $w_{j,t}$ is the portfolio weight in asset $j$ at time $t$ + 1 but before rebalancing and $w_{j,t+1}$ is the desired portfolio weight in asset $j$ at time $t$ + 1. Therefore, the portfolio turnover is a measure of the variability in the portfolio holdings and can indirectly indicate the magnitude if the transaction costs associated to each strategy. Clearly, the smaller the turnover, the smaller the transaction costs associated to the implementation of the strategy.

## Research Hypothesis

The research hypothesis is that the appropriate application of robust strategies in the construction of mean-variance portfolios allows the achievement of better investment results (measured with mean return and risk) in comparison to classical portfolios (benchmark). Hence, it is verified whether the given method allows one to obtain higher mean return compared to the classical method using the Wilcoxon signed rank test at significance level of 5%. Similarly, it is examined whether the robust methods will have lower risk (measured by standard deviation) compared to the classical method (see Kaszuba, 2013).

## Results of Empirical Study

In the ninth column of Table 2, it can be observed that most of the return data were not normally distributed except AKRA, INTP, and UNVR. Also, UNVR had the best performance for having the highest mean return and the lowest risk (measured by standard deviation) compared to other stocks.

Presented in Table 3 are the out-of-sample performance of the classical and all robust approaches for each time window *win* in which the former serves as a benchmark. The results presented in Table 3 concern only portfolios for which risk aversion is equal to 10. Other risk aversion parameters were tested, such as $\gamma$ = 1, 100, and 1000; the summary of these results are presented in Table 4.

It can be seen that the mean returns are higher in all seven robust approaches compared to the classical approach. An examination in the out-of-sample performance of portfolio returns indicated that the highest mean returns are obtained by robust portfolio estimation generated using *CM*-estimators (as presented in Table 3).

**Table 2.** Summary statistics of the 20 stocks used in the dataset

| | Min | Max | Mean | Std. Dev | Var | Skew | Kurtosis | K.Smirnov |
|---|---|---|---|---|---|---|---|---|
| AALI | -0.4329 | 0.3459 | -0.0007 | 0.0723 | 0.0052 | -0.5740 | 6.4580 | 0.0004 |
| AKRA | -0.2673 | 0.1982 | 0.0029 | 0.0612 | 0.0038 | -0.1660 | 1.6770 | 0.4150 |
| BBCA | -0.7071 | 0.1588 | 0.0017 | 0.0579 | 0.0034 | -5.0540 | 62.0380 | 0.0004 |
| BBNI | -0.4362 | 0.3920 | 0.0033 | 0.0634 | 0.0040 | 0.2190 | 11.6080 | 0.0002 |
| BBRI | -0.6434 | 0.2975 | 0.0015 | 0.0655 | 0.0043 | -2.6460 | 27.4440 | 0.0048 |
| BMRI | -0.2744 | 0.2380 | 0.0033 | 0.0548 | 0.0030 | -0.2460 | 4.2520 | 0.0293 |
| CPIN | -1.5404 | 0.3868 | 0.0033 | 0.1109 | 0.0123 | -7.4410 | 105.2200 | 0.0000 |
| INDF | -0.2542 | 0.2654 | 0.0027 | 0.0556 | 0.0031 | -0.1560 | 3.9300 | 0.0034 |
| INTP | -0.4418 | 0.2747 | 0.0032 | 0.0579 | 0.0033 | -0.9070 | 10.6050 | 0.0527 |
| ITMG | -0.5557 | 0.3153 | -0.0008 | 0.0773 | 0.0060 | -0.9150 | 8.7970 | 0.0012 |
| JSMR | -0.2942 | 0.1842 | 0.0036 | 0.0449 | 0.0020 | -0.4700 | 6.3350 | 0.0409 |
| KLBF | -1.5991 | 0.4970 | 0.0010 | 0.1038 | 0.0108 | -10.0080 | 159.2970 | 0.0000 |
| LPKR | -0.2587 | 0.3520 | 0.0011 | 0.0598 | 0.0036 | 0.4410 | 4.6370 | 0.0112 |
| MNCN | -0.2801 | 0.5994 | 0.0032 | 0.0786 | 0.0062 | 1.3750 | 10.3110 | 0.0026 |
| PGAS | -1.5549 | 0.2841 | -0.0023 | 0.0974 | 0.0095 | -11.3460 | 180.6230 | 0.0000 |
| PTBA | -0.5771 | 0.2451 | 0.0002 | 0.0685 | 0.0047 | -1.5180 | 14.2460 | 0.0007 |
| SMGR | -0.6012 | 0.2766 | 0.0030 | 0.0591 | 0.0035 | -2.5900 | 31.3040 | 0.0042 |
| TLKM | -1.5864 | 0.1382 | -0.0035 | 0.0930 | 0.0086 | -13.7920 | 234.9950 | 0.0000 |
| UNTR | -0.4215 | 0.2895 | 0.0009 | 0.0699 | 0.0049 | -0.8050 | 7.9740 | 0.0018 |
| UNVR | -0.1676 | 0.1436 | **0.0042** | **0.0402** | 0.0016 | 0.0500 | 1.5960 | 0.0590 |

