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Efficient Class of Estimators for Finite Population Mean using Auxiliary Information in Two-Occasion Successive Sampling

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In the case of sampling on two occasions, a class of estimators is considered which uses information on the first occasion as well as the second occasion in order to estimate the population means on the current (second) occasion. The usefulness of auxiliary information in enhancing the efficiency of this estimation is examined through the class of proposed estimators. Some properties of the class of estimators and a strategy of optimum replacement are discussed. The proposed class of estimators were empirically compared with the sample mean estimator in the case of no matching. The established optimum estimator, which is a linear combination of the means of the matched and unmatched portions of the sample at the current occasion, was empirically compared with the proposed class of estimators. Mutual comparisons of the proposed estimator were carried out. Suitable recommendations are made to the survey statistician for practical applications.

Keywords: Successive sampling, auxiliary information, bias, mean square error, optimum replacement strategy

Introduction

The theory of successive sampling has a long and rich history in sample surveys. When the value of the characteristics of interest of a finite population are subject to change over time, a survey carried out on a single occasion gives an idea only about the given occasion and can’t give an idea about the (a) nature or rate of change of the characteristic at different occasions and (b) the average value of the characteristic over all occasions or most the recent occasions. Successive sampling, however, provides a tool for determining robust estimates. Successive (rotation)

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sampling was first coined by Jessen (1942) in the analysis of a survey on farm data. This theory was expanded by Patterson (1950), Eckler (1955), Cochran (1977), Gupta (1979), Das (1982), and Chaturvedi and Tripathi (1983).

Sen (1971) used this concept in designing the estimator of the current population mean on the current occasion by using information on two auxiliary variables which were easily available on the previous occasion. Sen (1972, 1973) extended this work for more than two auxiliary variables on the current occasion. V. K. Singh, Singh, and Shukla (1991) and G. N. Singh and Singh (2001) used the auxiliary information on the current occasion and proposed an estimator for the current population in two-occasion successive sampling. G. N. Singh (2003) extended this methodology for h-occasions successive sampling in the estimation of the current population mean in two-occasion successive sampling.

Sometimes, information on an auxiliary variable may be readily available on the first as well as the second occasion; for example, tonnage (or seat capacity) of each vehicle or ship is known in transportation surveys. More examples are available in survey literature, and may be used where the information on auxiliary variables are available on both occasions in two occasions successive sampling. Feng and Zou (1997), Biradar and Singh (2001), G. N. Singh (2005), G. N. Singh and Priyanka (2008), G. N. Singh and Karna (2009), H. P. Singh and Vishwakarma (2009), G. N. Singh and Prasad (2010), G. N. Singh, Prasad, and Karna (2011), G. N. Singh, Homa, and Murya (2013), H. P. Singh et al. (2015), and H. P. Singh, Kim, and Tarray (2016), among others, used auxiliary information on both occasions for estimating the population mean on the current (second) occasion in two-occasion successive sampling.

In some situations, information on several auxiliary variables may be readily available or may be made easily available by diverting some amounts of funds available in surveys. For example, to study the case of public health and welfare of a state or country, several factors are known that can be treated as an auxiliary variable, such as the number of beds, the number of doctors and supporting staff in different hospitals, amount of funds available for medicine, etc., may be known. Similarly, there are several pieces of information available, which if utilized appropriately can be led to improve the precision of the suggested estimators. The purpose of this study is to propose some relevant class of estimators for estimating the current population mean in two-occasion sampling by utilizing the dynamic auxiliary variables available on both the occasions.

Formulation of the Proposed Estimators with Notation and Sample Structure

Let U = (U_1, U_2,…, U_N) be a finite population of size N which has been sampled over two occasions and the character under study be denoted by x (y) on the first (second) occasion. Assume the information on a varying auxiliary variable z_1 (z_2), whose population mean is known and is closely related to x and y, is available on the first (second) occasion. A simple random sample (without replacement) of n units is drawn on the first (second) occasion. When selecting the sample on the second (current) occasion, retain a (matched) random sub-sample of size $m = n\lambda$ from the sample selected on the first occasion and the remaining $u = n\mu$ units are replaced by a fresh selection under simple random sampling without replacement from the entire population so the sample size on the second (current) occasion is also n. λ and μ ($\lambda + \mu = 1$) are the fractions of the matched and fresh samples, respectively, on the second (current) occasion. Onwards, note the following notations:

$\bar{X}$, $\bar{Y}$, $\bar{Z}$: The population means of the variables x, y, and z, respectively.
$\bar{Z}_1$, $\bar{Z}_2$: The population means of the variables z_1 and z_2, respectively.
$\bar{x}_n$, $\bar{x}_m$, $\bar{y}_u$, $\bar{y}_m$, $\bar{z}_u$, $\bar{z}_m$, $\bar{z}_n$, $\bar{z}_{1n}$, $\bar{z}_{2u}$, $\bar{z}_{2m}$: The sample means of the respective variables based on the sample size shown in subscripts.
$C_x, C_y, C_z, C_{z_1}, C_{z_2}$: The coefficient of variation of the variables given in the subscripts.
$\beta_{yx}, \beta_{yz}, \beta_{xz}, \beta_{xz_1}, \beta_{xz_2}, \beta_{yz_1}, \beta_{yz_2}$: The population regression coefficient of the variables shown in the subscripts.
$\rho_{yx}, \rho_{yz}, \rho_{xz}, \rho_{yz_1}, \rho_{yz_2}, \rho_{xz_1}, \rho_{xz_2}, \rho_{z_1z_2}$: The correlation coefficients between the variables shown in the subscripts.
$S_x^2, S_y^2, S_z^2, S_{z_1}^2, S_{z_2}^2$: The population variances of the variables x, y, z, z_1, and z_2, respectively

Proposed Estimators

To estimate the population mean $\bar{Y}$ on the current (second) occasion, two sets of estimators are proposed. The first estimator is based on the sample size u (= $n\mu$) drawn afresh on the second occasion and second estimator is based on the matched sample of size m (= $n\lambda$) common to both occasions. The two sets of proposed estimators are defined below.

The estimator based on the fresh sample is defined as

$$T_{iu} = \bar{y}_u + k_i\left(\bar{Z}_2 - \bar{z}_{2u}\right); \quad i = 1, 2 \tag{1}$$

and, by setting $i = 1$ and $k_1 = b_{yz_2}$, equation (1) becomes

$$T_{1u} = \bar{y}_u + b_{yz_2}\left(\bar{Z}_2 - \bar{z}_{2u}\right). \tag{2}$$

By setting $i = 2$ and $k_2 = \bar{y}_u / \bar{z}_{2u}$, equation (1) becomes

$$T_{2u} = \frac{\bar{y}_u}{\bar{z}_{2u}}\bar{Z}_2 \tag{3}$$

The estimators based on the matched sample are defined as

$$T_{jm} = \bar{y}_m + b_{yx}\left(x_n^* - \bar{x}_n\right) + b_{yz_2}\left(\bar{Z}_2 - \bar{z}_{2m}\right), \tag{4}$$

where $\bar{x}_n^* = \bar{x}_n + k_j^*\left(\bar{Z}_1 - \bar{z}_{1n}\right)$, j = 1, 2, 3, 4.

Setting $j = 1$ and $k_1^* = 0$, equation (4) becomes

$$T_{1m} = \bar{y}_m + b_{yx}\left(\bar{x}_n - \bar{x}_m\right) + b_{yz_2}\left(\bar{Z}_2 - \bar{z}_{2m}\right). \tag{5}$$

Setting $j = 2$ and $k_2^* = b_{xz_1}$, equation (4) becomes

$$T_{2m} = \bar{y}_m + b_{yx}^m\left(\left(\bar{x}_n + b_{xz_1}^n\left(\bar{Z}_1 - \bar{z}_{1n}\right)\right) - \bar{x}_m\right) + b_{yz_2}^m\left(\bar{Z}_2 - \bar{z}_{2m}\right). \tag{6}$$

Setting $j = 3$ and $k_3^* = \bar{x}_n / \bar{z}_{1n}$, equation (4) becomes

$$T_{3m} = \bar{y}_m + b_{yx}^m\left(\frac{\bar{x}_n}{\bar{z}_{1n}}\bar{Z}_1 - \bar{x}_m\right) + b_{yz_2}^m\left(\bar{Z}_2 - \bar{z}_{2m}\right), \tag{7}$$

and by setting $j = 4$ and $k_3^* = \bar{x}_n / \bar{Z}_1$, equation (4) becomes

$$T_{4m} = \bar{y}_m + b_{yx}^m \left(\frac{\bar{x}_n}{\bar{Z}_1} \bar{z}_{1n} - \bar{x}_m \right) + b_{yz_2}^m \left(\bar{Z}_2 - \bar{z}_{2m} \right). \tag{8}$$

Considering the combination of the two sets of estimators T_{iu} (i = 1, 2) and T_{jm} (j = 1, 2, 3, 4), the final estimators of the population mean on the current occasion are

$$T_{ij} = \phi_{ij} T_{iu} + \left(1 - \phi_{ij}\right) T_{jm}, \quad \left(i = 1, 2 \quad j = 1, 2, 3, 4\right), \tag{9}$$

where the ϕ_{ij} are unknown constants to be determined so as to minimize the mean square error of the estimators T_{ij} under certain criterions.

Remark 1. For estimating the population mean on each occasion, of the estimators, two sets T_{iu} (i = 1, 2) are suitable, which implies that more belief on T_{iu} could be shown by choosing ϕ_{ij} (i = 1, 2) as 1 (or close to 1). For estimating the change from one occasion to the next, the estimator T_{jm} (j = 1, 2, 3, 4) could be more useful so ϕ_{ij} (j = 1, 2, 3, 4) might be chosen as 0 (or close to 0). For asserting both the problems simultaneously, the suitable (optimum) choices of ϕ_{ij} (i = 1, 2; j = 1, 2, 3, 4) are desired.

Properties of the Proposed Estimators T_{ij}

The estimators T_{iu} and T_{jm} are regression, ratio, regression to regression, regression-cum ratio, and regression-cum product type in their respective structures, and they are biased estimators of the population mean $\bar{Y}$. Therefore, the final estimator T_{ij} presented in equation (9) is also a biased estimator of the population mean $\bar{Y}$. The bias B(.) and mean square error M(.) of the proposed estimators T_{ij} are derived up to the first order of sample size under the large sample assumption and using the following transformations:

$$\bar{y}_u = \bar{Y}\left(1+e_1\right), \quad \bar{y}_m = \bar{Y}\left(1+e_2\right), \quad \bar{x}_m = \bar{X}\left(1+e_3\right), \quad \bar{x}_n = \bar{X}\left(1+e_4\right),$$
$$\bar{z}_{2u} = \bar{Z}_2\left(1+e_5\right), \quad \bar{z}_{2m} = \bar{Z}_2\left(1+e_6\right), \quad \bar{z}_{1n} = \bar{Z}_1\left(1+e_7\right), \quad s_{yz_2}\left(u\right) = S_{yz_2}\left(1+e_8\right),$$
$$s_{z_2}^2\left(u\right) = S_{z_2}^2\left(1+e_9\right), \quad s_{yx}\left(m\right) = S_{yx}\left(1+e_{10}\right), \quad s_x^2\left(m\right) = S_x^2\left(1+e_{11}\right),$$
$$s_{yz_2}\left(m\right) = S_{yz_2}\left(1+e_{12}\right), \quad s_{z_2}^2\left(m\right) = S_{z_2}^2\left(1+e_{13}\right), \quad s_{xz_1}\left(n\right) = S_{yz_2}\left(1+e_{14}\right),$$
$$s_{z_1}^2\left(n\right) = S_{z_2}^2\left(1+e_{15}\right)$$

such that $E(e_k) = 0$ and $|e_k| \leq 1 \ \forall \ k = 1, 2, \ldots, 15$.

