

On Dynamics of $\xi^{(s)}$ Quadratic Stochastic Operators

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Abstract: A quadratic stochastic operator (in short QSO) is usually used to present the time evolution of differing species in biology. Some quadratic stochastic operators have been studied by Lotka and Volterra. The general problem in the nonlinear operator theory is to study the behavior of operators. This problem was not fully finished even for quadratic stochastic operators which are the simplest nonlinear operators. To study this problem, several classes of QSO were investigated. We study one class of ξ^s -QSO defined on 2D simplex. We study the dynamics of this class by analytic way.

Introduction

Nonlinear operators are characterizing many systems . The simplest one of the nonlinear operator is quadratic one. The Quadratic dynamical systems considered as an important source of analysis for the study of dynamical properties and modeling in various fields. A quadratic stochastic operator (in short QSO) is usually used to present the time evolution of species in biology. More precisely, the QSO describes a distribution of the next generation if the current distribution of the generation was given. The fascinating applications of the QSO to population genetics were given in Lyubich, 1992. In Ganikhodzhaev et al., 2011, it was given along self-contained exposition of the recent achievements and open problems in the theory of the QSO. The main

problem in the nonlinear operator theory is to study the behavior of nonlinear operators. This problem was not fully finished even in the class of QSO (the QSO is the simplest nonlinear operator). An asymptotic behavior of the QSO even on the small dimensional simplex is complicated (see Saburov, 2007 and Saburov 2012). In order to solve this problem, many researchers always introduced a certain class of QSO and studied their behaviors. For more information, one may refer to Ganikhodzhaev et al., 2011. However, all these classes of QSO together would not cover all QSO. Therefore, there are many classes of QSO which were not studied yet. Recently, in the paper of Mukhamedov et al., 2011, it was introduced a new class of QSO. This class was called $\xi^{(s)}$ -QSO.

[1] Definitions:

Definition 1.1. The quadratic stochastic operator (QSO) is a mapping of the simplex

$$S^{m-1} = \{x = (x_1, \dots, x_m) \in R^m: x_i \geq 0, \sum_{i=1}^m x_i = 1\} \quad (1.1)$$

into itself, of the form

$$V: x'_k = \sum_{i,j=1}^m P_{i,j,k} x_i x_j, \quad (1.2)$$

Where $P_{ij,k}$ are coefficient of heredity and

$$P_{ij,k} \geq 0, \quad P_{ij,k} = P_{ji,k}, \quad \sum_{k=1}^m P_{ij,k} = 1 \quad (1.3)$$

Thus, each quadratic stochastic operator V can be uniquely defined by a cubic matrix $\mathbf{P} = (P_{ij,k})_{i,j,k=1}^m$ with conditions (1.3).

Note that each element $x \in S^{m-1}$ is a probability distribution on $E = \{1, \dots, m\}$.

For a given $x^{(0)} \in S^{m-1}$ the trajectory $\{x^{(n)}\}$, $n = 0, 1, 2, \dots$ of $x^{(0)}$ under the action of QSO (1.2) is defined by

$$x^{(n+1)} = V(x^{(n)}), \quad \text{where } n = 0, 1, 2, \dots$$

Definition 1.2. ($\xi^{(s)}$ operator)

In order to introduce a new class of QSO. We need some helping notations.

Let us consider $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_m)$. We say that x is equivalent to y ($x \sim y$) if

- (i) $x < y$ (x is absolutely continuous with respect to y). If $y_k = 0 \Rightarrow x_k = 0$,
- (ii) $y < x$ if $x_k = 0 \Rightarrow y_k = 0$.

Let $I = \{1, 2, \dots, m\}$ and denoted $supp(x) = \{i \in I : x_i \neq 0\}$. Then we say that x is singular to y ($x \perp y$) if $supp(x) \cap supp(y) = \emptyset$.