Note: The bold values indicate the best performance of out-of-sample portfolio.

Also, it can be seen that MVE portfolios obtained higher Sharpe ratio than the ones obtained with the classical or other robust approaches. Whereas, in the context of risk, MCD generated using the fast algorithm exhibited the lowest risks. Meanwhile, MVE portfolios achieved the lowest turnover. Therefore, portfolio robust estimation (Rob.Est) created using a two-step approach (*CM*, *S*, MCD, and MVE portfolios) outperformed the classical approach for this case.

It can also be noticed that by analyzing the performance of Rob.Opt portfolios one can observe that increasing the investors' tolerance for estimation error $\alpha$ can decrease the performance of all out-of-sample for this portfolios.

Presented in Table 4 are the out-of-sample performance's portfolio, i.e., mean returns ( $\hat{\mu}^s$ ), risk ( $\hat{\sigma}^s$ ), Sharp Ratio (SR), and portfolio turnover (TO) at a number of different risk aversions, as well as different $p$-values of the Wilcoxon test for differences between the portfolios returns calculated with the given method and classical portfolios. The presented $p$-values for Wilcoxon test for observation pairs allows us to see whether the average weekly returns for the investigated portfolios were significantly higher than the average returns for classical portfolios.

**Table 3.** The out-of-sample performance of portfolio return for each time window *win* at $\gamma$ = 10

| | | Rob.Est | | | | Rob.Opt | | |
|---|---|---|---|---|---|---|---|---|
| ***win*** | **Classic** | **CM** | **S** | **MCD** | **MVE** | **$\alpha$=5%** | **$\alpha$=10%** | **$\alpha$=20%** |
| 1 | 0.0167 | 0.0641 | 0.0576 | 0.0523 | 0.0556 | 0.0080 | 0.0088 | 0.0077 |
| 2 | 0.0370 | 0.0450 | 0.0635 | 0.0477 | 0.0450 | 0.0485 | 0.0496 | 0.0500 |
| 3 | -0.0017 | 0.0139 | 0.0135 | 0.0110 | 0.0099 | -0.0032 | -0.0033 | -0.0037 |
| 4 | 0.0017 | 0.0060 | 0.0190 | -0.0078 | 0.0151 | 0.0155 | 0.0137 | 0.0117 |
| 5 | -0.0577 | -0.0769 | -0.0775 | -0.0662 | -0.0609 | -0.0593 | -0.0584 | -0.0582 |
| 6 | 0.0434 | 0.0160 | 0.0317 | 0.0181 | 0.0087 | 0.0753 | 0.0742 | 0.0687 |
| 7 | -0.0099 | -0.0121 | 0.0012 | -0.0030 | 0.0023 | -0.0186 | -0.0222 | -0.0181 |
