Journal of Modern Applied Statistical Methods

Volume 16 | Issue 2 Article 21

December 2017

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Recommended Citation

Grine, R., & Zeghdoudi, H. (2017). On Poisson Quasi-Lindley Distribution and its Applications. Journal of Modern Applied Statistical Methods, 16(2), 403-417. doi: 10.22237/jmasm/1509495660

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Cover Page Footnote
The authors acknowledge Editor for the constant encouragement to finalize the paper. Further, the authors acknowledge profound thanks to an anonymous referee for giving critical comments which have immensely improved the presentation of the paper.

Journal of Modern Applied Statistical Methods
November 2017, Vol. 16, No. 2, 403-417.
doi: 10.22237/jmasm/1509495660

ISSN 1538 − 9472

On Poisson Quasi-Lindley Distribution and its Applications

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This paper proposes a recent version of compound Poisson distributions named the Poisson quasi-Lindley (PQL) distribution by compounding Poisson and quasi-Lindley distributions. Some properties of the distributions are given with estimation and some illustrative examples.

Keywords: Lindley distribution, Poisson distribution, Poisson-Lindley distribution, gamma Lindley distribution, maximum-likelihood estimation

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Introduction

Statistical distributions are commonly applied to describe real-world phenomena and are most frequently used in different fields such as medicine, finance, biological engineering sciences, and actuarial science. The one-parameter Lindley distribution is used in modeling lifetime data, and appears to perform well. To obtain it, let X be a random variable following the one-parameter distribution with the density function (Lindley, 1958)

$$\mathrm{f}\left(x,\theta\right)=\begin{cases}\dfrac{\theta^2\left(1+x\right)\mathrm{e}^{-\theta x}}{1+\theta}, & x,\theta>0\\ 0, & \text{otherwise}\end{cases} \tag{1}$$

Sankaran (1970) used (1) assuming that the parameter of a Poisson distribution has Lindley distribution, and it was named the Poisson- Lindley distribution.

Asgharzadeh, Bakouch, and Esmaeili (2013), Ghitany, Al-Mutairi, and Nadarajah (2008), and Ghitany, Atieh, and Nadarajah (2008) studied the new

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distribution bounded to (1) and derived the zero-truncated Poisson-Lindley and Pareto Poisson-Lindley distributions. Sankaran (1970) introduced the discrete Poisson-Lindley distribution by combining the Poisson and Lindley distributions. Mahmoudi and Zakerzadeh (2010) proposed an extended version of the compound Poisson distribution, which was obtained by compounding the Poisson distribution with the generalized Lindley distribution, which was further analyzed by Zakerzadeh and Dolati (2009). Zeghdoudi and Nedjar (2016a) and Shanker and Mishra (2013) introduced the pseudo-Lindley and quasi-Lindley distributions, based on mixtures of gamma (2, θ) and exponential (θ) distributions, where the density function of the random variable X is given by

$$\mathrm{f}_{\mathrm{QL}}\left(x;\theta,\alpha\right)=\begin{cases}\dfrac{\theta\left(\alpha+\theta x\right)\mathrm{e}^{-\theta x}}{\alpha+1}, & x,\theta>0;\alpha>-1\\ 0, & \text{otherwise}\end{cases}\tag{2}$$

Zeghdoudi and Nedjar (2016b, 2017) proposed compound Poisson distributions, named the Poisson Gamma Lindley (PGaL) distribution and Poisson pseudo-Lindley, by compounding Poisson and gamma Lindley (pseudo-Lindley) Nedjar and Zeghdoudi (2016a, 2016b) distributions. The purpose of this study is to introduce a new lifetime distribution by compounding Poisson and quasi-Lindley distributions, which may be useful in modeling lifetime data and biological sciences.