Under the above transformations the proposed estimators take the following form:

$$T_{1u} = \bar{Y}\left(1+e_1\right) + \beta_{yz_2}\bar{Z}_2\left(-e_5 + e_5e_9 - e_5e_8\right) \tag{10}$$

$$T_{2u} = \bar{Y}\left(1+e_1\right)\left(1+e_5\right)^{-1} \tag{11}$$

$$\begin{aligned} T_{1m} = \bar{Y}\left(1+e_2\right) &+ \beta_{yx}\bar{X}\left(e_4 - e_3 - e_4e_{11} + e_3e_{11} + e_4e_{10} - e_3e_{10}\right) \\ &- \beta_{yz_2}\bar{Z}_2\left(e_6 - e_6e_9 + e_6e_8\right) \end{aligned} \tag{12}$$

$$\begin{aligned} T_{2m} = \bar{Y}\left(1+e_2\right) &+ \beta_{yx}\bar{X}\left(e_4 - e_3 - e_4e_{11} + e_3e_{11} + e_4e_{10} - e_3e_{10}\right) \\ &- \beta_{yx}\beta_{xz_1}\bar{Z}_1\left(e_7 - e_7e_{11} + e_7e_{10}\right) - \beta_{yz_2}\bar{Z}_2\left(e_6 - e_6e_9 + e_6e_8\right) \end{aligned} \tag{13}$$

$$\begin{aligned} T_{3m} = \bar{Y}\left(1+e_2\right) &+ \beta_{yx}\bar{X}\left(e_4 - e_3 - e_4e_{11} + e_3e_{11} + e_4e_{10} - e_3e_{10}\right. \\ &\left. - e_7 - e_7e_{11} + e_7e_{10} - e_4e_7 + e_7^2\right) - \beta_{yz_2}\bar{Z}_2\left(e_6 - e_6e_9 + e_6e_8\right) \end{aligned} \tag{14}$$

$$\begin{aligned} T_{4m} = \bar{Y}\left(1+e_2\right) &+ \beta_{yx}\bar{X}\left(e_4 - e_3 + e_7 - e_4e_{11} + e_3e_{11} + e_4e_7 - e_7e_{11}\right. \\ &\left. + e_4e_{10} - e_3e_{10} + e_7e_{10}\right) - \beta_{yz_2}\bar{Z}_2\left(e_6 - e_6e_9 + e_6e_8\right) \end{aligned} \tag{15}$$

Theorem 1. The bias of the estimators T_{ij} (i =1, 2; j = 1, 2, 3, 4) up to the first order approximation is obtained as

$$\mathrm{B}\left(T_{ij}\right) = \phi_{ij}\,\mathrm{B}\left(T_{iu}\right) + \left(1-\phi_{ij}\right)\mathrm{B}\left(T_{jm}\right), \quad i = 1,2;\ j = 1,2,3,4\,, \tag{16}$$

where

$$\mathrm{B}\left(T_{1u}\right) = \left(\frac{1}{u} - \frac{1}{N}\right)\left(\frac{\mu_{0003}\mu_{0101}}{\mu_{0002}^2} - \frac{\mu_{0102}}{\mu_{0002}}\right) \tag{17}$$

$$\mathrm{B}\left(T_{2u}\right) = \left(\frac{1}{u} - \frac{1}{N}\right)\left(\frac{\bar{Y}\mu_{0002}}{\bar{Z}_2^2} - \frac{\mu_{0101}}{\bar{Z}_2}\right) \tag{18}$$

$$\mathrm{B}\left(T_{1m}\right)=\left(\frac{1}{n}-\frac{1}{m}\right)\left(\frac{\mu_{2100}}{\mu_{2000}}-\frac{\mu_{3000}\mu_{1100}}{\mu_{2000}^{2}}\right)+\left(\frac{1}{m}-\frac{1}{N}\right)\left(\frac{\mu_{0003}\mu_{0101}}{\mu_{0002}^{2}}-\frac{\mu_{0102}}{\mu_{0002}}\right) \quad (19)$$

$$\begin{aligned}\mathrm{B}\left(T_{2m}\right)=&\left(\frac{1}{n}-\frac{1}{N}\right)\left(\frac{\mu_{0110}\mu_{1100}}{\mu_{2000}\mu_{0020}}-\frac{\mu_{1020}\mu_{1100}\mu_{0110}}{\mu_{0020}^{2}\mu_{2000}}-\frac{\mu_{1110}\mu_{1010}}{\mu_{2000}\mu_{0020}}\right)\\&+\left(\frac{1}{n}-\frac{1}{m}\right)\left(\frac{\mu_{2100}}{\mu_{2000}}-\frac{\mu_{3000}\mu_{1100}}{\mu_{2000}^{2}}\right)\end{aligned} \quad (20)$$

$$\begin{aligned}\mathrm{B}\left(T_{3m}\right)=&\left(\frac{1}{n}-\frac{1}{N}\right)\left(\frac{\bar{X}}{\bar{Z}_1^{2}}\frac{\mu_{0020}\mu_{1100}}{\mu_{2000}}-\frac{\mu_{1010}\mu_{1100}}{\bar{Z}_1\mu_{2000}}+\frac{\mu_{1010}\mu_{1100}}{\bar{Z}_1\mu_{2000}^{2}}-\frac{\mu_{1110}}{\bar{Z}_1\mu_{2000}}\right)\\&+\left(\frac{1}{n}-\frac{1}{m}\right)\left(\frac{\mu_{2100}}{\mu_{2000}}-\frac{\mu_{3000}\mu_{1100}}{\mu_{2000}^{2}}\right)+\left(\frac{1}{m}-\frac{1}{N}\right)\left(\frac{\mu_{0003}\mu_{0101}}{\mu_{0001}^{3}}-\frac{\mu_{0102}}{\mu_{0002}}\right)\end{aligned} \quad (21)$$

$$\begin{aligned}\mathrm{B}\left(T_{4m}\right)=&\left(\frac{1}{n}-\frac{1}{N}\right)\left(\frac{\mu_{1010}\mu_{1100}}{\bar{Z}_1\mu_{2000}}-\frac{\bar{X}}{\bar{Z}_1}\frac{\mu_{2010}\mu_{1100}}{\mu_{2000}}+\frac{\bar{X}}{\bar{Z}_1}\frac{\mu_{1110}}{\mu_{2000}}\right)\\&+\left(\frac{1}{m}-\frac{1}{n}\right)\left(\frac{\mu_{3000}\mu_{1100}}{\mu_{2000}^{2}}-\frac{\mu_{2100}}{\mu_{2000}}\right)+\left(\frac{1}{m}-\frac{1}{N}\right)\left(\frac{\mu_{0003}\mu_{0101}}{\mu_{0001}^{3}}-\frac{\mu_{0102}}{\mu_{0002}}\right)\end{aligned} \quad (22)$$

where

$$\mu_{pqrs}=\mathrm{E}\left[\left(x_i-\bar{X}\right)^{p}\left(y_i-\bar{Y}\right)^{q}\left(z_{1_i}-\bar{Z}\right)^{r}\left(z_{2_i}-\bar{Z}\right)^{s}\right],$$

$p, q, r, s \geq 0$ are integers.

Proof: The bias of an estimator T_{ij} is shown below:

$$\begin{aligned}\mathrm{B}\left(T_{ij}\right)&=\mathrm{E}\left[T_{ij}-\bar{Y}\right]=\phi_{ij}\,\mathrm{E}\left[T_{iu}-\bar{Y}\right]+\left(1-\phi_{ij}\right)\mathrm{E}\left[T_{jm}-\bar{Y}\right]\\&=\phi_{ij}\,\mathrm{B}\left(T_{iu}\right)+\left(1-\phi_{ij}\right)\mathrm{B}\left(T_{jm}\right)\end{aligned}$$

Using the transformations and retaining the terms up to the first order approximation, the expressions of $\mathrm{B}(T_{iu})$ and $\mathrm{B}(T_{jm})$ are obtained as in equations

(17)-(22). Hence, the expression of the bias of the estimator T_{ij} are obtained as in equation (16).

Theorem 2. The mean square error of the estimators T_{ij} (i =1, 2; j = 1, 2, 3, 4) up to the first order approximation is derived as

$$\mathrm{M}\left(T_{ij}\right) = \phi_{ij}^2\,\mathrm{M}\left(T_{iu}\right) + \left(1-\phi_{ij}\right)^2 \mathrm{M}\left(T_{jm}\right) + 2\phi_{ij}\left(1-\phi_{ij}\right)\mathrm{C}\left(T_{iu}, T_{jm}\right), \tag{23}$$

where

$$\mathrm{M}\left(T_{1u}\right) = \left[\left(\frac{1}{u}-\frac{1}{N}\right)\left(1-\rho_{yz_2}^2\right)\right]S_y^2 \tag{24}$$

$$\mathrm{M}\left(T_{2u}\right) = \left[\left(\frac{1}{u}-\frac{1}{N}\right)\left(2-2\rho_{yz_2}\right)\right]S_y^2 \tag{25}$$

$$\begin{aligned}\mathrm{M}\left(T_{1m}\right) = \Bigg[&\left(\frac{1}{m}-\frac{1}{N}\right)+\left(\frac{1}{n}-\frac{1}{m}\right)\rho_{yx}^2-\left(\frac{1}{m}-\frac{1}{N}\right)\rho_{yz_2}^2 \\ &-2\left(\frac{1}{n}-\frac{1}{m}\right)\rho_{xz_2}\rho_{yz_2}\rho_{yx}\Bigg]S_y^2\end{aligned} \tag{26}$$

$$\begin{aligned}\mathrm{M}\left(T_{2m}\right) = \Bigg[&\left(\frac{1}{m}-\frac{1}{N}\right)-\left(\frac{1}{m}-\frac{1}{n}\right)\rho_{yx}^2+\left(\frac{1}{n}-\frac{1}{N}\right)\rho_{yz_1}^2\rho_{yx}^2-\left(\frac{1}{m}-\frac{1}{N}\right)\rho_{yz_2}^2 \\ &-2\left(\frac{1}{n}-\frac{1}{N}\right)\rho_{xz_1}\rho_{xz_1}\rho_{yx}-2\left(\frac{1}{n}-\frac{1}{m}\right)\rho_{yz_2}\rho_{xz_2}\rho_{yx} \\ &+2\left(\frac{1}{n}-\frac{1}{N}\right)\rho_{yz_2}\rho_{xz_1}\rho_{yx}\rho_{z_1z_2}\Bigg]S_y^2\end{aligned} \tag{27}$$

$$\mathrm{M}\left(T_{3m}\right)=\left[\left(\frac{1}{m}-\frac{1}{N}\right)+\left(\frac{1}{m}-\frac{1}{N}\right)\rho_{yx}^{2}+2\left(\frac{1}{n}-\frac{1}{m}\right)\rho_{yx}^{2}-\left(\frac{1}{m}-\frac{1}{N}\right)\rho_{yz_2}^{2}\right.$$
$$-2\left(\frac{1}{n}-\frac{1}{N}\right)\rho_{yz_1}\rho_{yx}-2\left(\frac{1}{n}-\frac{1}{m}\right)\rho_{yz_2}\rho_{xz_2}\rho_{yx} \tag{28}$$
$$\left.+2\left(\frac{1}{n}-\frac{1}{N}\right)\rho_{yz_2}\rho_{yx}\rho_{z_1z_2}\right]S_y^2$$

$$\mathrm{M}\left(T_{4m}\right)=\left[\left(\frac{1}{m}-\frac{1}{N}\right)-\left(\frac{1}{m}-\frac{1}{N}\right)\rho_{yx}^{2}+2\left(\frac{1}{n}-\frac{1}{m}\right)\rho_{yx}^{2}-\left(\frac{1}{m}-\frac{1}{N}\right)\rho_{yz_2}^{2}\right.$$
$$+2\left(\frac{1}{n}-\frac{1}{N}\right)\rho_{yz_1}\rho_{yx}-2\left(\frac{1}{n}-\frac{1}{m}\right)\rho_{yz_2}\rho_{xz_2}\rho_{yx} \tag{29}$$
$$\left.-2\left(\frac{1}{n}-\frac{1}{N}\right)\rho_{yz_2}\rho_{yx}\rho_{z_1z_2}\right]S_y^2$$

$$\mathrm{C}\left(T_{1u},T_{1m}\right)=\mathrm{E}\left[\left(T_{1u}-\bar{Y}\right)\left(T_{1m}-\bar{Y}\right)\right]=\frac{S_y^2}{N}\left(1-\rho_{yz_2}^{2}\right) \tag{30}$$

$$\mathrm{C}\left(T_{1u},T_{2m}\right)=\mathrm{E}\left[\left(T_{1u}-\bar{Y}\right)\left(T_{2m}-\bar{Y}\right)\right]$$
$$=\frac{S_y^2}{N}\left(1-\rho_{yx}\rho_{xz_1}\rho_{yz_1}-\rho_{yz_2}^{2}+\rho_{yx}\rho_{yz_2}\rho_{xz_1}\rho_{z_1z_2}\right) \tag{31}$$

$$\mathrm{C}\left(T_{1u},T_{3m}\right)=\mathrm{E}\left[\left(T_{1u}-\bar{Y}\right)\left(T_{3m}-\bar{Y}\right)\right]=\frac{S_y^2}{N}\left(1-\rho_{yx}\rho_{xz_1}-\rho_{yz_2}^{2}+\rho_{yx}\rho_{yz_2}\rho_{z_1z_2}\right) \tag{32}$$

$$\mathrm{C}\left(T_{1u},T_{4m}\right)=\mathrm{E}\left[\left(T_{1u}-\bar{Y}\right)\left(T_{4m}-\bar{Y}\right)\right]=\frac{S_y^2}{N}\left(1+\rho_{yx}\rho_{yz_1}-\rho_{yz_2}^{2}+\rho_{yx}\rho_{yz_2}\rho_{z_1z_2}\right) \tag{33}$$

$$\mathrm{C}\left(T_{2u},T_{1m}\right)=\mathrm{E}\left[\left(T_{2u}-\bar{Y}\right)\left(T_{1m}-\bar{Y}\right)\right]=\frac{S_y^2}{N}\left(1-\rho_{yz_2}^{2}\right) \tag{34}$$

$$\mathrm{C}\left(T_{2u},T_{2m}\right)=\mathrm{E}\left[\left(T_{2u}-\bar{Y}\right)\left(T_{2m}-\bar{Y}\right)\right]$$
$$=\frac{S_y^2}{N}\left(1-\rho_{yx}\rho_{xz_1}\rho_{yz_1}-\rho_{yz_2}^{2}+\rho_{yx}\rho_{xz_1}\rho_{z_1z_2}\right) \tag{35}$$

$$C\left(T_{2u},T_{3m}\right)=E\left[\left(T_{2u}-\bar{Y}\right)\left(T_{3m}-\bar{Y}\right)\right]=\frac{S_y^2}{N}\left(1-\rho_{yx}\rho_{yz_1}+\rho_{yx}\rho_{z_1z_2}-\rho_{yz_2}^2\right) \quad (36)$$

and

$$C\left(T_{2u},T_{4m}\right)=E\left[\left(T_{2u}-\bar{Y}\right)\left(T_{4m}-\bar{Y}\right)\right]=\frac{S_y^2}{N}\left(1+\rho_{yx}\rho_{yz_1}-\rho_{yx}\rho_{z_1z_2}-\rho_{yz_2}^2\right). \quad (37)$$

Proof: The mean square error of the estimators T_{ij} ($i = 1, 2$; $j = 1, 2, 3, 4$) are given by

$$\begin{aligned} M\left(T_{ij}\right)=E\left[T_{ij}-\bar{Y}\right]&=E\left[\phi_{ij}\left(T_{iu}-\bar{Y}\right)+\left(1-\phi_{ij}\right)\left(T_{jm}-\bar{Y}\right)\right]^2 \\ &=\phi_{ij}^2\, M\left(T_{iu}\right)+\left(1-\phi_{ij}\right)^2 M\left(T_{jm}\right)+2\phi_{ij}\left(1-\phi_{ij}\right)C\left(T_{iu},T_{jm}\right) \end{aligned}$$

where

$$\begin{aligned} M\left(T_{iu}\right)=E\left[T_{iu}-\bar{Y}\right]^2, \quad M\left(T_{jm}\right)=E\left[T_{jm}-\bar{Y}\right]^2, \\ C\left(T_{iu},T_{jm}\right)=E\left[\left(T_{iu}-\bar{Y}\right)\left(T_{jm}-\bar{Y}\right)\right] \end{aligned}$$

Using the transformation and retaining the terms up to the first order approximation, the expressions of M(T_{iu}), M(T_{jm}), and C(T_{iu}, T_{jm}) are obtained as in equation (24)-(37). Hence, the expression of the bias of the estimators T_{ij} are obtained as in equation (23).