| 8 | 0.0684 | 0.1429 | 0.1275 | -0.0030 | 0.1220 | 0.0248 | 0.0270 | 0.0314 |
| 9 | 0.0046 | 0.0310 | 0.0242 | 0.0178 | 0.0291 | 0.0139 | 0.0137 | 0.0122 |
| 10 | 0.0100 | 0.0313 | 0.0236 | 0.0266 | 0.0257 | 0.0043 | 0.0038 | 0.0045 |
| 11 | -0.0122 | -0.0247 | -0.0228 | -0.0238 | -0.0237 | -0.0155 | -0.0144 | -0.0149 |
| 12 | 0.0135 | 0.0277 | 0.0189 | 0.0229 | 0.0327 | 0.0088 | 0.0099 | 0.0118 |
| 13 | -0.0281 | 0.0085 | -0.0025 | -0.0162 | 0.0050 | -0.0166 | -0.0167 | -0.0227 |
| 14 | 0.0006 | -0.0025 | 0.0074 | -0.0051 | -0.0017 | -0.0090 | -0.0083 | -0.0070 |
| 15 | 0.0054 | 0.0056 | -0.0193 | 0.0073 | 0.0083 | 0.0195 | 0.0186 | 0.0168 |
| 16 | -0.0060 | -0.0107 | -0.0217 | -0.0087 | -0.0163 | -0.0131 | -0.0125 | -0.0112 |
| 17 | -0.0222 | -0.0123 | -0.0304 | -0.0092 | -0.0086 | -0.0113 | -0.0158 | -0.0169 |
| 18 | -0.0267 | -0.0112 | -0.0131 | -0.0036 | -0.0016 | -0.0125 | -0.0118 | -0.0164 |
| 19 | 0.0006 | 0.0050 | 0.0079 | 0.0065 | 0.0060 | 0.0171 | 0.0149 | 0.0143 |
| 20 | 0.0389 | 0.0236 | 0.0223 | 0.0256 | 0.0309 | 0.0272 | 0.0285 | 0.0289 |
| 21 | -0.0081 | -0.0132 | -0.0198 | -0.0098 | -0.0106 | -0.0059 | -0.0049 | -0.0062 |
| 22 | -0.0249 | -0.0088 | 0.0136 | -0.0112 | 0.0004 | 0.0103 | 0.0058 | 0.0024 |
| 23 | -0.0248 | -0.0398 | -0.0594 | -0.0338 | -0.0479 | -0.0171 | -0.0181 | -0.0193 |
| 24 | -0.0011 | 0.0129 | 0.0079 | 0.0105 | 0.0093 | 0.0130 | 0.0109 | 0.0105 |
| 25 | 0.0200 | 0.0203 | 0.0456 | 0.0198 | 0.0037 | 0.0169 | 0.0150 | 0.0193 |
| $\hat{\mu}^s$ | 0.0015 | **0.0097** | 0.0088 | 0.0026 | 0.0096 | 0.0048 | 0.0043 | 0.0038 |
| $\hat{\sigma}^s$ | 0.0269 | 0.0396 | 0.0409 | **0.0249** | 0.0347 | 0.0257 | 0.0258 | 0.0256 |
| SR | 0.0555 | 0.2442 | 0.2142 | 0.1038 | **0.2751** | 0.1881 | 0.1678 | 0.1491 |
| TO | 1.5891 | 1.1840 | **1.1097** | 1.1527 | 1.2927 | 1.6178 | 2.0235 | 2.0165 |

Note: The bold values indicate the best performance

An examination in the out-of-sample performance of portfolio returns indicated that the highest mean returns were obtained by robust portfolios. Of the robust approaches, portfolios generated with *CM*-estimators achieved the higher mean returns at $\gamma = 1$ and 10. Meanwhile, Rob.Opt portfolios obtained higher mean returns at $\gamma$ = 100 and 1000.