Poisson Quasi-Lindley Distribution

Consider $d\mathrm{F}(\lambda) = \mathrm{e}^{\lambda\Phi}\mathrm{h}(\lambda)\mathrm{B}(\Phi)d\lambda$, where $\mathrm{h}(\lambda) = \alpha + \theta\lambda$ and $\mathrm{B}(\Phi) = -\Phi / (\alpha + 1)$, then the compound Poisson distribution is (Sankaran, 1970)

$$\begin{aligned}\mathrm{P}_x\left(\Phi\right)&=\int_0^\infty \frac{\mathrm{e}^{-\lambda}\lambda^x}{x!}\,d\,\mathrm{F}\left(\lambda\right)\\ &=\frac{\mathrm{B}\left(\Phi\right)}{x!}\left[\alpha\int_0^\infty \mathrm{e}^{(\Phi-1)\lambda}\lambda^x d\lambda+\left(-\Phi\right)\int_0^\infty \mathrm{e}^{(\Phi-1)\lambda}\lambda^{x+1}d\lambda\right]\\ &=\frac{-\Phi}{\alpha+1}\left(\frac{\alpha\left(1-\Phi\right)-\Phi\left(x+1\right)}{\left(1-\Phi\right)^{x+2}}\right)\end{aligned}\tag{3}$$

Then, replace Φ with $-\theta$:

$$\mathrm{P}_x(\theta) = \frac{\theta}{\alpha+1}\left(\frac{\alpha(1+\theta)+\theta(x+1)}{(1+\theta)^{x+2}}\right) \tag{4}$$

Now, the density function of Poisson quasi-Lindley (PQL) is given by

$$\mathrm{f}_{\mathrm{PQL}}(x;\alpha,\theta) = \frac{\theta(\alpha+\theta+\alpha\theta+\theta x)}{(1+\alpha)(1+\theta)^{x+2}}, \quad x = 0,1,\ldots,\theta > 0, \alpha > -1 \tag{5}$$

Remark 1: If $\alpha = \theta$, this distribution is the Poisson-Lindley distribution.

The first and second derivatives of $\mathrm{f}_{\mathrm{PQL}}(x)$ are

$$\frac{d}{dx}\mathrm{f}_{\mathrm{PQL}}(x) = -\frac{\theta\left(\theta\ln(\theta+1)-\theta+\alpha\ln(\theta+1)+\theta\alpha\ln(\theta+1)+x\theta\ln(\theta+1)\right)}{(\theta+1)^{x+2}(\alpha+1)} \tag{6}$$

and

$$\begin{aligned}&\frac{d^2}{dx^2}\mathrm{f}_{\mathrm{PQL}}(x)\\&= -\frac{\theta\ln(\theta+1)\left[\theta\ln(\theta+1)-2\theta+\alpha\ln(\theta+1)+\theta\alpha\ln(\theta+1)+x\theta\ln(\theta+1)\right]}{(\theta+1)^{x+2}(\alpha+1)}\end{aligned} \tag{7}$$

When $\frac{d}{dx}\mathrm{f}_{\mathrm{PQL}}(x) = 0$, the solution is

$$\hat{x} = \frac{1}{\ln(\theta+1)} - \frac{(\theta+\alpha+\theta\alpha)}{\theta}$$

and

$$\frac{d^2}{dx^2}\mathrm{f}_{\mathrm{PQL}}(\hat{x}) = \frac{-\theta^2\ln(\theta+1)(\theta+1)^{\frac{1}{\theta\ln(\theta+1)}\left[\theta\ln(\theta+1)-\theta+\alpha\ln(\theta+1)+\theta\ln(\theta+1)\right]-2}}{(\alpha+1)} < 0$$

For θ, α, $\hat{x} > 0$,

$$\hat{x} = \frac{1}{\ln(\theta+1)} - \frac{(\theta+\alpha+\theta\alpha)}{\theta}$$

is the unique critical point at which $f_{PQL}(x;\theta,\alpha)$ is maximum and $f_{PQL}(x)$ is concave. But if $\hat{x} < 0$, the density function $f_{PQL}(x)$ is decreasing in x. Therefore, the mode of PQL is given by

$$\text{Mode}(X) = \begin{cases} \dfrac{1}{\ln(\theta+1)} - \dfrac{(\theta+\alpha+\theta\alpha)}{\theta}, & \forall \theta > 0, \alpha > -1 \\ 0, & \text{otherwise} \end{cases} \tag{8}$$

The cumulative distribution function (cdf) of the PQL is

$$F_{PQL}(x) = 1 - \frac{\alpha + 2\theta + \alpha\theta + \theta x + 1}{(1+\alpha)(1+\theta)^{x+2}}, \quad x = 0,1,\ldots,\theta > 0, \alpha > -1 \tag{9}$$

The plots of density and distribution for some value of α and θ are given in Figures 1 and 2.