Note that if $x, y \in S^n$, then $x \perp y$ if and only if $(x, y) = 0$. By \mathbf{P} we denoted the set of ordered pairs of I , i.e. $\mathbf{P} = \{(ij) : i < j, i, j \in I\}$ let $\xi = \{A_i\}_{i=1}^m$ be a partition of \mathbf{P} , i.e. $A_i \cap A_j = \emptyset, \cup_{i=1}^m A_i = \mathbf{P}$. A quadratic stochastic operator V given by (1.2), (1.3) is called $\xi^{(s)}$ – QSO if the following conditions are satisfied:

- (i) For every $(ij), (uv) \in A_k, \forall k = \{1, \dots, n\}$ one has $(P_{ij,1}, P_{ij,2}, \dots, P_{ij,n}) \sim (P_{uv,1}, P_{uv,2}, \dots, P_{uv,n})$
- (ii) For every $(ij) \in A_k, (uv) \in A_l, k \neq l$ one has $(P_{ij,1}, P_{ij,2}, \dots, P_{ij,n}) \perp (P_{uv,1}, P_{uv,2}, \dots, P_{uv,n})$
- (iii) For all $i, j \in I$, one has $(P_{ii,1}, P_{ii,2}, \dots, P_{ii,n}) \perp (P_{jj,1}, P_{jj,2}, \dots, P_{jj,n})$.

Definition 1.3. Let $\{x^{(n)}\}_{n=1}^\infty$ be the trajectory of the point $x^{(0)} \in S^{m-1}$ under QSO. Denoted by $w(x^{(0)})$ the set of limit points of the trajectory. Since $x^{(n)} \subset S^{m-1}$ and S^{m-1} is compact, it follows that $w(x^{(0)}) \neq \emptyset$. Obviously, if $w(x^{(0)})$ consists of a single point, then the trajectory converges, and $w(x^{(0)})$ is a fixed point of QSO.

In this paper we are going to study the dynamics of $\xi^{(s)}$ –QSO on 2D simplex which appeared from this partition $\xi_1 = \{(1,2), \{(1,3), (2,3)\}$ Namely, we consider the following operator

$$V_a: S^2 \rightarrow S^2$$

$$V_a: \begin{cases} x' = x^2 + 2ax(1-x), \\ y' = y(1-x+z), \\ z' = z^2 + 2(1-a)x(1-x), \end{cases} \quad (2)$$

Where $0 < a < 1$ Let e_1, e_2, e_3 be vertices of the simplex S^2

Theorem 1.4. Let $V_a: S^2 \rightarrow S^2$ be a $\xi^{(s)}$ -QSO given by (2). Then the following statement holds true :

- (i) One has that $Fix(V_a) = \{e_1, e_2, e_3\}$ if $a \neq \frac{1}{2}$, $\{y = 0\} \cup \{x = z\}$ if $a = \frac{1}{2}$
- (ii) Let $0 < a < \frac{1}{2}$, Let $x^0 = (x^0, y^0, z^0) \in S^2$ be an initial point. If $y^0 = 0$ then $w(x^0) = e_3$ If $y^0 \neq 0$ then $w(x^0) = e_2$;
- (iii) Let $0 < a < \frac{1}{2}$, Let $x^0 = (x^0, y^0, z^0) \in S^2$ be an initial point. If $x^0 = 0$ then $w(x^0) = e_2$. If $x^0 \neq 0$ then $w(x^0) = e_1$;
- (iv) Let $a = \frac{1}{2}$, Let $x^0 = (x^0, y^0, z^0) \in S^2$ be an initial point. If $x^0 \geq \frac{1}{2}$ then $w(x^0) = (x^0, 0, 1 - x^0)$ If $x^0 < \frac{1}{2}$ then $w(x^0) = (x^0, 1 - 2x^0, x^0)$

Proof [i]: to find fixed points of V_a we need to solve the equation, $Vx = x$, namely

$$x^2 + 2ax(1 - x) = x$$

$$y(1 - x + z) = y$$

$$z^2 + 2(1 - a)x(1 - x) = z$$

From here we find

$$x = 0, x = 1, a = \frac{1}{2};$$

Therefore, we have two cases :

$$I) a \neq \frac{1}{2};$$

$$II) a = \frac{1}{2};$$

Let $a \neq \frac{1}{2}$. If $x = 0$, then $z^2 = z$. So, $z = 0, z = 1$.

If $z = 0$, then $y = 1$. Therefore, $(0,1,0)$ is a fixed point.

If $z = 1$, then $y = 0$. Therefore $(0,0,1)$ is a fixed point.

If $x = 1$, then $z^2 = z$. Therefore $z = 0, z = 1$.

The equation $z = 1$ is insolvable .

If $z = 0$, then $y = 0$. Hence $(1,0,0)$ is a fixed point

This means that when $a \neq \frac{1}{2}$, the fixed points are $\{(1,0,0), (0,1,0), (0,0,1)\}$.