**Table 4.** Out-of-sample performance's portfolio i.e. mean returns ($\hat{\mu}^s$), risk ($\hat{\sigma}^s$), Sharpe ratio (SR) and portfolio turnover (TO) at different of risk aversions

| | | | **Rob.Est** | | | | **Rob.Opt** | | |
|---|---|---|---|---|---|---|---|---|---|
| | | **Classic** | **CM** | **S** | **MCD** | **MVE** | **$\alpha$=5%** | **$\alpha$=10%** | **$\alpha$=20%** |
| $\gamma = 1$ | $\hat{\mu}^s$ | -0.0076 | **0.0160** | 0.0128 | 0.0142 | 0.0159 | -0.0003 | -0.0033 | -0.0073 |
| | *p*-value | 1.0000 | 0.0773 | 0.0773 | 0.1759 | 0.0954 | 0.4410 | 0.6169 | 0.8289 |
| | $\hat{\sigma}^s$ | 0.0435 | 0.0572 | 0.0592 | 0.0752 | 0.0595 | **0.0341** | 0.0351 | 0.0385 |
| | *p*-value | 1.0000 | 0.0075* | 0.001* | 0.0012* | 0.0274* | 0.0004* | 0.0000* | 0.0000* |
| | SR | -0.1751 | **0.2801** | 0.2163 | 0.1891 | 0.2671 | -0.0093 | -0.0929 | -0.1898 |
| | TO | 1.9026 | 1.9372 | 2.0000 | 1.8958 | 1.9282 | **1.6731** | 1.6461 | 2.0955 |
| $\gamma = 10$ | $\hat{\mu}^s$ | 0.0015 | **0.0097** | 0.0088 | 0.0026 | 0.0096 | 0.0048 | 0.0043 | 0.0038 |
| | *p*-value | 1.0000 | 0.3859 | 0.3350 | 0.5004 | 0.2887 | 0.5379 | 0.5900 | 0.7148 |
| | $\hat{\sigma}^s$ | 0.0269 | 0.0396 | 0.0409 | **0.0249** | 0.0347 | 0.0257 | 0.0258 | 0.0256 |
| | *p*-value | 1.0000 | 0.0000* | 0.0004* | **0.0000*** | 0.0000* | 0.0000* | 0.0000 | 0.0000* |
| | SR | 0.0555 | **0.2442** | 0.2142 | 0.1038 | 0.2751 | 0.1881 | 0.1678 | 0.1491 |
| | TO | 1.5891 | 1.1840 | **1.1097** | 1.1527 | 1.2927 | 1.6178 | 2.0235 | 2.0165 |
| $\gamma = 100$ | $\hat{\mu}^s$ | 0.0058 | 0.0062 | 0.0054 | 0.0049 | 0.0064 | **0.0067** | 0.0066 | 0.0065 |
| | *p*-value | 0.9693 | 0.9540 | 0.8929 | 0.9234 | 0.9234 | 0.9234 | 0.9234 | 0.9234 |
| | $\hat{\sigma}^s$ | 0.0290 | 0.0276 | 0.0267 | **0.0252** | 0.0262 | 0.0292 | 0.0290 | 0.0288 |
| | *p*-value | 1.0000 | 0.0000* | 0.0000* | 0.0000* | 0.0000* | 0.0000* | 0.0000* | 0.0000* |
| | SR | 0.2010 | 0.2239 | 0.2033 | 0.1953 | **0.2449** | 0.2308 | 0.2283 | 0.2273 |
| | TO | 1.4530 | 1.0739 | **0.9222** | 1.0650 | 1.0682 | 1.6414 | 2.0457 | 2.0276 |
| $\gamma = 1000$ | $\hat{\mu}^s$ | 0.0057 | 0.0042 | 0.0041 | 0.0045 | 0.0053 | **0.0061** | 0.0060 | 0.0060 |
| | *p*-value | 1.0000 | 0.8626 | 0.8929 | 0.8929 | 0.9693 | 0.9847 | 0.9847 | 0.9847 |
| | $\hat{\sigma}^s$ | 0.0286 | 0.0243 | 0.0244 | 0.0245 | **0.0240** | 0.0290 | 0.0288 | 0.0290 |
| | *p*-value | 1.0000 | 0.0000* | 0.0000* | 0.0000* | 0.0000* | 0.0000* | 0.0000* | 0.0000* |
| | SR | 0.1978 | 0.1718 | 0.1678 | 0.1835 | **0.2208** | 0.2103 | 0.2083 | 0.2069 |
| | TO | 1.4585 | **1.0547** | 1.0112 | 1.0072 | 1.3808 | 2.0360 | 2.0317 | 2.0270 |