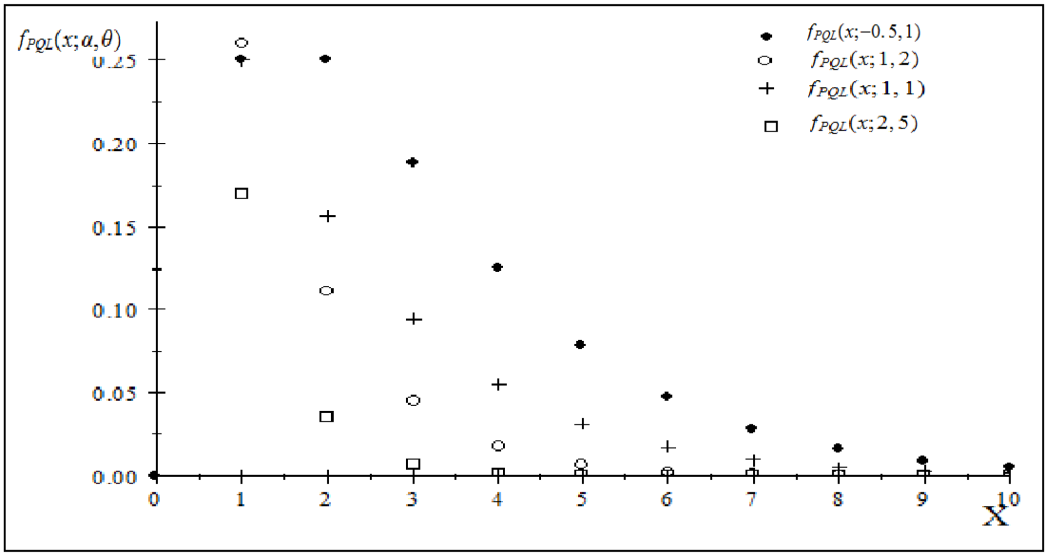

Figure 1. Plots of the density function for some parameter values

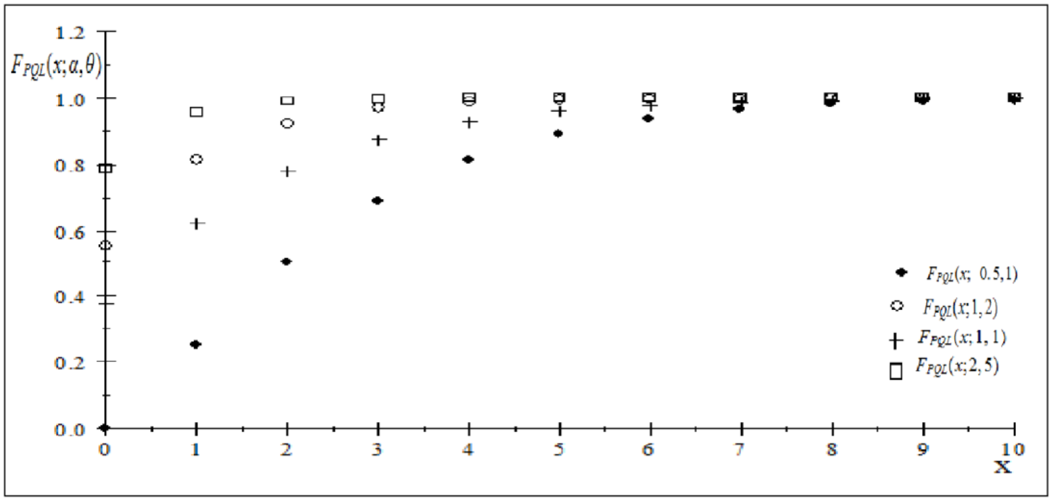

Figure 2. Plots of the distribution function for some parameter values

Survival and Hazard Rate Function

Let

$$\mathrm{S}_{\mathrm{PQL}}(x) = 1 - \mathrm{F}_{\mathrm{PQL}}(x) = \frac{\alpha + 2\theta + \alpha\theta + \theta x + 1}{(1+\alpha)(1+\theta)^{x+2}} \tag{10}$$