Remark 2. Because x and y denote the same study variable over two occasions and z_1, z_2 are auxiliary variables correlated to x and y, looking on the stability nature *viz*. Reddy (1978), the coefficients of variation of y, x, z_1, and z_2 in the population are considered to be approximately equal, i.e. $C_x = C_{z_1} = C_{z_2} = C_y$.

Minimum Mean Square Error of the Estimators T_{ij}

Because the mean square error of the estimators T_{ij} in equation (23) are a function of the unknown constants ϕ_{ij} ($i = 1, 2$; $j = 1, 2, 3, 4$), it is minimized with respect to ϕ_{ij} and subsequently the optimum value of ϕ_{ij}, say $(\phi_{ij})_{\text{opt}}$, is obtained as

$$\left(\phi_{ij}\right)_{\text{opt}} = \frac{\mathrm{M}\left(T_{jm}\right) - \mathrm{C}\left(T_{iu}, T_{jm}\right)}{\mathrm{M}\left(T_{iu}\right) + \mathrm{M}\left(T_{jm}\right) - 2\,\mathrm{C}\left(T_{iu}, T_{jm}\right)}, \quad i = 1, 2;\ j = 1, 2, 3, 4. \tag{38}$$

Substituting the value of $(\phi_{ij})_{\text{opt}}$ from equation (38) in equation (23) yields the optimum (minimum) mean square error of the estimators T_{ij},

$$\mathrm{M}\left(T_{ij}\right)_{\text{opt}} = \frac{\mathrm{M}\left(T_{iu}\right)\mathrm{M}\left(T_{jm}\right) - \left[\mathrm{C}\left(T_{iu}, T_{jm}\right)\right]^2}{\mathrm{M}\left(T_{iu}\right) + \mathrm{M}\left(T_{jm}\right) - 2\,\mathrm{C}\left(T_{iu}, T_{jm}\right)}, \quad i = 1, 2;\ j = 1, 2, 3, 4. \tag{39}$$

Substituting the values of $\mathrm{M}(T_{iu})$, $\mathrm{M}(T_{jm})$, and $\mathrm{C}(T_{iu}, T_{jm})$ from equations (24)-(37) in equations (38) and (39), we get the simplified values of $(\phi_{ij})_{\text{opt}}$ and $\mathrm{M}(T_{ij})_{\text{opt}}$ as

$$\left(\phi_{11}\right)_{\text{opt}} = \frac{\mu_{11}\left[A_4 - \mu_{11}A_2\right]}{A_3 + \mu_{11}A_5 + \mu_{11}^2 A_2} \tag{40}$$

$$\mathrm{M}\left(T_{11}\right)_{\text{opt}} = \frac{\left[A_6 + \mu_{11}A_7 + \mu_{11}^2 A_8\right]}{\left[A_3 + \mu_{11}A_5 + \mu_{11}^2 A_2\right]} \frac{S_y^2}{n} \tag{41}$$

$$\left(\phi_{12}\right)_{\text{opt}} = \frac{\mu_{12}\left[A_{12} - \mu_{12}A_{13}\right]}{A_3 + \mu_{12}A_{14} + \mu_{12}^2 A_{15}} \tag{42}$$

$$\mathrm{M}\left(T_{12}\right)_{\text{opt}} = \frac{\left[A_{16} + \mu_{12}A_{17} + \mu_{12}^2 A_{18}\right]}{\left[A_3 + \mu_{12}A_{14} + \mu_{12}^2 A_{15}\right]} \frac{S_y^2}{n} \tag{43}$$

$$\left(\phi_{13}\right)_{\text{opt}} = \frac{\mu_{13}\left[A_{22} - \mu_{13}A_{23}\right]}{A_3 + \mu_{12}A_{24} + \mu_{13}^2 A_{25}} \tag{44}$$

$$\mathrm{M}\left(T_{13}\right)_{\text{opt}} = \frac{\left[A_{26} + \mu_{13}A_{27} + \mu_{13}^2 A_{28}\right]}{\left[A_3 + \mu_{13}A_{24} + \mu_{13}^2 A_{25}\right]} \frac{S_y^2}{n} \tag{45}$$

$$\left(\phi_{14}\right)_{\text{opt}} = \frac{\mu_{14}\left[A_{33} - \mu_{14}A_{34}\right]}{A_{3} + \mu_{14}A_{35} + \mu_{14}^{2}A_{36}} \tag{46}$$

$$\mathrm{M}\left(T_{14}\right)_{\text{opt}} = \frac{\left[A_{37} + \mu_{14}A_{38} + \mu_{14}^{2}A_{39}\right]}{\left[A_{3} + \mu_{14}A_{35} + \mu_{14}^{2}A_{36}\right]}\frac{S_{y}^{2}}{n} \tag{47}$$

$$\left(\phi_{21}\right)_{\text{opt}} = \frac{\mu_{21}\left[A_{4} - \mu_{21}A_{2}\right]}{A_{40} + \mu_{21}A_{41} + \mu_{21}^{2}A_{42}} \tag{48}$$

$$\mathrm{M}\left(T_{21}\right)_{\text{opt}} = \frac{\left[A_{43} + \mu_{21}A_{44} + \mu_{21}^{2}A_{45}\right]}{\left[A_{40} + \mu_{21}A_{41} + \mu_{21}^{2}A_{42}\right]}\frac{S_{y}^{2}}{n} \tag{49}$$

$$\left(\phi_{22}\right)_{\text{opt}} = \frac{\mu_{22}\left[A_{47} - \mu_{22}A_{48}\right]}{A_{40} + \mu_{22}A_{49} + \mu_{22}^{2}A_{50}} \tag{50}$$

$$\mathrm{M}\left(T_{22}\right)_{\text{opt}} = \frac{\left[A_{51} + \mu_{22}A_{52} + \mu_{22}^{2}A_{53}\right]}{\left[A_{40} + \mu_{22}A_{49} + \mu_{22}^{2}A_{50}\right]}\frac{S_{y}^{2}}{n} \tag{51}$$

$$\left(\phi_{23}\right)_{\text{opt}} = \frac{\mu_{23}\left[A_{55} - \mu_{23}A_{56}\right]}{A_{40} + \mu_{23}A_{57} + \mu_{23}^{2}A_{58}} \tag{52}$$

$$\mathrm{M}\left(T_{23}\right)_{\text{opt}} = \frac{\left[A_{59} + \mu_{23}A_{60} + \mu_{23}^{2}A_{61}\right]}{\left[A_{40} + \mu_{23}A_{57} + \mu_{23}^{2}A_{58}\right]}\frac{S_{y}^{2}}{n} \tag{53}$$

$$\left(\phi_{24}\right)_{\text{opt}} = \frac{\mu_{24}\left[A_{63} - \mu_{24}A_{64}\right]}{A_{40} + \mu_{24}A_{65} + \mu_{24}^{2}A_{66}} \tag{54}$$

$$\mathrm{M}\left(T_{24}\right)_{\text{opt}} = \frac{\left[A_{67} + \mu_{24}A_{68} + \mu_{24}^{2}A_{69}\right]}{\left[A_{40} + \mu_{24}A_{65} + \mu_{24}^{2}A_{66}\right]}\frac{S_{y}^{2}}{n} \tag{55}$$

where $\mu_{ij} = u / n$ and

$$A_1 = 1-\rho_{yx}^2-\rho_{yz_2}^2+2\rho_{yz_2}^2\rho_{yx}\rho_{xz_2},\, A_2=\rho_{yx}^2-2\rho_{yx}\rho_{yz_2}\rho_{xz_2},\, A_3=1-\rho_{yz_2}^2,\, A_4=A_1+A_2,$$
$$A_5=A_4-A_3,\, A_6=A_3\left(A_4-A_3 f\right),\, A_7=-A_1A_3 f-A_2A_3-A_2A_3 f+A_3^2 f,\, A_8=A_2A_3 f,$$
$$A_9=\rho_{yx}^2+\rho_{yx}^2\rho_{xz_1}^2-2\rho_{yx}\rho_{yz_1}\rho_{xz_1}-2\rho_{yx}\rho_{xz_1}\rho_{xz_2}+2\rho_{yx}\rho_{yz_2}\rho_{xz_1}\rho_{z_1z_2},$$
$$A_{10}=1+\rho_{yx}^2\rho_{xz_1}^2-\rho_{yz_2}^2-2\rho_{xz_1}\rho_{yx}\rho_{yz_1}+2\rho_{yz_2}\rho_{yx}\rho_{xz_1}\rho_{z_1z_2},$$
$$A_{11}=1-\rho_{xz_1}\rho_{yx}\rho_{yz_1}-\rho_{yz_2}^2+2\rho_{yz_2}\rho_{yx}\rho_{xz_1}\rho_{z_1z_2},\, A_{12}=A_1+A_9-A_{10} f+A_{11} f,$$
$$A_{13}=A_9-A_{10} f+A_{11} f,\, A_{14}=-A_3 f+A_1+A_9-A_{10} f+2A_{11} f,$$
$$A_{15}=A_3 f-A_9+A_{10} f-2A_{11} f,\, A_{16}=A_1A_3+A_9A_3-A_{10}A_3 f,$$
$$A_{17}=-A_3A_9+A_3A_{10} f-A_3A_1 f-A_3A_9 f+A_{10}A_3 f^2-A_{11}^2 f^2,$$
$$A_{18}=A_3A_9 f-A_{10}A_3 f^2+A_{11}^2 f^2,\, A_{19}=2\rho_{yx}^2-2\rho_{yx}\rho_{yz_1}-2\rho_{yx}\rho_{yz_2}\rho_{xz_2}+2\rho_{yz_2}\rho_{yx}\rho_{z_1z_2},$$
$$A_{20}=1+\rho_{yx}^2-\rho_{yz_2}^2-2\rho_{yx}\rho_{yz_1}+2\rho_{yz_2}\rho_{yx}\rho_{xz_2},\, A_{21}=1-\rho_{yx}\rho_{yz_1}-\rho_{yz_2}^2+\rho_{yz_2}\rho_{yx}\rho_{xz_2},$$
$$A_{22}=A_1+A_{19}-A_{20} f+A_{21} f,\, A_{23}=A_{19}-A_{20} f+A_{21} f,$$
$$A_{24}=-A_3-A_3 f+A_1+A_{19}-A_{20} f+2A_{21} f,\, A_{25}=A_3 f-A_{19}+A_{20} f-2A_{21} f,$$
$$A_{26}=A_1A_3+A_3A_{19}-A_3A_{20} f,$$
$$A_{27}=-A_3A_{19}+A_3A_{20} f-A_3A_1 f-A_3A_{19} f+A_{20}A_3 f^2-A_{21}^2 f^2,$$
$$A_{28}=A_3A_{19} f-A_{20}A_3 f^2+A_{11}^2 f^2,\, A_{29}=1-3\rho_{yx}^2-\rho_{yz_2}^2+2\rho_{yz_2}^2\rho_{yx}\rho_{xz_2},$$
$$A_{30}=2\rho_{yx}^2+2\rho_{yx}\rho_{yz_1}-2\rho_{yx}\rho_{yz_2}\rho_{xz_2}-2\rho_{yz_2}\rho_{yx}\rho_{z_1z_2},$$
$$A_{31}=1-\rho_{yx}^2-\rho_{yz_2}^2+2\rho_{yx}\rho_{yz_1}-2\rho_{yz_2}\rho_{yx}\rho_{z_1z_2},\, A_{32}=1+\rho_{yx}\rho_{yz_1}-\rho_{yz_2}^2+\rho_{yz_2}\rho_{yx}\rho_{z_1z_2},$$
$$A_{33}=A_{29}+A_{30}-A_{31} f+A_{32} f,\, A_{34}=A_{30}-A_{31} f+A_{32} f,$$
$$A_{35}=-A_3-A_3 f+A_{29}+A_{30}-A_{31} f+2A_{32} f,\, A_{36}=A_3 f-A_{30}A_{31} f-2A_{32} f,$$
$$A_{37}=A_{29}A_3+A_{30}A_3-A_{31}A_3 f,$$
$$A_{38}=-A_3A_{30}+A_3A_{31} f-A_3A_{29} f-A_3A_{30} f+A_{31}A_3 f^2-A_{32}^2 f^2,$$
$$A_{39}=A_3A_{30} f-A_{31}A_3 f^2+A_{32}^2 f^2,\, A_{40}=2\left(1-\rho_{yz_2}\right),$$
$$A_{41}=-A_{40}-A_{40} f+A_1+A_2+A_3 f,\, A_{42}=A_{40} f-A_2-A_3 f,$$
$$A_{43}=A_1A_{40}+A_2A_{40}-A_3A_{40} f,$$
$$A_{44}=-A_{40}A_2+A_{40}A_3 f-A_{40}A_1 f-A_{40}A_2 f+A_3A_{40} f^2-A_3^2 f^2,$$
$$A_{45}=A_{40}A_2 f-A_3A_{40} f^2+A_3^2 f^2,\, A_{46}=1-\rho_{yz_2}^2-\rho_{yx}\rho_{yz_1}\rho_{xz_1}+\rho_{yx}\rho_{z_1z_2},$$
$$A_{47}=A_1+A_9-A_{10} f+A_{46} f,\, A_{48}=A_9-A_{10} f+A_{46} f,$$
$$A_{49}=-A_{40}-A_{40} f+A_1+A_9-A_{10} f+2A_{46} f,\, A_{50}=A_{40} f-A_9+A_{10} f-2A_{46} f,$$