Note: The bold values indicate the best performance; an asterisk (*) indicates *p*-values at a significance level of 0.05

The corresponding results for the portfolio risk showed that the Rob.Est portfolios were better than two portfolio approaches (i.e. classical and Rob.Opt). The lowest portfolio risk was achieved by Rob.Est in the majority of the scenarios ($\gamma$ = 10, 100 and 1000). The research demonstrated that portfolios generated with MCD and MVE achieved a lower portfolio risk compared to *S*- and *CM*-

estimators. Therefore, it is obvious if the largest Sharpe ratios are obtained by Rob.Est in all cases.

Comparing portfolio turnover values, one can observe that for all portfolios, increasing the risk aversion value from 1 to 1000 has caused these values to decrease. Portfolios created using robust estimators (*CM* and *S*) had the lowest turnover except at $\gamma = 1$.

An empirical study using the real market data indicated that, for all robust portfolios with robust estimation and robust optimization on portfolio weights, there were statistically significant improvements in the risk. The classical portfolios were characterized by a much higher risk than robust portfolios. However, in the context of mean return, the difference in performances between robust techniques and classical techniques did not seem to be statistically significant ($p$-value > 0.05), the robust estimation techniques were able to deliver more stability in the portfolio weights in comparison to the classical approach. The main implication of this finding is that, if we assume equal performance across techniques, investors will be better off by choosing a strategy that does not require any radical changes in the portfolio composition over time. These substantial changes in portfolio composition are rather difficult to be implemented in practice due to (i) management costs; and (ii) negative cognitive aspects perceived by investors and/or investment managers (see Santos, 2010).

Because the aim was to examine portfolios regarding their robustness properties, a small turnover indicates the stability of portfolio, which means it is more robust. From the point of view of an investor, the stability of weights in a portfolio constructed by them throughout the entire duration of the investment is a significant element. In this case, as seen in Table 4, the smallest turnover is achieved by Rob.Est. These findings are corroborated by the visual inspection of Figure 1 and Figure 2, which show the time-varying portfolio weights and boxplots of each portfolio technique.