and

$$\mathrm{h}_{\mathrm{PQL}}(x) = \frac{\mathrm{f}_{\mathrm{PQL}}(x)}{1 - \mathrm{F}_{\mathrm{PQL}}(x)} = \frac{\theta(\alpha + \theta + \alpha\theta + \theta x)}{\alpha + 2\theta + \alpha\theta + \theta x + 1} \tag{11}$$

be the survival and hazard rate function of PQL, respectively.

Proposition 1: Let $\mathrm{h}_{\mathrm{PQL}}(x)$ be the hazard rate function of X. Then $\mathrm{h}_{\mathrm{PQL}}(x)$ is increasing.

Proof: According to Glaser (1980) and from the density function of PQL,

$$\begin{aligned}\rho(x) &= \frac{f'_{\mathrm{PQL}}(x)}{f_{\mathrm{PQL}}(x)} \\ &= \frac{\theta\ln(\theta+1)-\theta+\alpha\ln(\theta+1)+\theta\alpha\ln(\theta+1)+x\theta\ln(\theta+1)}{\alpha+\theta+\alpha\theta+\theta x}\end{aligned} \tag{12}$$

it follows that

$$\rho'(x)=\frac{\theta^2}{(\theta+\alpha+x\theta+\theta\alpha)^2}>0$$

∀ x, α, θ, implying that $h_{\mathrm{PQL}}(x)$ is increasing.

Maximum Likelihood Estimates

Consider the point estimation of the parameters that index the PQL(θ, α). Let the log-likelihood function of a single observation (say x_i) for the vector of parameters (θ, α) be written as

$$\begin{aligned}&\ln l(x;\alpha,\theta) \\ &= \ln\theta+\ln(\alpha+\theta+\alpha\theta+\theta x)-\ln(1+\alpha)-2\ln(1+\theta)-x\ln(1+\theta)\end{aligned} \tag{13}$$

The derivatives of ln l(x; θ, α) with respect to θ and α are:

$$\frac{\partial \ln l(x;\alpha,\theta)}{\partial\theta}=\frac{1}{\theta}+\left(\frac{1+\alpha+x}{\alpha+\theta+\alpha\theta+\theta x}\right)-\frac{2+x}{1+\theta} \tag{14}$$

$$\frac{\partial \ln l(x;\alpha,\theta)}{\partial\alpha}=\left(\frac{1+\theta}{\alpha+\theta+\alpha\theta+\theta x}\right)-\frac{1}{1+\alpha} \tag{15}$$

The maximum likelihood estimators $\hat{\theta}$ of θ and $\hat{\alpha}$ of α are obtained by solving non-linear equations

$$\hat{\theta}=\frac{1}{x},\quad \hat{\alpha}=\frac{-x}{1+x} \tag{16}$$

and

$$\begin{cases} \mathrm{E}\left(\hat{\theta}\right) = \dfrac{\theta\left(\alpha+\theta+\alpha\theta\right)}{\left(1+\alpha\right)\left(1+\theta\right)^3} \\ -1 \le \mathrm{E}\left(\hat{\alpha}\right) \le 0 \end{cases} \tag{17}$$

