

# ON QUASI TOTALLY $m$ -CLASS $A_k^*$ OPERATORS

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**ABSTRACT.** In this present paper, we will introduce a new class of operators that we call Quasi Totally  $m$ -Class  $A_k^*$  operator in Hilbert spaces. It is a generalization of some previous studies in the field of classes of operators, especially for this especially for a Quasi  $m$ -Class  $A_k^*$ . We will study some properties, provide example and discuss tensor product of this class of operators. example and discuss tensor product of this class of operators.

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**Key words :** Class  $A_k^*$  operator; Quasi  $M$ -Class  $A_k^*$  operator ;Class  $A_k$  operator.

Received:20-12-2024.

Accepted:05.02.2025.

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## 1. INTRODUCTION

The spectral properties of linear operators on Hilbert's space are one of the important tools in quantum mechanics. In fact, linear operators and quantum mechanics have interrelationships. The first goal of this article is to extend operator to a new class of operators and present some properties. In 1998, T.Furuta and m.Itho

have defined the well-known class A operator as  $(\Gamma^{\hat{a}}\Gamma)^2 \leq \Gamma^{\hat{a}^2}\Gamma^{-2}$  see [1]. In 2011, Young min Han and Ju Hee Son have defined quasi-m-hyponormal as  $\Gamma^{\hat{a}}(m^2|\Gamma-\eta|\Gamma) \geq \Gamma^{\hat{a}}|(\Gamma-\eta)^{\hat{a}}|\Gamma$  for all  $\eta \in \square$  and some positive integer m and have studied some properties of this class see [2]. In 2012, S. Panayappan, N. Jayanthi has defined class  $A_k$  operator as  $|\Gamma|^2 \leq |\Gamma^{k+1}|^{\frac{2}{k+1}}$  for some positive integer k see [6]. In 2013, S. Panayappan and Jayanthi introduced and some properties of the class  $A_k^*$  operator as  $|\Gamma^{\hat{a}}|^2 \leq |\Gamma^{k+1}|^{\frac{2}{k+1}}$  some positive integer k and studied some properties and tensor product of this class see [7]. In 2019, P. Shanmugapriya and P. Maheswari Naik has defined m-Class  $A_k^*$  operators as  $|\Gamma^{\hat{a}}|^2 \leq m|\Gamma^{k+1}|^{\frac{2}{k+1}}$  for some m and k are positive integers and studied some spectral properties and tensor product of this class see [10].

Throughout this note we assume that  $H$  is an infinite dimensional separable Hilbert space. Let  $B(H)$  know the algebra of bounded linear operators that act on  $H$ , If  $\Gamma \in B(H)$  the nul space of  $\Gamma$  we will refer to it  $N(\Gamma)$  and the range space of  $\Gamma$  we will refer to it  $R(\Gamma)$ .

**Definition 1.1.** [7] Let  $\Gamma \in B(H)$ . an operator  $\Gamma$  is said to be class  $A_k^*$  if there is a positive integer  $k$  such that

$$\left(\Gamma^{*k}\Gamma^k\right)^{\frac{1}{k}} \geq \Gamma\Gamma^* \quad \text{i.e.} \quad \left|\Gamma^k\right|^{\frac{2}{k}} \geq \left|\Gamma^*\right|^2$$

**Definition 1.2.** Let  $\Gamma \in B(H)$ . an operator  $\Gamma$  is said to be Totally m-Class  $A_k^*$  if there are two positive integers  $k$  and  $m$  such that

$$m\left[\left(\Gamma^k - \eta\right)^* \left(\Gamma^k - \eta\right)\right]^{\frac{1}{k}} \geq (\Gamma - \eta)(\Gamma - \eta)^*; \quad \forall \eta \in \square$$

$$\text{i.e.:} \quad m\left|\Gamma^k - \eta\right|^{\frac{2}{k}} \geq \left|(\Gamma - \eta)^*\right|^2 \quad \forall \eta \in \square$$

**Definition 1.3.** Let  $\Gamma \in B(H)$ . an operator  $\Gamma$  is said to be Quasi Totally m-Class  $A_k^*$  if there are two positive integers  $k$  and  $m$  such that

$$\Gamma^* \left( m \left[ \left( \Gamma^k - \eta \right)^* \left( \Gamma^k - \eta \right) \right]^{\frac{1}{k}} \right) \Gamma \geq \Gamma^* (\Gamma - \eta) (\Gamma - \eta)^* \Gamma \quad \forall \eta \in \square$$

$$\text{i.e } \Gamma^* \left( m|\Gamma^k - \eta|^{\frac{2}{k}} - |(\Gamma - \eta)^*|^2 \right) \Gamma \geq 0; \quad \forall \eta \in \square$$

In particular (choose  $\eta = 0$ ) an operator  $\Gamma$  is called Quasi m-Class  $A_k^*$  [9] if

$$\Gamma^* \left( m(\Gamma^* \Gamma^k)^{\frac{1}{k}} \right) \Gamma \geq \Gamma^* (\Gamma \Gamma^*) \Gamma$$

In general, the following implication holds:

Hyponormal operator [11,12]  $\Rightarrow$  Class  $A_k^*$  operator [7,8]

$\Rightarrow$  Totally m-Class  $A_k^*$  operator

$\Rightarrow$  Quasi Totally m-Class  $A_k^*$  operator.