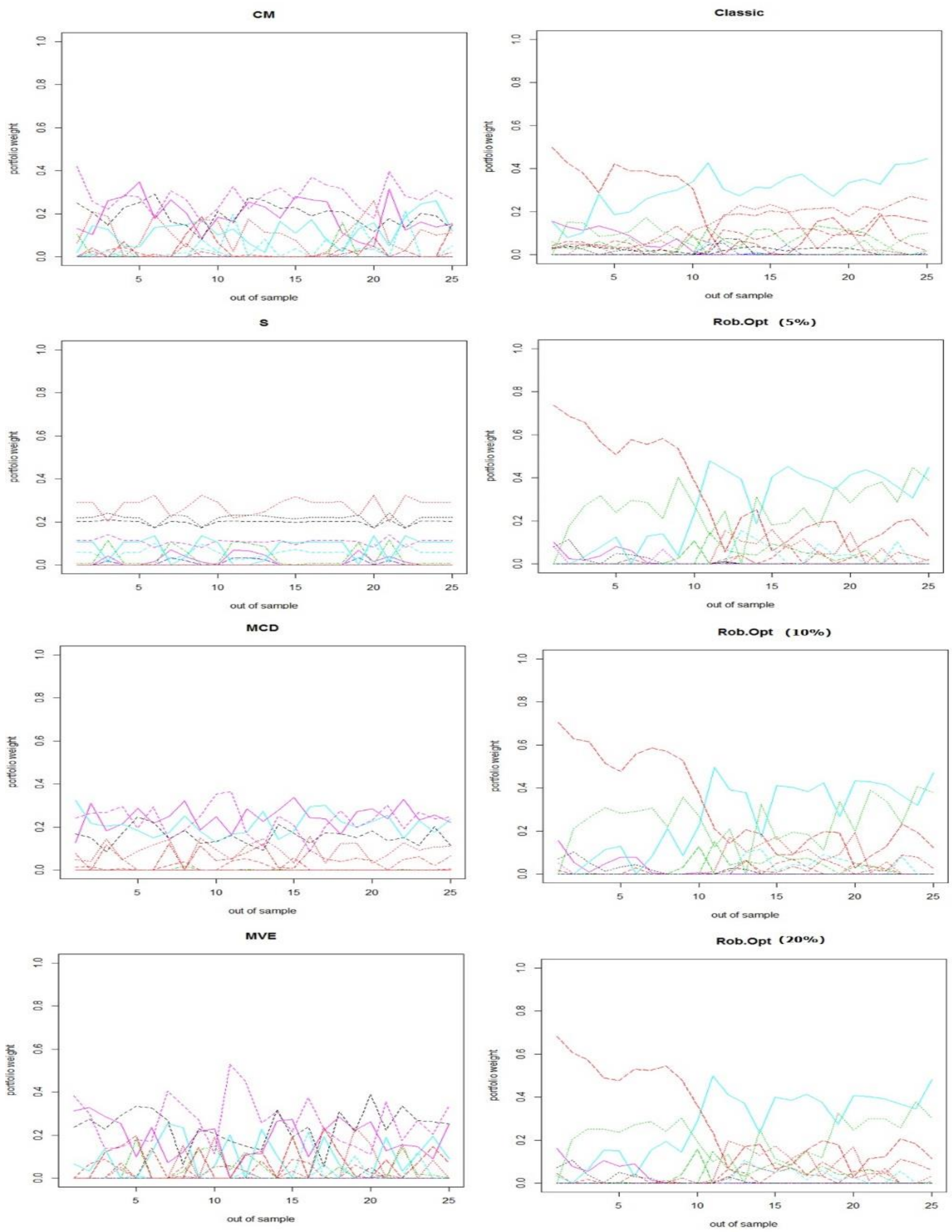


**Figure 1.** Time-varying portfolio weights for classical portfolio and robust portfolios for the case of $\gamma = 10$

Plotted in Figure 1 are the time-varying portfolio weights for the classical portfolio, robust portfolio optimization (right column of graphs), and robust portfolio estimation (left column of graphs) at risk aversion is equal to 10. All of the eight graphs map the time window *win* on the *x*-coordinate, while the *y*-coordinate maps the portfolio weights. Other risk aversion parameters were tested, such as $\gamma = 1$, 100, and 1000, but the insights from the results were similar, and thus the results are presented only for the case $\gamma = 10$.

It can be seen that Figure 1 corroborates the main findings by showing the high instability associated to the time-varying portfolio weights (compositions) of classic and Rob.Opt in contrast to the relative stability in the composition of Rob.Est.

Figure 2 gives the boxplots of the portfolio weights of classical portfolio, robust portfolio estimation, and robust portfolio optimization for the case of $\gamma = 10$.