Proposition 2: Let $X_1, X_2, \ldots, X_n$ be n independent random variables from the PQL(α, θ) distribution. Then the moment generating function (mgf) of $S = \Sigma_{i=1}^{n} X_i$ is given by

$$\mathrm{M_S}\left(t\right) = \frac{\left(\alpha\theta \mathrm{e}^t + \theta^2\right)^n}{\left(1+\alpha\right)^n \left(\mathrm{e}^t + \theta \mathrm{e}^t - \mathrm{e}^{2t}\right)^n} \tag{18}$$

and

$$\mathrm{M_X}\left(t\right) = \mathrm{E}\left(\mathrm{e}^{tX}\right) \frac{\alpha\theta \mathrm{e}^t + \theta^2}{\left(1+\alpha\right)\left(\mathrm{e}^t + \theta \mathrm{e}^t - \mathrm{e}^{2t}\right)} \tag{19}$$

Proof: The mgf of X is

$$\mathrm{M_X}\left(t\right) = \mathrm{E}\left(\mathrm{e}^{tX}\right) \frac{\alpha\theta \mathrm{e}^t + \theta^2}{\left(1+\alpha\right)\left(\mathrm{e}^t + \theta \mathrm{e}^t - \mathrm{e}^{2t}\right)}$$

According to (19) and using the independent random variables $X_1, X_2, \ldots, X_n$, the mgf of $S = \Sigma_{i=1}^{n} X_i$. Also, successive derivation is used and, by recurrence, find (18).

Corollary 1: Let $X \sim$ PQL(θ, α). Then the mean and variance for X are

$$\mathrm{E}\left(X\right) = \frac{2+\alpha}{\left(1+\alpha\right)\theta} \tag{20}$$

$$\mathrm{V}\left(X\right) = \frac{2+4\alpha+\alpha^2+\theta\left(\alpha+2\right)\left(1+\alpha\right)}{\left(1+\alpha\right)^2 \theta^2} \tag{21}$$

Proof: $\mathrm{E}\left(X\right)=\mathrm{M}'_X\left(t=0\right)$, $\mathrm{E}\left(X^2\right)=\mathrm{M}''_X\left(t=0\right)$, and V($X$) = E($X^2$) – E^2($X$). Then

$$\begin{aligned}\mathrm{E}\left(X\right)&=\frac{2+\alpha}{\left(1+\alpha\right)\theta}\\ \mathrm{E}\left(X^2\right)&=\frac{6+2\theta+\alpha\left(\theta+2\right)}{\left(1+\alpha\right)\theta^2}\end{aligned}\tag{22}$$

which achieves the proof.

Moments Estimates

Using the first moment m and second moment m_2 about the PQL distribution, we have

$$\begin{cases} m=\dfrac{2+\alpha}{\left(1+\alpha\right)\theta}\\ m_2=\dfrac{6+2\theta+\alpha\left(2+\theta\right)}{\left(1+\alpha\right)\theta^2}\end{cases}\tag{23}$$

where $m_2 = S^2 + m^2$ and S^2 is the variance. Solve this non-linear system and find the couple (θ, α), where $(\theta, \alpha) \succ 0$ for all $S \succ 0$, $m \succ 0$. The solving of the non-linear system (23) gives

$$\left(m_2-m\right)\theta^2-4m\theta+2=0 \quad \text{and} \quad \alpha=\frac{2-\theta m}{\theta m-1}\tag{24}$$

The solution of $(m_2 - m)\theta^2 - 4m\theta + 2 = 0$ is

$$-\frac{1}{m-m_2}\left(2m+\sqrt{2}\sqrt{m-m_2+2m^2}\right) \quad \text{if } m-m_2\neq 0\tag{25}$$

because

$$\hat{\theta}=-\frac{1}{m-m_2}\left(2m+\sqrt{2}\sqrt{m-m_2+2m^2}\right) \quad \text{and} \quad \hat{\alpha}=\frac{2-\hat{\theta}m}{\hat{\theta}m-1}\tag{26}$$

The Quantile Function of the Poisson Quasi-Lindley Distribution

Lambert *W* Function

The Lambert *W* function is a standard due to its implementation in the computer algebra system Maple in the 1980s (Conte & de Boor, 1980) and, subsequently, Corless, Gonnet, Hare, Jeffrey, and Knuth (1996) provided a comprehensive survey of the history, theory, and applications of this function. The Lambert *W* function is defined as the solution of the equation:

$$W\left(z\right)\exp\left(W\left(z\right)\right)=z,\quad z \text{ is a complex number} \tag{27}$$