$$A_{51} = A_1 A_{40} + A_9 A_{10} - A_{10} A_{40} f,$$
$$A_{52} = -A_{40} A_9 + A_{40} A_{10} f - A_{40} A_1 f - A_{40} A_9 f + A_{10} A_{40} f^2 - A_{46}^2 f^2,$$
$$A_{53} = A_{40} A_9 f - A_{10} A_{40} f^2 + A_{46}^2 f^2, A_{54} = 1 - \rho_{yz_2}^2 - \rho_{yx}\rho_{yz_1} + \rho_{yx}\rho_{z_1 z_2},$$
$$A_{55} = A_1 + A_{19} - A_{20} f + A_{54} f, A_{56} = A_{19} - A_{20} f + A_{54} f,$$
$$A_{57} = -A_{40} - A_{40} f + A_1 + A_{19} - A_{20} f + 2A_{54} f, A_{58} = A_{40} f - A_{19} + A_{20} f - 2A_{54} f,$$
$$A_{59} = A_1 A_{40} + A_{19} A_{40} - A_{20} A_{40} f,$$
$$A_{60} = -A_{40} A_{19} + A_{40} A_{20} f - A_{40} A_1 f - A_{40} A_{19} f + A_{20} A_{40} f^2 - A_{54}^2 f^2,$$
$$A_{61} = A_{40} A_{19} f - A_{20} A_{40} f^2 + A_{54}^2 f^2, A_{62} = 1 - \rho_{yz_2}^2 + \rho_{yx}\rho_{yz_1} - \rho_{yx}\rho_{z_1 z_2},$$
$$A_{63} = A_{29} + A_{30} - A_{31} f + A_{62} f, A_{64} = A_{30} - A_{31} f + A_{62} f,$$
$$A_{65} = -A_{40} - A_{40} f + A_{29} + A_{30} - A_{31} f + 2A_{62} f, A_{66} = A_{40} f - A_{30} + A_{31} f - 2A_{62} f,$$
$$A_{67} = A_{29} A_{40} + A_{30} A_{40} - A_{31} A_{40} f,$$
$$A_{68} = -A_{40} A_{30} + A_{40} A_{31} f - A_{40} A_{29} f - A_{40} A_{30} f + A_{31} A_{40} f^2 - A_{62}^2 f^2,$$
$$A_{69} = A_{40} A_{30} f - A_{31} A_{40} f^2 + A_{62}^2 f^2,$$

with $f = n / N$.

Optimum Replacement Strategy of the Estimator T_{ij}

To obtain the optimum values of μ_{ij} (i = 1, 2; j = 1, 2, 3, 4) (the fraction of the sample to be drawn afresh on the current occasion) so that the population mean $\bar{Y}$ may be estimated with maximum precision and minimum cost, minimize the mean square error of T_{ij} given in equations (41), (43), (45), (47), (49), (51), (53), and (55) with respect to μ_{ij}, which yields a quadratic equation in μ_{ij}. The respective solutions of μ_{ij}, say $\hat{\mu}_{ij}$, are given below:

$$T_1 \hat{\mu}_{11}^2 + 2T_2 \hat{\mu}_{11} + T_3 = 0 \tag{56}$$

$$\hat{\mu}_{11} = \frac{-T_2 \pm \sqrt{T_2^2 - T_1 T_3}}{T_1} \tag{57}$$

$$P_1 \hat{\mu}_{12}^2 + 2P_2 \hat{\mu}_{12} + P_3 = 0 \tag{58}$$

$$\hat{\mu}_{12} = \frac{-P_2 \pm \sqrt{P_2^2 - P_1 P_3}}{P_1} \tag{59}$$

$$Q_1 \hat{\mu}_{13}^2 + 2Q_2 \hat{\mu}_{13} + Q_3 = 0 \tag{60}$$

$$\hat{\mu}_{13} = \frac{-Q_2 \pm \sqrt{Q_2^2 - Q_1 Q_3}}{Q_1} \tag{61}$$

$$R_1 \hat{\mu}_{14}^2 + 2R_2 \hat{\mu}_{14} + R_3 = 0 \tag{62}$$

$$\hat{\mu}_{14} = \frac{-R_2 \pm \sqrt{R_2^2 - R_1 R_3}}{R_1} \tag{63}$$

$$S_1 \hat{\mu}_{21}^2 + 2S_2 \hat{\mu}_{21} + S_3 = 0 \tag{64}$$

$$\hat{\mu}_{21} = \frac{-S_2 \pm \sqrt{S_2^2 - S_1 S_3}}{S_1} \tag{65}$$

$$U_1 \hat{\mu}_{22}^2 + 2U_2 \hat{\mu}_{22} + U_3 = 0 \tag{66}$$

$$\hat{\mu}_{22} = \frac{-U_2 \pm \sqrt{U_2^2 - U_1 U_3}}{U_1} \tag{67}$$

$$V_1 \hat{\mu}_{23}^2 + 2V_2 \hat{\mu}_{23} + V_3 = 0 \tag{68}$$

$$\hat{\mu}_{23} = \frac{-V_2 \pm \sqrt{V_2^2 - V_1 V_3}}{V_1} \tag{69}$$

$$W_1 \hat{\mu}_{24}^2 + 2W_2 \hat{\mu}_{24} + W_3 = 0 \tag{70}$$

$$\hat{\mu}_{24} = \frac{-W_2 \pm \sqrt{W_2^2 - W_1 W_3}}{W_1} \tag{71}$$

where

$$\begin{aligned}
&T_1 = A_5A_8 + A_2A_7, \quad T_2 = A_3A_8 + A_2A_6, \quad T_3 = A_3A_7 - A_5A_6, \\
&P_1 = A_{18}A_{14} - A_{17}A_{15}, \quad P_2 = A_3A_{18} - A_{16}A_{15}, \quad P_3 = A_3A_{17} - A_{16}A_{14}, \\
&Q_1 = A_{28}A_{24} - A_{25}A_{27}, \quad Q_2 = A_3A_{28} - A_{26}A_{25}, \quad Q_3 = A_3A_{27} - A_{27}A_{24}, \\
&R_1 = A_{39}A_{35} - A_{38}A_{36}, \quad R_2 = A_3A_{38} - A_{37}A_{36}, \quad R_3 = A_3A_{38} - A_{37}A_{35}, \\
&S_1 = A_{45}A_{41} - A_{44}A_{42}, \quad S_2 = A_{40}A_{45} - A_{43}A_{42}, \quad S_3 = A_{40}A_{44} - A_{43}A_{41}, \\
&U_1 = A_{53}A_{49} - A_{50}A_{52}, \quad U_2 = A_{40}A_{53} - A_{51}A_{50}, \quad U_3 = A_{40}A_{52} - A_{51}A_{49}, \\
&V_1 = A_{61}A_{57} - A_{58}A_{60}, \quad V_2 = A_{40}A_{61} - A_{58}A_{59}, \quad V_3 = A_{40}A_{60} - A_{57}A_{59}, \\
&W_1 = A_{69}A_{65} - A_{66}A_{68}, \quad W_2 = A_{40}A_{69} - A_{66}A_{67}, \quad W_3 = A_{40}A_{68} - A_{65}A_{67}
\end{aligned}$$

The real values of $\hat{\mu}$ exit if and only if the quantity under the square root is greater than or equal to zero, i.e.

$$\begin{aligned}
&\left(T_2^2 - T_1T_3\right) \geq 0, \left(P_2^2 - P_1P_3\right) \geq 0, \left(Q_2^2 - Q_1Q_3\right) \geq 0, \left(R_2^2 - R_1R_3\right) \geq 0, \\
&\left(S_2^2 - S_1S_3\right) \geq 0, \left(U_2^2 - U_1U_3\right) \geq 0, \left(V_2^2 - V_1V_3\right) \geq 0, \left(W_2^2 - W_1W_3\right) \geq 0
\end{aligned}$$

For any situation which satisfies these conditions, two real values of $\hat{\mu}$ are possible. Hence, while choosing the value of $\hat{\mu}$, it is mentioned that $0 \leq \hat{\mu} \leq 1$ and all others values of $\hat{\mu}$ are inadmissible. In the case when both the values are admissible, the lowest one will be the best choice because it reduces the cost of the survey. Substituting the admissible value of $\hat{\mu}$, say $\mu^{(0)}$, from equations (57), (59), (61), (63), (65), (67), (69), and (71) into equations (41), (43), (45), (47), (49), (51), (53), and (55), respectively, gives the optimum value of the mean square error of the estimators T_{ij} as

$$\mathrm{M}\left(T_{11}\right)^*_{\mathrm{opt}} = \frac{\left[A_6 + \mu_{11}^* A_7 + \mu_{11}^{*2} A_8\right]}{\left[A_3 + \mu_{11}^* A_5 + \mu_{11}^{*2} A_2\right]} \frac{S_y^2}{n} \tag{72}$$

$$\mathrm{M}\left(T_{12}\right)^*_{\mathrm{opt}} = \frac{\left[A_{16} + \mu_{12}^* A_{17} + \mu_{12}^{*2} A_{18}\right]}{\left[A_3 + \mu_{12}^* A_{14} + \mu_{12}^{*2} A_{15}\right]} \frac{S_y^2}{n} \tag{73}$$

$$\mathrm{M}\left(T_{13}\right)^{*}_{\text{opt}} = \frac{\left[A_{26} + \mu^{*}_{13} A_{27} + \mu^{*2}_{13} A_{28}\right]}{\left[A_{3} + \mu^{*}_{13} A_{24} + \mu^{*2}_{13} A_{25}\right]} \frac{S_y^2}{n} \tag{74}$$

$$\mathrm{M}\left(T_{14}\right)^{*}_{\text{opt}} = \frac{\left[A_{37} + \mu^{*}_{14} A_{38} + \mu^{*2}_{14} A_{39}\right]}{\left[A_{3} + \mu^{*}_{14} A_{35} + \mu^{*2}_{14} A_{36}\right]} \frac{S_y^2}{n} \tag{75}$$

$$\mathrm{M}\left(T_{21}\right)^{*}_{\text{opt}} = \frac{\left[A_{43} + \mu^{*}_{21} A_{44} + \mu^{*2}_{21} A_{45}\right]}{\left[A_{40} + \mu^{*}_{21} A_{41} + \mu^{*2}_{21} A_{42}\right]} \frac{S_y^2}{n} \tag{76}$$

$$\mathrm{M}\left(T_{22}\right)^{*}_{\text{opt}} = \frac{\left[A_{51} + \mu^{*}_{22} A_{52} + \mu^{*2}_{22} A_{53}\right]}{\left[A_{40} + \mu^{*}_{22} A_{49} + \mu^{*2}_{22} A_{50}\right]} \frac{S_y^2}{n} \tag{77}$$

$$\mathrm{M}\left(T_{23}\right)^{*}_{\text{opt}} = \frac{\left[A_{59} + \mu^{*}_{23} A_{60} + \mu^{*2}_{23} A_{61}\right]}{\left[A_{40} + \mu^{*}_{23} A_{57} + \mu^{*2}_{23} A_{58}\right]} \frac{S_y^2}{n} \tag{78}$$

$$\mathrm{M}\left(T_{24}\right)^{*}_{\text{opt}} = \frac{\left[A_{67} + \mu^{*}_{24} A_{68} + \mu^{*2}_{24} A_{69}\right]}{\left[A_{40} + \mu^{*}_{24} A_{65} + \mu^{*2}_{24} A_{66}\right]} \frac{S_y^2}{n} \tag{79}$$

Efficiency Comparisons

The percent relative efficiencies of the estimators T_{ij} ($i = 1, 2$; $j = 1, 2, 3, 4$) are calculated with respect to (i) sample mean estimator $\bar{y}_n$ when there is no matching and (ii) natural successive sampling estimator $\hat{\hat{\bar{Y}}} = \phi^{*}\bar{y}_u + \left(1-\phi^{*}\right)\bar{y}'_m$ when no auxiliary information is used on any occasion, where $\bar{y}'_m = \bar{y}_m + \beta_{yx}\left(\bar{x}_n - \bar{x}_m\right)$. Because $\bar{y}_n$ and $\hat{\hat{\bar{Y}}}$ are unbiased estimators of population $\bar{Y}$, therefore, following Sukhatme, Sukhatme, Sukhatme, and Ashok (1984), the variance of $\bar{y}_n$ and the optimum variance of $\hat{\hat{\bar{Y}}}$ are given below:

$$\mathrm{V}\left(\bar{y}_n\right) = \left(\frac{1}{n} - \frac{1}{N}\right) S_y^2 \tag{80}$$

and

$$\mathrm{V}\left(\hat{\bar{Y}}\right)_{\text{opt}} = \left[1+\sqrt{1-\rho_{yx}^2}\right]\frac{S_y^2}{2n} - \frac{S_y^2}{N} \tag{81}$$

The optimum mean square errors of the estimators T_{ij} are derived in equations (72)-(79), which involved six correlations. For the efficiency comparison of the suggested estimators, we introduce the assumption $\rho_{yz_1} = \rho_{yz_2} = \rho_{xz_1} = \rho_{xz_2} = \rho_0$, which is considered by Cochran (1977) and Feng and Zou (1997). Using the above assumptions, we have three correlations: ρ_{yx}, ρ_0, and $\rho_{z_1z_2}$.