Now, consider the case when $a = \frac{1}{2}$,

Let us take

$$y = y(1 - x + z)$$

If $y = 0$, then $x = x$ and $z = z$. Therefore x, z are arbitrary so, all points in the line $y = 0$ are fixed points.

Now, if $y \neq 0$, then $x = z$. So, $x = x, z = z$. This means that the line $x = z$, is fixed point.

Consequently, the fixed points of the V_a are

$$\begin{cases} \{(1,0,0), (0,1,0), (0,0,1)\} & a \neq \frac{1}{2}. \\ \{(x, 1 - 2x, x), (x, 0, 1 - x)\}, x \in [0,1] & a = \frac{1}{2}. \end{cases}$$

Proof [ii]:

It is easy to see that the line $y = 0$ is invariant w.r.t. V_a .

Also, let $x = 0$. Then $x' = 0$. Therefore, the line $x = 0$ is invariant under V_a .

Now, if $y = 0$. Then V_a becomes

$$V_a: \begin{cases} x' = x^2 + 2ax(1 - x) \\ y' = 0 \\ z' = (1 - x)^2 + 2(1 - a)x(1 - x) \end{cases}$$

Now, consider the function $f(x) = (1 - 2a)x^2 + 2ax$.

Now, pick up $x_0, 0 < x_0 < 1$. We want to show that $f(x_0) \leq x_0$. Indeed, from

$$(1 - 2a)x^2 + 2ax \leq x, \text{ one find } (1 - 2a)x^2 \leq (1 - 2a)x.$$

if, $0 \leq a \leq \frac{1}{2}$. Then $(1 - 2a) > 0$. Thus, $x^2 \leq x$ therefore $x - x^2 \geq 0$.

So, if $0 < x_0 < 1$ then $f(x_0) \leq x_0$. This means that for any $n \in \mathbb{N}$ $f^{(n+1)}(x_0) \leq f^{(n)}(x_0)$. So, the sequence $x^{(n)} = f^{(n)}(x_0)$ is decreasing and bounded moreover $\{x^{(n)}\}$ converges to x^* . One can see that x^* is fixed point of $f(x)$. The only possibility is $x^* = 0$.

Now, since $\{x^{(n)}\}$ converges to 0, and $z^n = 1 - x^n \rightarrow 1$.

So, if $0 < x_0 < 1, y = 0$, then $(x_0) = (0,0,1)$.

Now, when $x = 0$. Then V_a becomes

$$V_a: \begin{cases} x' = 0 \\ y' = y(1 + z) \\ z' = z^2 \end{cases}$$

Then one can establish that

$$\lim_{n \rightarrow \infty} z^n = 0.$$

So, $y^{(n)} = 1 - z^{(n)}$. Therefore, when $x = 0$. then $w(x^0) = (0,1,0)$.

Now, let $z = 0$. Then V_a becomes

$$V_a: \begin{cases} x' = x^2 + 2ax(1 - x) \\ y' = y(1 - x) \\ z' = 2(1 - a)x(1 - x) \end{cases}$$

if, $x' \neq 0, y' \neq 0, z' \neq 0$, sthen $w(x^0)$ goes to the interior of the simplex S^2

Now, we want to show that $w(x^0) = (0,1,0)$, when $x^0 \in \text{int}(S^2)$. In order to do that let us divided $\text{int}(S^2)$ into two regions. Namely,

$$S_1 = \{(x, y, z): x > z\}, \quad S_2 = \{(x, y, z): x < z\}.$$

First we want to show that S_2 is invariant w.r.t. V_a . Let $(x, y, z) \in S_2, i. e. x < z$. then,

$$z' = z_0^2 + 2(1 - a)x_0(1 - x_0) > x_0^2 + 2ax_0(1 - x_0) = x' , \text{ when } 0 < a < \frac{1}{2}.$$

This implies the assertion.

Claim: if $x_0 > z_0$ then there exists an n such that $x^n < z^n$.