If z is a real number such that $z \geq -1 / \mathrm{e}$ then $W(z)$ becomes a real function and there are two possible real branches. The real branch taking on values in $(-\infty, -1]$ is called the negative branch and denoted by W_{-1}. The real branch taking on values in $[-1, \infty)$ is called the principal branch and denoted by W_0. Equation (8) has two possible solutions if $z \in (-1 / \mathrm{e}, 0]$ and a unique solution if $z \geq 0$. For the results in this note, use the negative branch W_{-1}, which satisfies the following elementary properties: $W_{-1}(-1 / \mathrm{e}) = -1$, $W_{-1}(z)$ decreasing as z increases to 0, and $W_{-1}(z) \to -\infty$ as $z \to 0$.

Lemma 1: Let a, b, c, and d be fixed complex numbers. The solution of the equation $z + abz = c$ with respect to $z \in \mathbb{C}$ is

$$z=c-\frac{1}{d\ln\left(b\right)}W\left(ab^{c}\ln\left(b\right)\right) \tag{28}$$

For details of the proof, see Jodrá (2010).

The quantile function of X is $\mathrm{Q}_X\left(u\right)=\mathrm{F}_X^{-1}\left(u\right), 0 \prec u \prec 1$. An explicit expression for Q_X in terms of the Lambert *W* function follows.

Theorem 1: For any θ, $\alpha \succ 0$, the quantile function of the PQL distribution X is

$$\mathrm{Q}_X\left(u\right)=-\frac{\alpha+2\theta+\alpha\theta+1}{\theta}-\frac{1}{\ln\left(1+\theta\right)}W_{-1}\left(\frac{\ln\left(1+\theta\right)}{\theta\left(1+\theta\right)^{\frac{\alpha+\alpha\theta+1}{\theta}}}\left(u-1\right)\right) \tag{29}$$

where W_{-1} denotes negative branch of Lambert W function.

Proof: For any fixed θ, α, let $u \in (0, 1)$.We have to solve the equation $F_X(x) = u$ with respect to x, for $x \succ 0$. Solve the following equation; the first quantiles are obtained by substituting u = 1/4, 1/2, 3/4 in equation (29)

$$\begin{aligned} Q_1 &= F^{-1}\left(\frac{1}{4},\theta,\alpha\right) \\ &= -\frac{\alpha+2\theta+\alpha\theta+1}{\theta}-\frac{1}{\ln(1+\theta)}W_{-1}\left(\frac{\ln(1+\theta)}{\theta(1+\theta)^{\frac{\alpha+\alpha\theta+1}{\theta}}}\left(\frac{1}{4}-1\right)\right) \end{aligned} \tag{30}$$

$$\begin{aligned} Q_2 &= F^{-1}\left(\frac{1}{2},\theta,\alpha\right) \\ &= -\frac{\alpha+2\theta+\alpha\theta+1}{\theta}-\frac{1}{\ln(1+\theta)}W_{-1}\left(\frac{\ln(1+\theta)}{\theta(1+\theta)^{\frac{\alpha+\alpha\theta+1}{\theta}}}\left(\frac{1}{2}-1\right)\right) \end{aligned} \tag{31}$$

$$\begin{aligned} Q_3 &= F^{-1}\left(\frac{3}{4},\theta,\alpha\right) \\ &= -\frac{\alpha+2\theta+\alpha\theta+1}{\theta}-\frac{1}{\ln(1+\theta)}W_{-1}\left(\frac{\ln(1+\theta)}{\theta(1+\theta)^{\frac{\alpha+\alpha\theta+1}{\theta}}}\left(\frac{3}{4}-1\right)\right) \end{aligned} \tag{32}$$

Illustrative Examples

Example 1

Shown in Table 1 are some quantile of the PQL distribution, which were calculated from the closed-form expression for $Q_X(u)$ given in Theorem 1.