The percent relative efficiencies $E_{ij}^{(1)}$ and $E_{ij}^{(2)}$ of the estimators T_{ij} (under optimal conditions) with respect to $\bar{y}_n$ and $\hat{\bar{Y}}$ are defined below:

$$E_{ij}^{(1)} = \frac{\mathrm{V}\left(\bar{y}_n\right)}{\mathrm{M}\left(T_{ij}\right)_{\text{opt}}^*}\times 100 \quad \text{and} \quad E_{ij}^{(2)} = \frac{\mathrm{V}\left(\hat{\bar{Y}}\right)_{\text{opt}}}{\mathrm{M}\left(T_{ij}\right)_{\text{opt}}^*}\times 100, \quad i=1,2;\ j=1,2,3,4\ .$$

Empirical Study

The expressions of the optimum values of μ_{ij}, (i.e. $\mu_{ij}^{(0)}$) and the percent relative efficiencies $E_{ij}^{(1)}$ and $E_{ij}^{(2)}$ are in terms of population correlation coefficients ρ_{yx}, ρ_0, and $\rho_{z_1z_2}$. Tables 1-8 present the values of $\mu_{ij}^{(0)}$, $E_{ij}^{(1)}$, and $E_{ij}^{(2)}$ for a fixed value of f = 0.1 and different choices of ρ_{yx}, ρ_0, and $\rho_{z_1z_2}$.

Table 1. Optimum values of $\mu_{11}^{(0)}$ and percent relative efficiencies of the estimator T_{11} with respect to the estimators $\bar{y}_n$ and $\hat{\bar{Y}}$

ρ_{xy}	ρ_0	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**
0.3	$\mu_{11}^{(0)}$	0.4904	0.4776	0.4580	0.4259	0.3628
	$E_{11}^{(1)}$	130.4890	148.4939	177.9641	232.7923	370.6171
	$E_{11}^{(2)}$	127.1500	144.6940	173.4101	226.8353	361.1332
0.4	$\mu_{11}^{(0)}$	0.4935	0.4772	0.4533	0.4156	0.3461
	$E_{11}^{(1)}$	131.4110	148.3820	175.9258	226.6202	352.3091
	$E_{11}^{(2)}$	125.3160	141.5000	167.7663	216.1094	335.9688
0.5	$\mu_{11}^{(0)}$	*	0.4802	0.4519	0.4093	0.3348
	$E_{11}^{(1)}$	**	149.3999	175.3590	222.8784	339.9487
	$E_{11}^{(2)}$	**	138.2800	162.3069	206.2895	314.6462
0.6	$\mu_{11}^{(0)}$	0.5104	0.4867	0.4539	0.4064	0.3274
	$E_{11}^{(1)}$	136.4283	151.6376	176.2118	221.1809	331.8836
	$E_{11}^{(2)}$	121.2696	134.7890	156.6327	196.6052	295.0077
0.7	$\mu_{11}^{(0)}$	0.5258	0.4973	0.4594	0.4067	0.3231
	$E_{11}^{(1)}$	141.0230	155.3114	178.5626	221.3672	327.2309
	$E_{11}^{(2)}$	118.6272	130.6464	150.2051	186.2120	275.2635
0.8	$\mu_{11}^{(0)}$	0.5481	0.5132	0.4690	0.4102	0.3215
	$E_{11}^{(1)}$	147.7268	160.8353	182.6430	223.4546	325.5421
	$E_{11}^{(2)}$	114.8986	125.0941	142.0557	173.7981	253.1994
0.9	$\mu_{11}^{(0)}$	0.5810	0.5364	0.4835	0.4173	0.3226
	$E_{11}^{(1)}$	157.7795	168.9982	188.9145	227.6427	326.6647
	$E_{11}^{(2)}$	108.3323	116.0351	129.7097	156.3008	224.2897

Note: * does not exist
** no gain

Table 2. Optimum values of $\mu_{12}^{(0)}$ and percent relative efficiencies of the estimator T_{12} with respect to the estimators $\bar{y}_n$ and $\hat{\bar{Y}}$

ρ_{xy}	ρ_0	$\rho_{z_1z_2}$	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**
0.3	0.4	$\mu_{12}^{(0)}$	0.4925	0.4940	0.4955	0.4969	0.4984
		$E_{12}^{(1)}$	123.7835	122.6390	121.5178	120.4192	119.3426
		$E_{12}^{(2)}$	120.6160	119.5007	118.4082	117.3378	116.2887
	0.5	$\mu_{12}^{(0)}$	0.4786	0.4813	0.4840	0.4866	0.4891
		$E_{12}^{(1)}$	140.3211	137.9947	135.7530	133.5914	131.5053
		$E_{12}^{(2)}$	136.7304	134.4635	132.2792	130.1728	128.1402
	0.6	$\mu_{12}^{(0)}$	0.4571	0.4619	0.4666	0.4711	0.4754
		$E_{12}^{(1)}$	168.3291	163.4167	158.8167	154.4987	150.4364
		$E_{12}^{(2)}$	164.0217	159.2350	154.7527	150.5452	146.5869
	0.7	$\mu_{12}^{(0)}$	0.4211	0.4302	0.4388	0.4468	0.4544
		$E_{12}^{(1)}$	222.7075	210.6427	199.9742	190.4620	181.9193
		$E_{12}^{(2)}$	217.0085	205.2525	194.8570	185.5882	177.2641
	0.8	$\mu_{12}^{(0)}$	0.3485	0.3700	0.3885	0.4048	0.4192
		$E_{12}^{(1)}$	373.1511	326.5884	291.6780	264.3665	242.3182
		$E_{12}^{(2)}$	363.6024	318.2312	284.2141	257.6015	236.1174
0.4	0.4	$\mu_{12}^{(0)}$	0.4970	0.4990	0.5010	0.5029	0.5049
		$E_{12}^{(1)}$	126.3812	124.8064	123.2747	121.7842	120.3334
		$E_{12}^{(2)}$	120.5196	119.0178	117.5572	116.1358	114.7522
	0.5	$\mu_{12}^{(0)}$	0.4794	0.4831	0.4867	0.4901	0.4935
		$E_{12}^{(1)}$	143.3657	140.1532	137.0977	134.1874	131.4118
		$E_{12}^{(2)}$	136.7164	133.6528	130.7390	127.9637	125.3168
	0.6	$\mu_{12}^{(0)}$	0.4525	0.4592	0.4655	0.4715	0.4772
		$E_{12}^{(1)}$	172.5494	165.7021	159.4399	153.6874	148.3820
		$E_{12}^{(2)}$	164.5465	158.0168	152.0450	146.5593	141.5000
	0.7	$\mu_{12}^{(0)}$	0.4083	0.4210	0.4327	0.4434	0.4533
		$E_{12}^{(1)}$	231.1212	213.8394	199.2597	186.7676	175.9258
		$E_{12}^{(2)}$	220.4017	203.9214	190.0179	178.1052	167.7663

Table 2 (continued).

ρ_{xy}	ρ_0	$\rho_{z_1z_2}$	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**
0.4	0.8	$\mu_{12}^{(0)}$	0.3183	0.3500	0.3756	0.3971	0.4156
		$E_{12}^{(1)}$	412.7503	337.6417	288.4129	253.2293	226.6202
		$E_{12}^{(2)}$	393.6067	321.9818	275.0361	241.4844	216.1094
0.5	0.4	$\mu_{12}^{(0)}$	0.5055	0.5081	0.5105	0.5130	0.5153
		$E_{12}^{(1)}$	129.6446	127.6056	125.6365	123.7337	121.8937
		$E_{12}^{(2)}$	119.9951	118.1079	116.2854	114.5242	112.8211
	0.5	$\mu_{12}^{(0)}$	0.4845	0.4891	0.4936	0.4979	0.5021
		$E_{12}^{(1)}$	146.8017	142.6511	138.7540	135.0871	131.6295
		$E_{12}^{(2)}$	135.8752	132.0335	128.4265	125.0325	121.8323
	0.6	$\mu_{12}^{(0)}$	0.4529	0.4613	0.4692	0.4766	0.4836
		$E_{12}^{(1)}$	176.5595	167.7131	159.8085	152.6966	146.2588
		$E_{12}^{(2)}$	163.4181	155.2301	147.9139	141.3314	135.3727
	0.7	$\mu_{12}^{(0)}$	0.4018	0.4182	0.4327	0.4459	0.4578
		$E_{12}^{(1)}$	237.6206	215.0932	196.9329	181.9325	169.3007
		$E_{12}^{(2)}$	219.9344	199.0838	182.2751	168.3912	156.6996
	0.8	$\mu_{12}^{(0)}$	0.2987	0.3405	0.3724	0.3981	0.4196
		$E_{12}^{(1)}$	443.5963	338.7927	278.3913	238.3525	209.5417
		$E_{12}^{(2)}$	410.5792	313.5763	257.6706	220.6119	193.9454
0.6	0.4	$\mu_{12}^{(0)}$	0.5190	0.5220	0.5250	0.5279	0.5307
		$E_{12}^{(1)}$	133.8029	131.2542	128.8104	126.4650	124.2119
		$E_{12}^{(2)}$	118.9359	116.6704	114.4981	112.4133	110.4106
	0.5	$\mu_{12}^{(0)}$	0.4944	*	0.5053	0.5104	0.5153
		$E_{12}^{(1)}$	150.7748	**	140.8627	136.4283	132.2925
		$E_{12}^{(2)}$	134.0221	**	125.2113	121.2696	117.5934
	0.6	$\mu_{12}^{(0)}$	0.4585	0.4686	0.4780	0.4867	0.4948
		$E_{12}^{(1)}$	180.3263	169.4813	160.0028	151.6376	144.1925
		$E_{12}^{(2)}$	160.2900	150.6500	142.2247	134.7890	128.1711

Note: * does not exist
** no gain

Table 2 (continued).

ρ_{xy}	ρ_0	$\rho_{z_1z_2}$	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**
0.6	0.7	$\mu_{12}^{(0)}$	0.4020	0.4217	0.4388	0.4539	0.4675
		$E_{12}^{(1)}$	241.4739	214.2100	193.1063	176.2118	162.3347
		$E_{12}^{(2)}$	214.6435	190.4089	171.6501	156.6327	144.2975
	0.8	$\mu_{12}^{(0)}$	0.2912	0.3414	0.3778	0.4064	0.4299
		$E_{12}^{(1)}$	454.3984	328.5273	262.6792	221.1809	192.2438
		$E_{12}^{(2)}$	403.9097	292.0243	233.4927	196.6052	170.8834

Table 3. Optimum values of $\mu_{13}^{(0)}$ and percent relative efficiencies of the estimator T_{13} with respect to the estimators $\bar{y}_n$ and $\hat{\bar{Y}}$

ρ_{xy}	ρ_0	$\rho_{z_1z_2}$	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**
0.3	0.6	$\mu_{13}^{(0)}$	0.7037	0.6649	0.6114	0.5327	0.4054
		$E_{13}^{(1)}$	151.0761	150.5486	149.8111	148.7158	146.9328
		$E_{13}^{(2)}$	147.2102	146.6961	145.9776	144.9103	143.1729
	0.7	$\mu_{13}^{(0)}$	0.6675	0.6328	0.5881	0.5282	0.4440
		$E_{13}^{(1)}$	183.5163	182.4812	181.1345	179.3179	176.7449
		$E_{13}^{(2)}$	178.8202	177.8117	176.4994	174.7293	172.2221
	0.8	$\mu_{13}^{(0)}$	0.6281	0.5954	0.5550	0.5035	0.4359
		$E_{13}^{(1)}$	245.4888	243.2507	240.4555	236.8766	232.1453
		$E_{13}^{(2)}$	239.2069	237.0260	234.3024	230.8151	226.2048
	0.9	$\mu_{13}^{(0)}$	0.5689	0.5362	0.4968	0.4483	0.3870
		$E_{13}^{(1)}$	410.5928	403.8380	395.6042	385.3761	372.3671
		$E_{13}^{(2)}$	400.0861	393.5041	385.4810	375.5145	362.8385
0.4	0.6	$\mu_{13}^{(0)}$	0.6975	0.6444	0.5623	0.4187	0.1030
		$E_{13}^{(1)}$	150.6510	149.8794	148.6716	146.5400	141.8383
		$E_{13}^{(2)}$	143.6637	142.9279	141.7762	139.7434	135.2598

Table 3 (continued).