Proof: we suppose the contrary i.e. if $x_0 > z_0$ then $x^n \geq z^n$ for all n . Since $S_1 = \{(x, y, z): x \geq z\}$ is compact set then there is a subsequence such that $(x^{n_k}, y^{n_k}, z^{n_k})$ goes to $(x^*, y^*, z^*) \in S_1$, where $x^{n_k} \geq z^{n_k}$. We know that the sequence $\{x^{(n)}\}$ is decreasing, so it is convergent and its converges to 0. From $x^{(n)} \geq z^{(n)}$ one finds that $z^{(n)} \rightarrow 0$. Hence $(x^*, y^*, z^*) = (0,1,0)$.

On the other hand , from $y^{n+1} = y^n(1 - x^n + z^n)$, we have $y^{n+1} \leq y^n$, here we have used $x^{(n)} \geq z^{(n)}$. But the sequence $\{y^{(n)}\}$ converges to 1, se we come to a contradiction. This proves the claim.

Thus, it is enough to study the dynamics of V_a on S_2 .

Now, we are going to show that if $x_0 < z_0$ then $w(x^0) = (0,1,0)$.

If $x^0 < z^0$ then we have that $x^n < z^n, \forall n \geq 1$, and $y^{n+1} \leq y^n$.

Hence , using the same argument as in the proof of claim one sets $(x^{(n)}, y^{(n)}, z^{(n)}) \rightarrow (0,1,0), i. e. w(x_0) = \{(0,1,0)\}. \lim_{n \rightarrow \infty} y^n = 1$.

Proof [iii]:

Let us consider $y = 0$. then the V_a becomes

$$V_a: \begin{cases} x' = (1 - 2a)x^2 + 2ax \\ y' = 0 \\ z' = (2a - 1)z^2 + 2(1 - a)z \end{cases}$$

Consider the function $f(x) = (1 - 2a)x^2 + 2ax$.

We want to show that if $0 < x_0 < 1$, then $w(x^0) = \{(1,0,0)\}$.

One can see that

$f'(x) = 2(1 - 2a)x + 2a$. one can establish

$$2(1 - 2a)x + 2a > 0 \text{ if } 0 < x < 1.$$

Hence, $f(x)$ is increasing. Now, pick up x_0 , where $0 < x_0 < 1$. We want to prove that $f(x_0) > x_0$. Indeed, from $(1 - 2a)x^2 + 2ax > x$ one find $(1 - 2a)x^2 > (1 - 2a)x \dots (*)$

Since $(1 - 2a) < 0$ when $\frac{1}{2} < a < 1$ and $x^2 < x$, we get $(*)$ for all $0 \leq x \leq 1$.

So,

$$x^0 < f(x^0) = x' < f(x') = x'' < \dots < f(x^{(n-1)}) = x^{(n)}$$

Then,

$\lim_{n \rightarrow \infty} x^n = 1$. Therefore, $w(x^0) = \{(1,0,0)\}$.

Now, let us consider $x = 0$, then the V_a becomes

$$V_a: \begin{cases} x' = 0 \\ y' = y(1 + z) \\ z' = z^2 \end{cases}$$

One finds, $z' = f(z^0) = (z^0)^2 < z^0$, then $z^{(n)}$ is decreasing. Therefore, $\lim_{n \rightarrow \infty} z^n = 0$. Hence, $y^n \rightarrow 1$. So, if $0 < x_0 < 1$, then $w(x^0) = \{(0,1,0)\}$.

Now, let us consider $z = 0$, then the V_a becomes

$$V_a: \begin{cases} x' = (1 - 2a)x^2 + 2ax \\ y' = y(1 - x) \\ z' = 2(1 - a)x(1 - x) \end{cases}$$

So, $x' \neq 0, y' \neq 0, z' \neq 0$. then $w(x^0)$ goes to the interior of the simplex S^2

Now, we want to show that the behavior of $w(x^0) = \{(0,1,0)\}$ when $x^0 \in \text{int}(S^2)$. In order to do that let us divided $\text{int}(S^2)$ into two regions.

$$S_1 = \{(x, y, z): x > z\}, S_2 = \{(x, y, z): x < z\}$$

Let $x_0 > z_0$ then from.

$z' = z_0^2 + 2(1 - a)x_0(1 - x_0) < x_0^2 + 2ax_0(1 - x_0) = x'$, when $\frac{1}{2} < a < 1$, we have $z' < x'$. This means the region S_1 is invariant. we want to show that if $z_0 > x_0$, then one can find $n \in \mathbb{N}$ such that $z^n < x^n$. assume the contrary, i.e. if $x_0 < z_0$ then $x^n < z^n$ for all n . From

$y^{n+1} = y^n(1 - x^n + z^n)$ and $x^n < z^n$ we set $y^{n+1} > y^n$, therefore, y^n is increasing and y^n converges to 1 which is leading to a contradiction. Hence, we get the assertion.