Each graph in Figure 2 contains 20 boxplots corresponding to each of the twenty assets (for detail, see Table 1). Finally, the box for each portfolio weight has lines at the 25$^{th}$, 50$^{th}$, and 75$^{th}$ percentile values of the portfolio weights. The whiskers are lines extending from each end of the boxes to show the extent of the rest of the data. Extreme portfolio weights that have values beyond the whiskers are also depicted (as indicated by the white circles). We have tested other risk aversion parameters, such as $\gamma = 1$, 100 and 1000, but the insights from the results were similar and thus the results are presented only for the case $\gamma = 10$.

It can be observed from Figure 2 that the mean-variance portfolios (classical and Rob.Opt) are much more unstable than the Rob.Est portfolios. For instance, for $\gamma = 10$, it can be seen that the Rob.Opt portfolios generated using $\alpha = 5\%$ concentrate the allocation in only five assets of twenty available, and the allocation between these five assets radically changed in the period analyzed (see the second row of the second column in Figure 2). This is reflected in the high portfolio turnover as achieved by Rob.Opt (2.0235). As in the previous strategy, the changes in the portfolio weights associated to the Rob.Est were more stable over time since it produced little turnover.

A further step in the analysis was to check which observations are considered outliers and were responsible for this instability of the portfolios. To do so, we used a diagnostic tool called Mahalanobis distance. Briefly, the Mahalanobis distance can identify which observations are quite far from the bulk of data to be considered outliers (Werner, 2003).