ρ_{xy}	ρ_0	$\rho_{z_1z_2}$	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**
0.4	0.7	$\mu_{13}^{(0)}$	0.6559	0.6125	0.5535	0.4685	0.3359
		$E_{13}^{(1)}$	181.4624	180.0003	177.9936	175.0855	170.5205
		$E_{13}^{(2)}$	173.0461	171.6518	169.7381	166.9650	162.6117
	0.8	$\mu_{13}^{(0)}$	0.6120	0.5730	0.5232	0.4572	0.3659
		$E_{13}^{(1)}$	239.7321	236.7151	232.8269	227.6469	220.4315
		$E_{13}^{(2)}$	228.6132	225.7361	222.0283	217.0885	210.2078
	0.9	$\mu_{13}^{(0)}$	0.5474	0.5096	0.4630	0.4040	0.3271
		$E_{13}^{(1)}$	393.0190	384.4791	373.8422	360.2781	342.4515
		$E_{13}^{(2)}$	374.7906	366.6468	356.5033	343.5682	326.5684
0.5	0.6	$\mu_{13}^{(0)}$	0.6890	0.6058	0.4431	*	*
		$E_{13}^{(1)}$	150.7936	149.6352	147.3410	140.8400	**
		$E_{13}^{(2)}$	139.5700	138.4978	136.3743	130.3572	**
	0.7	$\mu_{13}^{(0)}$	0.6427	0.5857	0.5016	0.3651	0.1052
		$E_{13}^{(1)}$	180.1118	178.0750	175.0451	170.1042	160.6985
		$E_{13}^{(2)}$	166.7060	164.8208	162.0165	157.4433	148.7377
	0.8	$\mu_{13}^{(0)}$	0.5958	0.5482	0.4849	0.3963	0.2636
		$E_{13}^{(1)}$	235.1634	231.1981	225.8736	218.3839	207.1301
		$E_{13}^{(2)}$	217.6601	213.9900	209.0617	202.1296	191.7134
	0.9	$\mu_{13}^{(0)}$	0.5274	0.4831	0.4272	0.3542	0.2548
		$E_{13}^{(1)}$	378.6636	368.1038	354.6093	336.8406	312.4969
		$E_{13}^{(2)}$	350.4796	340.7057	328.2156	311.7694	289.2377
0.6	0.6	$\mu_{13}^{(0)}$	0.6703	0.4867	*	*	*
		$E_{13}^{(1)}$	151.5113	149.3725	138.9485	172.8835	160.7811
		$E_{13}^{(2)}$	134.6767	132.7756	123.5098	153.6742	142.9165
	0.7	$\mu_{13}^{(0)}$	0.6252	0.5438	0.4055	0.1192	*
		$E_{13}^{(1)}$	179.3545	176.4170	171.4019	161.0463	127.7445
		$E_{13}^{(2)}$	159.4263	156.8151	152.3572	143.1523	113.5507

Note: * does not exist
** no gain

Table 3 (continued).

ρ_{xy}	ρ_0	$\rho_{z_1 z_2}$	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**
0.6	0.8	$\mu_{13}^{(0)}$	0.5778	0.5179	0.4333	0.3048	0.0863
		$E_{13}^{(1)}$	231.4250	226.1950	218.7627	207.4412	188.2306
		$E_{13}^{(2)}$	205.7111	201.0622	194.4557	184.3922	167.3161
	0.9	$\mu_{13}^{(0)}$	0.5071	0.4546	0.3861	0.2927	0.1582
		$E_{13}^{(1)}$	366.2176	353.1933	336.0178	312.4628	278.3582
		$E_{13}^{(2)}$	325.5267	313.9496	298.6825	277.7447	247.4295

Table 4. Optimum values of $\mu_{14}^{(0)}$ and percent relative efficiencies of the estimator T_{14} with respect to the estimators $\bar{y}_n$ and $\hat{\hat{Y}}$

ρ_{xy}	ρ_0	$\rho_{z_1 z_2}$	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**
0.3	0.6	$\mu_{14}^{(0)}$	0.8051	0.7705	0.7147	0.6091	0.3344
		$E_{14}^{(1)}$	158.3813	158.6421	159.0791	159.9277	162.1820
		$E_{14}^{(2)}$	154.3284	154.5826	155.0083	155.8353	158.0319
	0.7	$\mu_{14}^{(0)}$	0.9949	*	*	*	0.6249
		$E_{14}^{(1)}$	196.0800	196.0860	196.1594	196.7016	194.8757
		$E_{14}^{(2)}$	191.0625	191.0683	191.1398	191.6681	189.8890
	0.8	$\mu_{14}^{(0)}$	*	*	*	*	0.4363
		$E_{14}^{(1)}$	291.1143	308.1703	**	241.4353	257.4570
		$E_{14}^{(2)}$	283.6649	300.2845	**	235.2572	250.8689
0.4	0.6	$\mu_{14}^{(0)}$	0.7018	0.6591	0.5995	0.5101	0.3613
		$E_{14}^{(1)}$	167.9310	169.3375	171.3226	174.3229	179.3608
		$E_{14}^{(2)}$	160.1423	161.4835	163.3766	166.2377	171.0420
	0.7	$\mu_{14}^{(0)}$	0.7946	0.7546	0.6880	0.5545	0.1528
		$E_{14}^{(1)}$	202.2086	203.0783	204.5727	207.6376	217.0745
		$E_{14}^{(2)}$	192.8300	193.6595	195.0846	198.0073	207.0065

Note: * does not exist
** no gain

Table 4 (continued).

ρ_{xy}	ρ_0	$\rho_{z_1z_2}$	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**
0.4	0.8	$\mu_{14}^{(0)}$	0.9823	0.9975	*	*	0.7975
		$E_{14}^{(1)}$	277.8293	277.7785	277.8647	279.6778	276.2330
		$E_{14}^{(2)}$	264.9434	264.8950	264.9772	266.7062	263.4212
0.5	0.6	$\mu_{14}^{(0)}$	0.6889	0.6531	0.6067	0.5445	0.4562
		$E_{14}^{(1)}$	192.4040	196.1477	201.0234	207.6273	217.0641
		$E_{14}^{(2)}$	178.0833	181.5484	186.0612	192.1736	200.9080
	0.7	$\mu_{14}^{(0)}$	0.7286	0.6858	0.6240	0.5269	0.3524
		$E_{14}^{(1)}$	225.7896	229.8347	235.7374	245.1169	262.2443
		$E_{14}^{(2)}$	208.9840	212.7280	218.1914	226.8728	242.7254
	0.8	$\mu_{14}^{(0)}$	0.8061	0.7633	0.6870	0.5122	*
		$E_{14}^{(1)}$	294.9041	297.7125	302.8927	315.1260	**
		$E_{14}^{(2)}$	272.9543	275.5537	280.3483	291.6711	**
0.6	0.6	$\mu_{14}^{(0)}$	*	*	*	*	*
		$E_{14}^{(1)}$	**	**	**	**	**
		$E_{14}^{(2)}$	**	**	**	**	**
	0.7	$\mu_{14}^{(0)}$	0.8733	0.8568	0.8349	0.8043	0.7588
		$E_{14}^{(1)}$	336.3958	354.1668	378.2031	412.4970	465.3477
		$E_{14}^{(2)}$	299.0185	314.8150	336.1806	366.6640	413.6424
	0.8	$\mu_{14}^{(0)}$	0.8278	0.7963	0.7486	0.6676	0.5000
		$E_{14}^{(1)}$	390.2813	409.1458	438.3804	489.5727	601.8214
		$E_{14}^{(2)}$	346.9168	363.6851	389.6715	435.1758	534.9524

Note: * does not exist
** no gain

Table 5. Optimum values of $\mu_{21}^{(0)}$ and percent relative efficiencies of the estimator T_{21} with respect to the estimators $\bar{y}_n$ and $\hat{\bar{Y}}$

ρ_{yx}	ρ_0	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**
0.3	$\mu_{21}^{(0)}$	*	0.6759	0.5064	0.3831	*
	$E_{21}^{(1)}$	124.9983	160.8880	219.5400	360.7346	124.9983
	$E_{21}^{(2)}$	121.7997	156.7710	213.9222	351.5036	121.7997
0.4	$\mu_{21}^{(0)}$	*	0.6497	0.4860	0.3639	*
	$E_{21}^{(1)}$	125.0000	159.3224	213.8407	342.9515	125.0000
	$E_{21}^{(2)}$	119.2024	151.9330	203.9227	327.0452	119.2024
0.5	$\mu_{21}^{(0)}$	*	0.6431	0.4747	0.3511	*
	$E_{21}^{(1)}$	124.8775	158.8769	210.3709	330.9411	124.8775
	$E_{21}^{(2)}$	115.5828	147.0516	194.7129	306.3091	115.5828
0.6	$\mu_{21}^{(0)}$	*	0.6531	0.4697	0.3428	*
	$E_{21}^{(1)}$	123.2817	159.5457	208.7938	323.1029	123.2817
	$E_{21}^{(2)}$	109.5837	141.8184	185.5945	287.2026	109.5837
0.7	$\mu_{21}^{(0)}$	*	0.6845	0.4703	0.3381	*
	$E_{21}^{(1)}$	101.9803	161.3349	208.9670	318.5806	101.9803
	$E_{21}^{(2)}$	*	135.7134	175.7811	267.9869	*
0.8	$\mu_{21}^{(0)}$	*	0.7589	0.4764	0.3364	*
	$E_{21}^{(1)}$	**	164.1551	210.9059	316.9390	**
	$E_{21}^{(2)}$	**	127.6762	164.0379	246.5081	**
0.9	$\mu_{21}^{(0)}$	0.1445	1.0008	0.4892	0.3375	0.1445
	$E_{21}^{(1)}$	156.9299	166.6666	214.7873	318.0302	156.9299
	$E_{21}^{(2)}$	156.9299	114.4342	147.4741	218.3613	156.9299

Note: * does not exist
** no gain

Table 6. Optimum values of $\mu_{22}^{(0)}$ and percent relative efficiencies of the estimator T_{22} with respect to the estimators $\bar{y}_n$ and $\hat{\bar{Y}}$

ρ_{yx}	ρ_0	$\rho_{z_1z_2}$	0.4	0.5	0.6	0.7	0.8
0.3	0.6	$\mu_{22}^{(0)}$	0.9900	0.9972	*	*	*
		$E_{22}^{(1)}$	124.9957	124.9997	124.9987	124.9908	124.9724
		$E_{22}^{(2)}$	121.7972	121.8010	121.8001	121.7923	121.7745
	0.7	$\mu_{22}^{(0)}$	0.7799	0.7656	0.7483	0.7267	0.6991
		$E_{22}^{(1)}$	162.3520	162.2089	162.0254	161.7867	161.4698
		$E_{22}^{(2)}$	158.1976	158.0581	157.8793	157.6467	157.3379
	0.8	$\mu_{22}^{(0)}$	0.6644	0.6406	0.6121	0.5771	0.5333
		$E_{22}^{(1)}$	227.6079	226.4697	225.0821	223.3607	221.1785
		$E_{22}^{(2)}$	221.7835	220.6745	219.3224	217.6451	215.5186
0.4	0.6	$\mu_{22}^{(0)}$	0.9823	0.9903	*	*	*
		$E_{22}^{(1)}$	124.9855	124.9962	125.0000	124.9931	124.9681
		$E_{22}^{(2)}$	119.1886	119.1988	119.2024	119.1958	119.1720
	0.7	$\mu_{22}^{(0)}$	0.7652	0.7470	0.7240	0.6943	0.6541
		$E_{22}^{(1)}$	161.2034	160.9714	160.6645	160.2494	159.6693
		$E_{22}^{(2)}$	153.7267	153.5054	153.2128	152.8170	152.2638
	0.8	$\mu_{22}^{(0)}$	0.6488	0.6207	0.5861	0.5426	0.4860
		$E_{22}^{(1)}$	223.0847	221.5307	219.5887	217.1067	213.8407
		$E_{22}^{(2)}$	212.7380	211.2560	209.4041	207.0371	203.9227
0.5	0.6	$\mu_{22}^{(0)}$	*	*	*	*	*
		$E_{22}^{(1)}$	124.9920	124.9665	124.9117	124.8030	124.5768
		$E_{22}^{(2)}$	115.6888	115.6652	115.6145	115.5139	115.3045
	0.7	$\mu_{22}^{(0)}$	0.7633	0.7412	0.7122	0.6720	0.6132
		$E_{22}^{(1)}$	160.8262	160.5258	160.1070	159.5026	158.5823
		$E_{22}^{(2)}$	148.8558	148.5778	148.1902	147.6308	146.7790

Note: * does not exist
** no gain

Table 6 (continued).

ρ_{yx}	ρ_0	$\rho_{z_1z_2}$	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**
0.5	0.8	$\mu_{22}^{(0)}$	0.6383	0.6052	0.5633	0.5084	0.4334
		$E_{22}^{(1)}$	220.0004	218.0108	215.4460	212.0384	207.3241
		$E_{22}^{(2)}$	203.6257	201.7842	199.4103	196.2563	191.8929
0.6	0.6	$\mu_{22}^{(0)}$	*	*	*	*	*
		$E_{22}^{(1)}$	124.6102	124.3316	123.8140	122.6460	118.2052
		$E_{22}^{(2)}$	110.7646	110.5170	110.0569	109.0186	105.0713
	0.7	$\mu_{22}^{(0)}$	0.7720	0.7459	0.7091	0.6531	0.5579
		$E_{22}^{(1)}$	161.2025	160.8682	160.3609	159.5457	158.0984
		$\mu_{22}^{(0)}$	143.2911	142.9940	142.5430	141.8184	140.5319
	0.8	$E_{22}^{(1)}$	0.6312	0.5918	0.5399	0.4684	0.3633
		$E_{22}^{(2)}$	218.1099	215.6326	212.3068	207.6494	200.7245
		$E_{22}^{(2)}$	193.8754	191.6734	188.7172	184.5772	178.4218

Note: * does not exist
** no gain

Table 7. Optimum values of $\mu_{23}^{(0)}$ and percent relative efficiencies of the estimator T_{23} with respect to the estimators $\bar{y}_n$ and $\hat{\bar{Y}}$

ρ_{xy}	ρ_0	$\rho_{z_1z_2}$	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**
0.3	0.6	$\mu_{23}^{(0)}$	0.9823	0.9922	*	*	*
		$E_{23}^{(1)}$	124.9849	124.9975	124.9987	124.9800	124.9210
		$E_{23}^{(2)}$	121.7866	121.7989	121.8001	121.7818	121.7244
	0.7	$\mu_{23}^{(0)}$	0.7909	0.7732	0.7500	0.7186	0.6734
		$E_{23}^{(1)}$	162.4125	162.2399	161.9962	161.6432	161.1092
		$E_{23}^{(2)}$	158.2565	158.0883	157.8509	157.5068	156.9865
	0.8	$\mu_{23}^{(0)}$	0.6770	0.6499	0.6161	0.5727	0.5151
		$E_{23}^{(1)}$	228.0593	226.7618	225.1137	222.9660	220.0717
		$E_{23}^{(2)}$	222.2234	220.9591	219.3532	217.2604	214.4402

Note: * does not exist

Table 7 (continued).