Table 1. Quantile of the PQL distribution

u	$\theta = 0.1, \alpha = 0.1$	$\theta = 0.1, \alpha = 0.5$	$\theta = 3.0, \alpha = 1.0$	$\theta = 5.0, \alpha = 1.0$
0.01	3.28840	7.87570	-0.33136	-0.50238
0.05	4.42420	8.60620	-0.29346	-0.47390
0.10	5.76500	9.54370	-0.24395	-0.43667
0.15	7.06270	10.51300	-0.19181	-0.39743
0.25	9.63630	12.56900	-7.83450×10^{-2}	-0.31190
0.30	10.94900	13.67100	-1.61900×10^{-2}	-0.26497
0.35	12.30000	14.83200	5.02850×10^{-2}	-0.21474
0.40	13.70500	16.06400	0.12176	-0.16067
0.45	15.18000	17.38100	0.19911	-0.10211
0.50	16.74500	18.79900	0.28342	-3.82080×10^{-2}
0.55	18.42500	20.34100	0.37617	3.21560×10^{-2}
0.60	20.25200	22.03700	0.47930	0.11049
0.65	22.26900	23.92800	0.59557	0.19889
0.70	24.53900	26.07600	0.72901	0.30046
0.75	27.15700	28.57300	0.88581	0.41995
0.80	30.28300	31.57700	1.07530	0.56534
0.85	34.21100	35.37800	1.32000	0.75150
0.90	39.59600	40.62500	1.66010	1.01170
0.95	48.50800	49.36800	2.23390	1.45160

Table 2. Mode, mean, and median for PQL

	$\theta = 0.01, \alpha = 0.10$	$\theta = 0.10, \alpha = 0.50$	$\theta = 0.05, \alpha = 1.00$
Q_1	104.660	12.569	30.601
Median = Q_2	172.920	18.799	41.819
Q_3	272.770	28.573	60.008
Mean	89.3990	3.9921	-1.5041
Mode	190.910	16.667	30.000

Displayed in Table 2 are the mode, mean and median for PQL distribution for different choices of parameters θ and α.

Simulation

The behavior of the MM estimators are examined for a finite sample size (n). A simulation study consisting of following steps is being carried out for each triplet (θ, α; n), where $\theta = 0.01$, $\alpha = 0.1, 0.01, 1$ and for $\alpha = 0.5$, $\theta = 0.05, 1, 5$, and $n = 10, 30, 50$. The steps are:

- Choose the initial values of θ_0, α_0 for the corresponding elements of the parameter vector $\mathbf{\Theta} = (\theta, \alpha)$ to specify PQL distribution;
- Choose sample size n;
- Generate N independent samples of size n from PQL (θ, α);
- Compute the MM estimate $\mathbf{\Theta}_n$ of $\mathbf{\Theta}_0$ for each of the N samples;
- Compute the mean of the obtained estimators over all N samples.

Note the

$$\text{average bias}\left(\theta\right)=\frac{1}{N}\sum_{i=1}^{n}\left(\mathbf{\Theta}_i-\mathbf{\Theta}_0\right) \tag{33}$$

and the average square error

$$\text{MSE}\left(\theta\right)=\frac{1}{N}\sum_{i=1}^{n}\left(\mathbf{\Theta}_i-\mathbf{\Theta}_0\right)^2 \tag{34}$$

Table 3. Average bias of the simulated estimates

	$\theta = 0.01$, $\alpha = 0.1$		**$\theta = 0.01$, $\alpha = 0.01$**		**$\theta = 0.01$, $\alpha = 1$**	
n	**bias(θ)**	**bias(α)**	**bias(θ)**	**bias(α)**	**bias(θ)**	**bias(α)**
10	0.000025560	-0.005612900	0.000002500	-0.000505090	0.000264000	-0.088393000
30	0.000008520	-0.001870960	0.000000833	-0.000168363	0.000088000	-0.029464330
50	0.000005112	-0.001122580	0.000000500	-0.000101018	0.000052800	-0.017678600