Now, we are going to prove that the trajectory of any initial point in the region S_1 converges to $(1,0,0)$. Since $z_0 < x_0$ we find $z^{(n)} < x^{(n)}$. On the other hand, $x^{(n+1)} > x^{(n)}$. Hence, $x^{(n)}$ converges to x^* , which a fixed point of $f(x)$. The only possibility is $x^* = 1$. Consequently, $w(x_0) = \{(1,0,0)\}$.

Proof [iv]:

Now, consider $a = \frac{1}{2}$.

$$V_a: \begin{cases} x' = x \\ y' = y(1 - x + z) \\ z' = z^2 + x(1 - x) \end{cases}$$

Let $y = 0$ then $y' = 0$ therefore, the line $y = 0$ is invariant.

If $x = z$ then $z^2 + z(1 - z) = z$ and we have $x^{n+1} = x^n$. From here $y^{n+1} = y^n(1 - x^n + z^n) = y^n(1 - x^n + z^n)$ if $x^0 \geq \frac{1}{2}$ then $x^n = x^0 \geq \frac{1}{2}$. Therefore, $y^n + z^n = 1 - x^n \leq \frac{1}{2}$, then $z^n \leq \frac{1}{2}$, thus $x^n \geq z^n$ then $1 - x^n + z^n \leq 1$ therefore, $y^{n+1} \leq y^n$ so, y^n is decreasing. This means that if $x^0 \geq \frac{1}{2}$, then $w(x^0) = \{(x^0, 0, 1 - x^0)\}$.

If $z^0 < x^0 < \frac{1}{2}$, then $x^n < \frac{1}{2}$, and $(z^0)^2 < (x^0)^2$ therefore, $z' = (z^0)^2 + x^0(1 - x^0) < (x^0)^2 + x^0(1 - x^0) = x'$. So, $z^n < x^n < \frac{1}{2}$.

Now, we want to show that z^n is increasing. From

$z' = (z^0)^2 + x^0(1 - x^0) > z^0$, one find $(z^0)^2 - z^0 + x^0 - (x^0)^2 > 0$. This means $(z^0 - x^0)(z^0 + x^0 - 1) > 0$, which is true. So, $z' > z^0$. Therefore, $\{z^{(n)}\}$ is increasing, $y^n + z^n = 1 - x^n = 1 - x^0$. So, $w(x^0) = \{(x^0, 1 - 2x^0, x^0)\}$.

Now, consider $x^0 < z^0$, then $x^0 \leq \frac{1}{2}$ now we want to show that $x^0 = x' < z' = (z^0)^2 + x^0(1 - x^0)$.

$(z^0)^2 > (x^0)^2$ one can find, $z' = (z^0)^2 + x^0(1 - x^0) > (x^0)^2 + x^0(1 - x^0) = x'$. Therefore, $z' > x' = x^0$.

Now, we are going to prove that z^n is decreasing.

$(z^0)^2 + x^0(1 - x^0) < z^0$, from here one can find $(z^0 - x^0)(z^0 + x^0 - 1) < 0$. Therefore, $z' < z^0$.

So, $\{z^{(n)}\}$ is decreasing and $w(x^0) = \{(x^0, 1 - 2x^0, x^0)\}$.

References

Ganikhodzhaev R. N., Mukhamedov F. M., and Rozikov U.A. Quadratic stochastic operators and processes: Results and Open Problems, *Infin. Dim. Anal. Quant. Prob. Rel. Topics* 14 (2) (2011) 279-335

Lyubich Yu.I. *Mathematical structures in population genetics*. Biomathematics, Springer-Verlag (1992).

Mukhamedov F., Saburov M., and Mohd Jamal A.H. On Dynamics of ξ^s quadratic stochastic operators, *Inter. Jour. of Modr. Phys.: Conf. Series* 9 (2012) 299-307.

Saburov M. On ergodic theorem for quadratic stochastic operators, *DANRUz.*, 6 (2007) 8-11.

Saburov M. The Li-Yorke chaos in quadratic stochastic Volterra operators, *OART2012*, Tashkent.