ρ_{xy}	ρ_0	$\rho_{z_1 z_2}$	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**
0.3	0.9	$\mu_{23}^{(0)}$	0.5824	0.5510	0.5130	0.4661	0.4065
		$E_{23}^{(1)}$	396.9078	391.1365	384.0539	375.1909	363.8247
		$E_{23}^{(2)}$	386.7512	381.1276	374.2262	365.5901	354.5147
0.4	0.6	$\mu_{23}^{(0)}$	0.9720	0.9832	*		*
		$E_{23}^{(1)}$	124.9612	124.9886	125.0000	124.9752	124.8351
		$E_{23}^{(2)}$	119.1655	119.1915	119.2024	119.1788	119.0452
	0.7	$\mu_{23}^{(0)}$	0.7695	0.7447	0.7104	0.6598	0.5780
		$E_{23}^{(1)}$	161.0824	160.7577	160.2779	159.5327	158.2762
		$E_{23}^{(2)}$	153.6113	153.3017	152.8441	152.1335	150.9352
	0.8	$\mu_{23}^{(0)}$	0.6562	0.6228	0.5797	0.5221	0.4409
		$E_{23}^{(1)}$	223.1409	221.2761	218.8258	215.4921	210.7338
		$E_{23}^{(2)}$	212.7915	211.0132	208.6765	205.4975	200.9598
	0.9	$\mu_{23}^{(0)}$	0.5598	0.5232	0.4779	0.4203	0.3448
		$E_{23}^{(1)}$	380.3266	372.9314	363.6460	351.6988	335.8323
		$E_{23}^{(2)}$	362.6869	355.6346	346.7799	335.3868	320.2562
0.5	0.6	$\mu_{23}^{(0)}$	0.9976	*	*	*	*
		$E_{23}^{(1)}$	124.9997	124.9813	124.8680	124.3137	**
		$E_{23}^{(2)}$	115.6960	115.6789	115.5741	115.0610	**
	0.7	$\mu_{23}^{(0)}$	0.7562	0.7220	0.6703	0.5831	0.4045
		$E_{23}^{(1)}$	160.3641	159.8555	159.0351	157.5812	154.4997
		$E_{23}^{(2)}$	148.4281	147.9574	147.1980	145.8524	143.0002
	0.8	$\mu_{23}^{(0)}$	0.6382	0.5967	0.5407	0.4612	0.3394
		$E_{23}^{(1)}$	219.3604	216.8137	213.3149	208.2627	200.4141
		$E_{23}^{(2)}$	203.0333	200.6762	197.4378	192.7616	185.4972
	0.9	$\mu_{23}^{(0)}$	0.5392	0.4962	0.4416	0.3698	0.2712
		$E_{23}^{(1)}$	366.8796	357.6462	345.7340	329.8765	307.8596
		$E_{23}^{(2)}$	339.5726	331.0264	320.0009	305.3236	284.9455

Note: * does not exist
** no gain

Table 7 (continued).

ρ_{xy}	ρ_0	$\rho_{z_1z_2}$	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**
0.6	0.6	$\mu_{23}^{(0)}$	*	*	*	0.3855	0.7019
		$E_{23}^{(1)}$	124.7031	124.0416	118.1805	128.9701	127.2655
		$E_{23}^{(2)}$	110.8472	110.2592	105.0493	114.6401	113.1249
	0.7	$\mu_{23}^{(0)}$	0.7472	0.6979	0.6105	0.4136	*
		$E_{23}^{(1)}$	160.2552	159.4993	158.0692	154.7028	139.7695
		$E_{23}^{(2)}$	142.4491	141.7772	140.5060	137.5136	124.2396
	0.8	$\mu_{23}^{(0)}$	0.6204	0.5673	0.4913	0.3731	0.1641
		$E_{23}^{(1)}$	216.4316	212.9830	207.9427	199.9911	185.7935
		$E_{23}^{(2)}$	192.3837	189.3182	184.8379	177.7699	165.1498
	0.9	$\mu_{23}^{(0)}$	0.5187	0.4675	0.4002	0.3078	0.1730
		$E_{23}^{(1)}$	355.3191	343.8360	328.5210	307.2312	275.8615
		$E_{23}^{(2)}$	315.8392	305.6320	292.0187	273.0944	245.2102

Note: * does not exist

Table 8. Optimum values of $\mu_{24}^{(0)}$ and percent relative efficiencies of the estimator T_{24} with respect to the estimators $\bar{y}_n$ and $\hat{\bar{Y}}$

ρ_{xy}	ρ_0	$\rho_{z_1z_2}$	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**
0.3	0.6	$\mu_{24}^{(0)}$	0.4532	0.3098	0.0440	*	*
		$E_{24}^{(1)}$	136.6962	139.2998	144.1264	*	*
		$E_{24}^{(2)}$	133.1982	135.7352	140.4383	*	*
	0.7	$\mu_{24}^{(0)}$	0.7551	0.6649	0.3941	*	*
		$E_{24}^{(1)}$	169.3926	170.0616	172.1917	*	163.2345
		$E_{24}^{(2)}$	165.0580	165.7099	167.7855	*	159.0575
	0.8	$\mu_{24}^{(0)}$	*	*	*	0.3754	0.6200
		$E_{24}^{(1)}$	256.1296	268.0813	195.2127	235.5873	240.5105
		$E_{24}^{(2)}$	249.5754	261.2213	190.2173	229.5587	234.3560

Note: * does not exist

Table 8 (continued).

ρ_{xy}	ρ_0	$\rho_{z_1z_2}$	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**
0.4	0.6	$\mu_{24}^{(0)}$	0.5321	0.4530	0.3359	0.1446	*
		$E_{24}^{(1)}$	146.2100	149.2274	153.7028	160.9946	**
		$E_{24}^{(2)}$	139.4287	142.3061	146.5740	153.5276	**
	0.7	$\mu_{24}^{(0)}$	0.6792	0.6045	0.4705	0.1598	*
		$E_{24}^{(1)}$	178.5578	180.7188	184.6569	193.8518	**
		$E_{24}^{(2)}$	170.2762	172.3370	176.0924	184.8608	**
	0.8	$\mu_{24}^{(0)}$	0.9197	0.9098	0.8804	*	*
		$E_{24}^{(1)}$	250.9050	250.7622	250.6741	**	250.0000
		$E_{24}^{(2)}$	239.2679	239.1317	239.0477	**	238.4049
0.5	0.6	$\mu_{24}^{(0)}$	0.6089	0.5596	0.4940	0.4022	0.2648
		$E_{24}^{(1)}$	168.3577	173.0219	179.2419	187.9328	200.9038
		$E_{24}^{(2)}$	155.8267	160.1439	165.9008	173.9449	185.9504
	0.7	$\mu_{24}^{(0)}$	0.6731	0.6175	0.5345	0.3974	0.1273
		$E_{24}^{(1)}$	201.7462	206.7289	214.2127	226.6348	251.1222
		$E_{24}^{(2)}$	186.7302	191.3420	198.2688	209.7663	232.4311
	0.8	$\mu_{24}^{(0)}$	0.7744	0.7212	0.6220	0.3724	*
		$E_{24}^{(1)}$	270.1604	273.6626	280.3900	297.7184	**
		$E_{24}^{(2)}$	250.0522	253.2938	259.5205	275.5591	**
0.6	0.6	$\mu_{24}^{(0)}$	*	*	*	*	*
		$E_{24}^{(1)}$	**	**	**	**	**
		$E_{24}^{(2)}$	**	**	**	**	**
	0.7	$\mu_{24}^{(0)}$	0.8636	0.8452	0.8205	0.7855	0.7317
		$E_{24}^{(1)}$	300.9492	317.4336	339.8368	371.9968	421.9851
		$E_{24}^{(2)}$	267.5104	282.1632	302.0772	330.6638	375.0979
	0.8	$\mu_{24}^{(0)}$	0.8186	0.7845	0.7321	0.6406	0.4406
		$E_{24}^{(1)}$	359.7332	377.8316	406.1680	456.5742	570.5409
		$E_{24}^{(2)}$	319.7629	335.8503	361.0382	405.8437	507.1475

Note: * does not exist
** no gain

Interpretations of Empirical Results

The following interpretations can be drawn from Tables 1-8:

Table 1

(a) For fixed values of ρ_{xy}, the values of $E_{11}^{(1)}$ and $E_{11}^{(2)}$ are increasing, whereas for increasing values of ρ_0, $\mu_{11}^{(0)}$ decreases. These trends are highly considerable in the estimation procedure of the population mean because it not only reduces the cost of the sample survey but also make the estimator more precise.
(b) For fixed values of ρ_0, the values of $\mu_{11}^{(0)}$ and $E_{11}^{(1)}$ slightly decrease at the very starting level and then significantly increase with increasing values of ρ_{xy}, while a decreasing pattern is observed in $E_{11}^{(2)}$.
(c) The lowest value of $\mu_{11}^{(0)}$ is 0.3215. This implies that only 30% of the sample on the current occasion has to be replaced by fresh units, which leads to a considerable depletion in the cost of the sample survey.

Table 2

(a) For fixed values of ρ_{yx} and ρ_0 the values of $\mu_{12}^{(0)}$, $E_{12}^{(1)}$, and $E_{12}^{(2)}$ are all observed in a decreasing pattern with very slow rate, with the increasing values of $\rho_{z_1 z_2}$.
(b) For fixed values of ρ_{yx} and $\rho_{z_1 z_2}$ the values of $\mu_{12}^{(0)}$ are found to be in a decreasing pattern while $E_{12}^{(1)}$ and $E_{12}^{(2)}$ are increasing with increasing values of ρ_0. These trends are also highly considerable in the estimation procedure of the population mean because it not only reduces the cost of the sample survey but also make the estimator more precise.
(c) For fixed values of ρ_0 and $\rho_{z_1 z_2}$ the values of $E_{12}^{(2)}$ are decreasing while the values of $\mu_{12}^{(0)}$ and $E_{12}^{(1)}$ do not show any regular pattern with increasing values of ρ_{yx}.
(e) The lowest value of $\mu_{12}^{(0)}$ is 0.1358. This implies that only 13% of the sample on the current occasion has to be replaced by fresh units, which leads to a considerable depletion in the cost of the sample survey.

Table 3

(a) For fixed values of ρ_{yx} and ρ_0 the values $\mu_{13}^{(0)}$, $E_{13}^{(1)}$, and $E_{13}^{(2)}$ are all in a decreasing pattern, observed with increasing values of $\rho_{z_1z_2}$.

(b) For fixed values of ρ_{yx} and $\rho_{z_1z_2}$ the values of $\mu_{13}^{(0)}$ are found to follow a decreasing pattern while $E_{13}^{(1)}$ and $E_{13}^{(2)}$ are increasing with the increasing value of ρ_0. This pattern indicates that the smaller fresh sample on the current occasion is required if information on the highly correlated auxiliary variables is available, these behaviors are highly correlated.

(c) For the fixed values of ρ_0 and $\rho_{z_1z_2}$ the values of $\mu_{13}^{(0)}$ and $E_{13}^{(1)}$ are decreasing with increasing values of ρ_{yx} while the values of $E_{13}^{(2)}$ are increasing with increasing values of ρ_{xy}.

(d) The lowest value of $\mu_{13}^{(0)}$ is 0.0863. This implies that only 8% of the sample on the current occasion has to be replaced by fresh units, which leads to a considerable depletion in the cost of the sample survey.

Table 4

a) For fixed values of ρ_{yx} and ρ_0 the optimum values of $\mu_{14}^{(0)}$ follow a decreasing pattern with the increasing values of $\rho_{z_1z_2}$, while the values of $E_{14}^{(1)}$ and $E_{14}^{(2)}$ are increasing with increasing values of $\rho_{z_1z_2}$. These trends are also highly considerable in the estimation procedure of the population mean because it not only reduces the cost of the sample survey but also makes the estimator more precise.

(b) For fixed values of ρ_{yx} and $\rho_{z_1z_2}$ the values $\mu_{14}^{(0)}$, $E_{14}^{(1)}$, and $E_{14}^{(2)}$ are all in an increasing pattern observed with the increasing values of ρ_0.

(c) For fixed values of ρ_0 and $\rho_{z_1z_2}$ the optimum values of $\mu_{14}^{(0)}$ are all decreasing at the very starting level and then a decremental pattern is observed with the increasing values of ρ_{yx}. The values of $E_{14}^{(1)}$ and $E_{14}^{(2)}$ are increasing at the very starting level and then a decremental pattern is observed with the increasing values of ρ_{yx}.

(d) The minimum value of $\mu_{14}^{(0)}$ is 0.1528. This implies that only 15% of the sample on the current occasion has to be replaced by fresh units, which leads to an appreciable reduction in the cost of the sample survey.