	$\theta = 0.05$, $\alpha = 0.5$		**$\theta = 1$, $\alpha = 0.5$**		**$\theta = 5$, $\alpha = 0.5$**	
n	**bias(θ)**	**bias(α)**	**bias(θ)**	**bias(α)**	**bias(θ)**	**bias(α)**
10	0.000684200	-0.038232000	0.027430000	-0.061022000	0.575600000	-0.111320000
30	0.000298060	-0.012744000	0.009143333	-0.020340660	0.191866600	-0.037106600
50	0.000136840	-0.007646400	0.005486000	-0.012204400	0.115120000	-0.022264000

Shown in Table 3, $\hat{\theta}$ is positively biased with bias(θ) → 0 for $\theta \to 0$, and $\hat{\alpha}$ is negatively biased with bias(α) → 0 for $\alpha \to 0$. Shown in Table 4, MSE(θ) and MSE(α) → 0 where $\theta \to 0$ and $n \to \infty$.

Example 2

Shown in Table 5 are some distributions of copying groups of random digits with expected frequencies obtained by fitting the Poisson, Poisson-Lindley, and PQL distributions.

Table 4. Average MSE of the simulated estimates

	$\theta = 0.01$, $\alpha = 0.1$		**$\theta = 0.01$, $\alpha = 0.01$**		**$\theta = 0.01$, $\alpha = 1$**	
n	**MSE(θ)**	**MSE(α)**	**MSE(θ)**	**MSE(α)**	**MSE(θ)**	**MSE(α)**
10	0.0000000066	0.0003150460	6.3×10^{-11}	0.0000025511	0.0000006970	0.0781332240
30	0.0000000022	0.0001050150	2×10^{-11}	0.0000008504	0.0000002323	0.0260444080
50	0.0000000013	0.0000630090	10^{-11}	0.0000005102	0.0000001394	0.0156266440

	$\theta = 0.05$, $\alpha = 0.5$		**$\theta = 1$, $\alpha = 0.5$**		**$\theta = 5$, $\alpha = 0.5$**	
n	**MSE(θ)**	**MSE(α)**	**MSE(θ)**	**MSE(α)**	**MSE(θ)**	**MSE(α)**
10	0.0000046813	0.0146168500	0.0075240400	0.0372368400	3.3131536000	0.1239214240
30	0.0000015604	0.0048722800	0.0025080160	0.0124228000	1.1043845300	0.0413071410
50	0.0000009363	0.0029233700	0.0015048090	0.0074473600	0.6626307200	0.0247842840

Table 5. Comparison between Poisson, Poisson-Lindley, and Poisson quasi-Lindley distributions

No. of errors per group	**Obs. freq.**	**Poisson**	**Poisson-Lindley**	**PQL**
	$m = 0.9$, $n = 2.451$	**$\hat{\theta} = 1.08$**	**$\hat{\theta} = 1.547$**	**$\hat{\theta} = 1.398$, $\hat{\beta} = 0.786$**
0	35	25.207	31.856	32.152
1	11	22.686	16.031	15.320
2	8	10.209	7.677	7.602
3	4	3.062	3.557	3.542
4	3	0.689	1.609	1.613
5	1	0.124	0.715	0.749
Total	62	62	62	62
χ^2	-	24.524	3.271	3.216

Conclusion

A new two-parameter distribution is proposed, referred to as the PQL distribution, which contains the Poisson Lindley distribution as special case. Various properties of the distribution are examined including the density function (pdf), cumulative distribution (cdf), survival and hazard rate function, moment generating function (mgf), mean, variance, and some results. Also, maximum likelihood estimates and moment estimates are discussed. The PQL model was fitted to several real data sets to show the potential of the new proposed distribution. The PQL distribution gives a much closer fit than the Poisson and Poisson-Lindley distributions, and thus can be considered as an important tool for modeling lifetime data. This suggests that the new model provides more accurate estimates as well as better fits.

Acknowledgements

The authors acknowledge Editor for the constant encouragement to finalize the paper. Further, the authors acknowledge profound thanks to an anonymous referee for giving critical comments which have immensely improved the presentation of the paper.

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