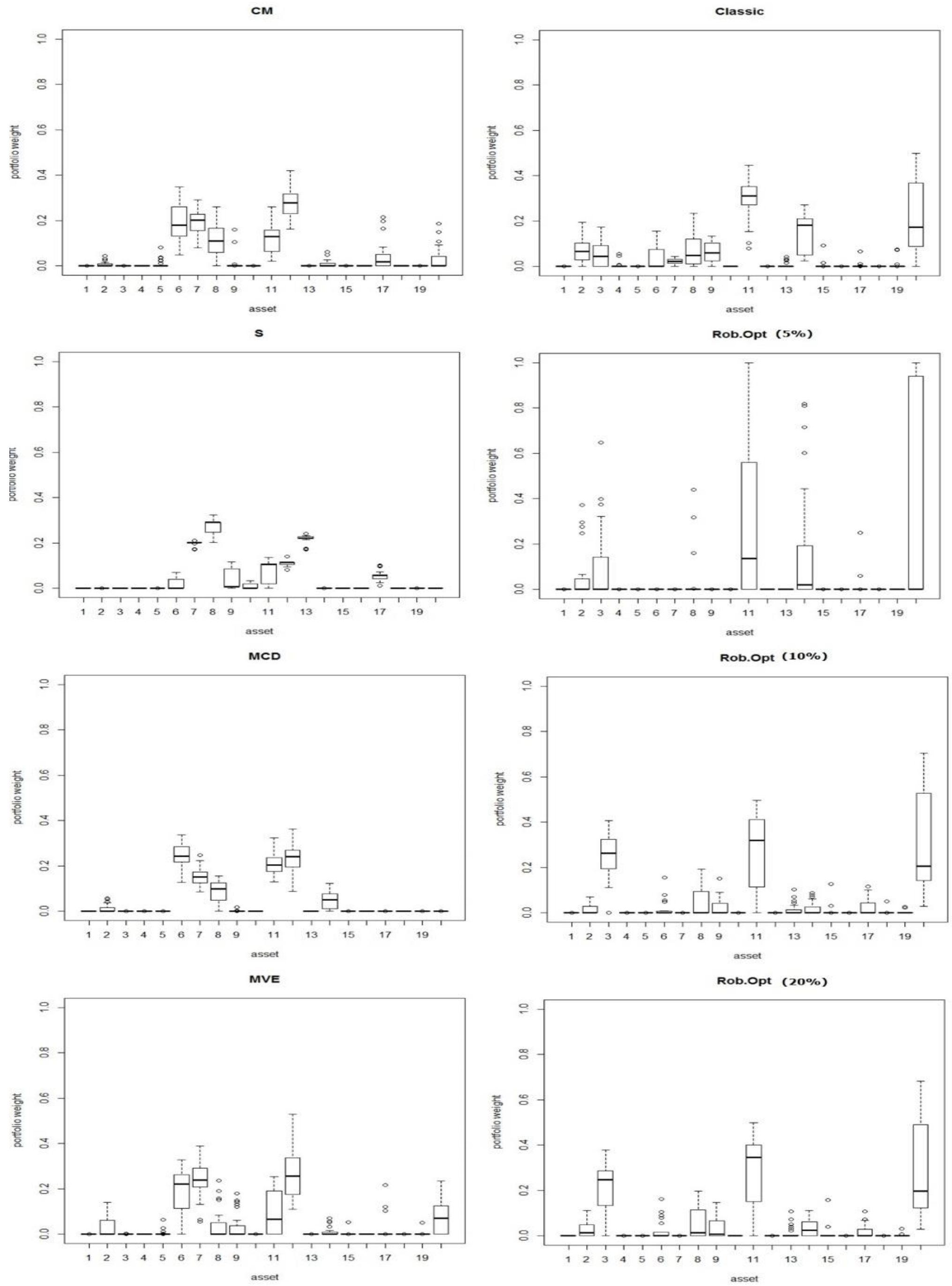


**Figure 2.** Boxplots of the portofolio weights for classical portfolio and robust portfolios for the case of $\gamma = 10$

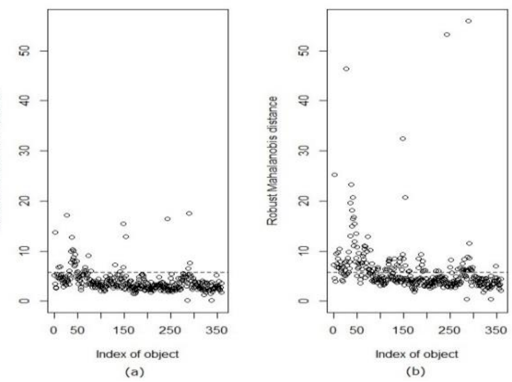


**Figure 3.** Mahalanobis distance of each of the 360 returns

Figure 3 shows the Mahalanobis distances of data using classical estimators in panel (a) and MCD estimators in panel (b). It is found that both pictures exhibited extreme return observations compared to the majority. They were detected to have a very strong influence on the classical estimates of the optimal portfolio weights (compositions). In short, it has been found that the outlying observations in the data have a strong influence on the composition of the resulting optimal portfolios.

In summary, the robust techniques lead to an improvement compared to the classical approach. Of the robust approaches, the robust estimation clearly outperforms the robust optimization approach. This improvement is possible due to the properties of robust estimator, which is not influenced by the presence of outliers.

## Conclusion

In this work, two different robust techniques, robust estimation and robust optimization, have been empirically tested and compared with a classical approach. From the results presented in the previous section, some important implications for investment decisions based on portfolio selection policies can be pointed out.

Based on an empirical analysis, it is shown that the robust portfolio estimation (Rob.Est) significantly outperformed the classical portfolio and robust portfolio optimization in terms of out-of-sample performance, i.e. mean excess return, risk, Sharpe ratio, and portfolio turnover, in the majority of the scenarios. The portfolio compositions of Rob.Est are shown to be more stable and consequently lead to a reduction of the transaction cost. This is simply because robustly estimated parameters will be closer to the true parameter values when there are some extreme observations (outliers) than their classical counterparts. Meanwhile, the portfolio compositions of Rob.Opt are heavily biased as this method works on a worst-case approach, so it can be detrimentally influenced by outliers in the data

Therefore, in this case, of the robust approaches the robust estimation clearly outperforms the robust optimization approach. In future research, the robust estimation should be combined with robust optimization in the formation of the optimal portfolio.

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