Table 5

(a) For fixed values of ρ_{yx}, the values of $E_{21}^{(1)}$ and $E_{21}^{(2)}$ are increasing, whereas for increasing values of ρ_0, $\mu_{21}^{(0)}$ decreases. This trend is also highly desirable in the efficient estimation procedure of the population mean because it not only reduces the cost of the sample survey, but also makes the estimator more precise.

(b) For fixed values of ρ_0, the optimum values of $\mu_{21}^{(0)}$ and $E_{21}^{(1)}$ are found in decreasing patterns at the very starting level and then an incremental pattern is observed, while the values of $E_{21}^{(2)}$ are decreasing with the increasing values of ρ_{yx}.

(c) The minimum value of $\mu_{21}^{(0)}$ is 0.3305. This implies that only 33% of the sample on the current occasion has to be replaced by fresh units, which leads to a considerable depletion in the cost of the sample survey.

Table 6

(a) For fixed values of ρ_{yx} and ρ_0 the values of $\mu_{22}^{(0)}$, $E_{22}^{(1)}$, and $E_{22}^{(2)}$ have a decreasing patterns observed with the increase in values of $\rho_{z_1z_2}$.

(b) For fixed values of ρ_{yx} and $\rho_{z_1z_2}$ the values $\mu_{22}^{(0)}$ are found in a decreasing pattern, and $E_{22}^{(1)}$ and $E_{22}^{(2)}$ are increasing with the increasing values of ρ_0. These patterns also indicate the smaller fresh sample at current occasion is required, if the information on highly correlated auxiliary variables is available, these behaviors are highly correlated.

(c) For fixed values of ρ_0 and $\rho_{z_1z_2}$ the values of $\mu_{22}^{(0)}$, $E_{22}^{(1)}$, and $E_{22}^{(2)}$ decreased with an increase in values of ρ_{yx}.

(d) The minimum value of $\mu_{22}^{(0)}$ is 0.4334. This shows that only 43% of the sample on the current occasion has to be replaced by fresh units, which leads to a considerable depletion in the cost of the sample survey.

Table 7

(a) For fixed values of ρ_{yx} and ρ_0 the values $\mu_{23}^{(0)}$, $E_{23}^{(1)}$, and $E_{23}^{(2)}$ are all decreasing with increasing values of $\rho_{z_1z_2}$.

(b) For fixed values of ρ_{yx} and $\rho_{z_1z_2}$ the values $\mu_{23}^{(0)}$ are found in a decreasing pattern while $E_{23}^{(1)}$ and $E_{23}^{(2)}$ are increasing with the increasing values of ρ_0. These sequences also indicate the smaller fresh sample on the current occasion is required, if the information on highly correlated auxiliary variables is available, these behaviors are highly correlated.

(d) For fixed values of ρ_0 and $\rho_{z_1z_2}$ the values of $\mu_{23}^{(0)}$, $E_{23}^{(1)}$, and $E_{23}^{(2)}$ are all decreasing with the increasing values of ρ_{yx}.

(e) The lowest value of $\mu_{23}^{(0)}$ is 0.2712. This shows that only 27% of the sample on the current occasion is substituted with fresh units, which leads to a considerable depletion in the cost of the sample survey.

Table 8

(a) For fixed values of ρ_{yx} and ρ_0 the optimum values of $\mu_{24}^{(0)}$ are found to be in an approximately decreasing pattern while $E_{24}^{(1)}$ and $E_{24}^{(2)}$ are increased with increasing values of $\rho_{z_1z_2}$. These results indicate the smaller fresh sample on the current occasion is required if the information on highly correlated auxiliary variables is available, these behaviors are highly correlated.

(b) For fixed values of ρ_0 and $\rho_{z_1z_2}$ the values of $\mu_{24}^{(0)}$ are found to be in an increasing pattern while $E_{24}^{(1)}$ and $E_{24}^{(2)}$ decreases with increasing values of ρ_{yx}. This pattern also indicates the smaller fresh sample at current occasion is required if the information on highly correlated auxiliary variables is available, these behaviors are highly correlated.

(c) For fixed values of ρ_{yx} and $\rho_{z_1z_2}$ the values of $\mu_{24}^{(0)}$, $E_{24}^{(1)}$, and $E_{24}^{(2)}$ increased with the increase in ρ_{yx}.

(d) The lowest value of $\mu_{24}^{(0)}$ is 0.1273. This shows that only 12% of the sample on the current occasion has to be replaced by fresh units, which leads to a considerable depletion in the cost of the sample survey.

Mutual Comparisons of the Proposed Estimators

Mutual comparisons of the proposed estimators T_{ij} (i = 1, 2; j = 1, 2, 3, 4) for their optimum mean square errors have been made with respect to the natural successive sampling estimator $\hat{\hat{Y}}$ (when there is no auxiliary information used on any occasion). For this comparison, we fix $\rho_{z_1z_2}$ and f for different choices of ρ_{yx} and ρ_0. The percent relative efficiencies $E_{ij}^{(2)}$ are given in Tables 9, 10, and 11.

Table 9. The percent relative efficiencies of the proposed estimators T_{ij} with respect to $\hat{\bar{Y}}$ for $\rho_{z_1 z_2} = 0.4$

ρ_{xy}	ρ_0	**0.6**	**0.7**	**0.8**
0.3	$E_{11}^{(2)}$	144.6940	173.4101	226.8353
	$E_{12}^{(2)}$	147.1126	178.4900	238.5757
	$E_{13}^{(2)}$	147.2100	178.8200	239.2069
	$E_{14}^{(2)}$	154.3280	191.0625	283.6649
	$E_{21}^{(2)}$	121.7979	156.7710	213.9272
	$E_{22}^{(2)}$	121.7972	158.0581	157.8791
	$E_{23}^{(2)}$	121.7866	158.2565	222.7264
	$E_{24}^{(2)}$	131.1982	165.0580	249.5754
0.4	$E_{11}^{(2)}$	141.5000	167.7663	216.1094
	$E_{12}^{(2)}$	144.0046	173.2253	228.5805
	$E_{13}^{(2)}$	143.6637	173.0461	228.6123
	$E_{14}^{(2)}$	160.1423	192.8300	264.9443
	$E_{21}^{(2)}$	119.2024	151.9330	203.9227
	$E_{22}^{(2)}$	119.1886	153.7261	212.7951
	$E_{23}^{(2)}$	119.1656	153.6113	212.7951
	$E_{24}^{(2)}$	139.4278	170.2762	239.2628
0.5	$E_{11}^{(2)}$	138.2800	162.3069	206.2895
	$E_{12}^{(2)}$	140.5170	167.6554	218.5372
	$E_{13}^{(2)}$	139.5700	166.7060	217.6601
	$E_{14}^{(2)}$	178.0833	208.9840	272.9543
	$E_{21}^{(2)}$	115.5828	147.0516	194.7129
	$E_{22}^{(2)}$	115.6888	148.5858	203.6252
	$E_{23}^{(2)}$	115.6960	148.4281	203.0330
	$E_{24}^{(2)}$	155.8267	186.7302	250.0522
0.6	$E_{11}^{(2)}$	134.7890	156.6327	196.6052
	$E_{12}^{(2)}$	136.3244	161.4230	207.8138

Table 9 (continued).

ρ_{xy}	ρ_0	**0.6**	**0.7**	**0.8**
0.6	$E_{13}^{(2)}$	134.6767	159.4263	205.7110
	$E_{14}^{(2)}$	**	229.0180	346.9100
	$E_{21}^{(2)}$	109.5831	141.8148	185.5940
	$E_{22}^{(2)}$	110.7646	143.2929	193.8170
	$E_{23}^{(2)}$	110.8472	142.4991	192.3600
	$E_{24}^{(2)}$	**	267.5109	319.7620

Note: ** no gain

Table 10. The percent relative efficiencies of the proposed estimators T_{ij} with respect to $\hat{\bar{Y}}$ for $\rho_{z_1 z_2} = 0.6$

ρ_{xy}	ρ_0	**0.6**	**0.7**	**0.8**
0.3	$E_{11}^{(2)}$	144.6940	173.4101	226.8353
	$E_{12}^{(2)}$	150.1449	176.7197	234.3901
	$E_{13}^{(2)}$	145.9776	176.4994	234.3024
	$E_{14}^{(2)}$	155.0083	191.1398	**
	$E_{21}^{(2)}$	121.7979	156.7710	213.9272
	$E_{22}^{(2)}$	121.8001	157.8791	219.3224
	$E_{23}^{(2)}$	121.8001	157.8509	219.3534
	$E_{24}^{(2)}$	140.4383	167.7855	190.2173
0.4	$E_{11}^{(2)}$	141.5000	167.7663	216.1094
	$E_{12}^{(2)}$	143.0230	170.9484	223.2368
	$E_{13}^{(2)}$	141.7762	169.7381	222.0283
	$E_{14}^{(2)}$	163.3766	195.0846	264.9772
	$E_{21}^{(2)}$	119.2024	151.9330	203.9227
	$E_{22}^{(2)}$	119.2024	153.2120	209.4014
	$E_{23}^{(2)}$	119.2024	152.8441	208.6765
	$E_{24}^{(2)}$	146.5740	176.0924	239.0477

Note: ** no gain

Table 10 (continued).

ρ_{xy}	ρ_0	**0.6**	**0.7**	**0.8**
0.5	$E_{11}^{(2)}$	138.2800	162.3069	206.2895
	$E_{12}^{(2)}$	139.4486	164.8864	211.9974
	$E_{13}^{(2)}$	136.3743	162.0156	209.0617
	$E_{14}^{(2)}$	186.0612	218.1914	280.3483
	$E_{21}^{(2)}$	115.5828	147.0516	194.7129
	$E_{22}^{(2)}$	115.6652	148.1902	199.4103
	$E_{23}^{(2)}$	115.5741	147.1980	197.4378
	$E_{24}^{(2)}$	165.9008	198.2688	259.5205
0.6	$E_{11}^{(2)}$	134.7890	156.6327	196.6052
	$E_{12}^{(2)}$	135.9615	160.0569	204.4225
	$E_{13}^{(2)}$	123.5098	152.3572	194.4557
	$E_{14}^{(2)}$	**	336.1806	389.6715
	$E_{21}^{(2)}$	109.5831	141.8148	185.5940
	$E_{22}^{(2)}$	110.0569	142.5430	188.7172
	$E_{23}^{(2)}$	105.0493	140.5060	184.8370
	$E_{24}^{(2)}$	**	302.0722	361.0382

Note: ** no gain

Table 11. The percent relative efficiencies of the proposed estimators T_{ij} with respect to $\hat{\bar{Y}}$ for $\rho_{z_1 z_2} = 0.8$

ρ_{xy}	ρ_0	**0.6**	**0.7**	**0.8**
0.3	$E_{11}^{(2)}$	144.6940	173.4101	226.8353
	$E_{12}^{(2)}$	144.9749	173.9691	228.0913
	$E_{13}^{(2)}$	143.1729	172.2221	226.2048
	$E_{14}^{(2)}$	158.0319	189.0319	250.8689
	$E_{21}^{(2)}$	121.7979	156.7710	213.9222
	$E_{22}^{(2)}$	121.7745	157.3379	215.5186
	$E_{23}^{(2)}$	121.7244	156.9865	214.4402

Table 11 (continued).

ρ_{xy}	ρ_0	0.6	0.7	0.8
0.3	$E_{24}^{(2)}$	**	159.0575	234.3560
0.4	$E_{11}^{(2)}$	141.5000	167.7663	216.1094
	$E_{12}^{(2)}$	141.1649	167.1260	214.7253
	$E_{13}^{(2)}$	135.2598	162.6117	210.2078
	$E_{14}^{(2)}$	171.0420	207.0065	263.4212
	$E_{21}^{(2)}$	119.2024	151.9330	203.9227
	$E_{22}^{(2)}$	119.1720	152.2683	203.9227
	$E_{23}^{(2)}$	119.0452	150.9352	200.9598
	$E_{24}^{(2)}$	**	**	238.4095
0.5	$E_{11}^{(2)}$	138.2800	162.3069	206.2895
	$E_{12}^{(2)}$	136.7974	159.6454	200.7785
	$E_{13}^{(2)}$	**	148.7377	191.7134
	$E_{14}^{(2)}$	200.9080	242.7254	**
	$E_{21}^{(2)}$	115.5828	147.0516	194.7129
	$E_{22}^{(2)}$	115.3045	146.7790	191.8929
	$E_{23}^{(2)}$	**	143.0020	185.4972
	$E_{24}^{(2)}$	232.4431	**	**
0.6	$E_{11}^{(2)}$	134.7890	156.6327	196.6052
	$E_{12}^{(2)}$	130.1055	150.6192	185.0459
	$E_{13}^{(2)}$	142.9165	113.5507	167.3161
	$E_{14}^{(2)}$	**	413.6424	534.9524
	$E_{21}^{(2)}$	109.5831	141.8148	185.5940
	$E_{22}^{(2)}$	105.0713	140.5319	178.4218
	$E_{23}^{(2)}$	113.1249	124.2396	165.1498
	$E_{24}^{(2)}$	**	375.0979	507.1475

Note: ** no gain

Conclusion

The use of modified ratio and regression type estimators for the estimation of the current population mean on the current occasion in two-occasion successive sampling is justified through empirical results. The use of highly correlated auxiliary variable z_1 and z_2 which are firmly fixed over time is greatly rewarding; it not only reduces the cost of the survey but also enhances the precision of the estimates as well. Also, it can be seen in the section of mutual comparisons that the proposed estimators T_{14} and T_{24} are more efficient than the other proposed estimators. The suggested class of estimators, especially T_{14} and T_{24}, are more rewarding in terms of precision and also reduce the cost of the survey. The proposed class of estimators may be recommended to survey practitioners for their real-life practical problems.

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