**Example1.4.** Let  $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in B(\square^2)$

Then

$$I_2^* \left( m|I_2^k - \eta|^{\frac{2}{k}} - |(I_2 - \eta)^*|^2 \right) I_2 = \begin{pmatrix} m|1-\eta|^{\frac{2}{k}} - |1-\eta|^2 & 0 \\ 0 & m|1-\eta|^{\frac{2}{k}} - |1-\eta|^2 \end{pmatrix} ; \forall \eta \in \square.$$

for  $k = 1$  and for  $m \geq 1$  then  $I_2$  is Quasi Totally m-Class  $A_k^*$  operator.

## 2. MAIN RESULTS

The following example shows that  $\Gamma_1$  and  $\Gamma_2$  are a Quasi Totally m-Class  $A_k^*$  operator but the sum  $\Gamma_1 + \Gamma_2$  isn't a Quasi Totally m-Class  $A_k^*$  operator.

**Example2.1:** Let  $\Gamma_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in B(\square^3)$  then

$$\Gamma_1^* \left( m|\Gamma_1^k - \eta|^{\frac{2}{k}} - |(\Gamma_1 - \eta)^*|^2 \right) \Gamma_1 = \begin{pmatrix} m|1-\eta|^{\frac{2}{k}} - |1-\eta|^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

for  $m|1-\eta|^{\frac{2}{k}} \geq |1-\eta|^2$  then  $\Gamma_1$  is a Quasi Totally  $m$ -Class  $A_k^*$  operator.

$$\text{and let } \Gamma_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in B(\mathbb{C}^3)$$

Then

$$\Gamma_2^* \left( m|\Gamma_2^k - \eta|^{\frac{2}{k}} - |(\Gamma_2 - \eta)^*|^2 \right) \Gamma_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & m|\eta|^{\frac{2}{k}} - |\eta|^2 - 1 \end{pmatrix}; \quad \forall \eta \in \mathbb{C}.$$

for  $m|\eta|^{\frac{2}{k}} \geq |\eta|^2 + 1$  then  $\Gamma_2$  is Quasi Totally  $m$ -Class  $A_k^*$  operator.

$$\text{and } \Gamma_1 + \Gamma_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in B(\mathbb{C}^3) \text{ then}$$

$$(\Gamma_1 + \Gamma_2)^* \left( m|(\Gamma_1 + \Gamma_2)^k - \eta|^{\frac{2}{k}} - |(\Gamma_1 + \Gamma_2 - \eta)^*|^2 \right) (\Gamma_1 + \Gamma_2) = \begin{pmatrix} m|1-\eta|^{\frac{2}{k}} - |1-\eta|^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & m|\eta|^{\frac{2}{k}} - |\eta|^2 - 1 \end{pmatrix}; \quad \forall \eta \in \mathbb{C}.$$

for  $\eta = 0$

$$(\Gamma_1 + \Gamma_2)^* \left( m|(\Gamma_1 + \Gamma_2)^k - \eta|^{\frac{2}{k}} - |(\Gamma_1 + \Gamma_2)^*|^2 \right) (\Gamma_1 + \Gamma_2) = \begin{pmatrix} m-1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

so  $\Gamma_1 + \Gamma_2$  isn't Quasi Totally  $m$ -Class  $A_k^*$  operator.

**Proposition 2.2.** Let  $\Gamma_1, \Gamma_2 \in B(H)$  are Quasi Totally  $m$ -Class  $A_k^*$  operator such that

$\Gamma_1^k (\Gamma_2 - \eta) = \Gamma_2^k (\Gamma_1 - \eta) = (\Gamma_2 - \eta)^* \Gamma_1^k = (\Gamma_1 - \eta)^* \Gamma_2^k = 0$ , for  $k \in \mathbb{N}^*$  and for all  $\eta \in \mathbb{C}$  then  $\Gamma_1 + \Gamma_2$  is Quasi Totally  $m$ -class  $A_k^*$  operator.

*Proof.* Let  $\Gamma_1, \Gamma_2 \in B(H)$  are Quasi Totally  $m$ -Class  $A_k^*$  operator then

$$\Gamma_1^* \left( m|\Gamma_1^k - \eta|^{\frac{2}{k}} - |(\Gamma_1 - \eta)^*|^2 \right) \Gamma_1 \geq 0 \quad \text{and} \quad \Gamma_2^* \left( m|\Gamma_2^k - \eta|^{\frac{2}{k}} - |(\Gamma_2 - \eta)^*|^2 \right) \Gamma_2 \geq 0; \quad \forall \eta \in \mathbb{C}$$

so

$$\begin{aligned}
 & (\Gamma_1 + \Gamma_2)^* \left( m' \left( (\Gamma_1 + \Gamma_2)^k - \eta \right)^{\frac{2}{k}} - \left| (\Gamma_1 + \Gamma_2 - \eta)^* \right|^2 \right) (\Gamma_1 + \Gamma_2) \\
 &= (\Gamma_1 + \Gamma_2)^* \left( m' \left| \Gamma_1^k + \Gamma_2^k + \sum_{i=1}^{k-1} \binom{i}{k-1} \Gamma_1^{k-1} \Gamma_2^{k-1-i} - \eta - \eta \right|^{\frac{2}{k}} - \left| \Gamma_1^* + \Gamma_2^* - \bar{\eta} - \bar{\eta} \right|^2 \right) (\Gamma_1 + \Gamma_2) \\
 &= (\Gamma_1 + \Gamma_2)^* \left( m' \left[ \left| \Gamma_1^k - \eta \right|^2 + \left| \Gamma_2^k - \eta \right|^2 \right]^{\frac{1}{k}} - \left| \Gamma_1^* - \bar{\eta} \right|^2 - \left| \Gamma_2^* - \bar{\eta} \right|^2 \right) (\Gamma_1 + \Gamma_2) \\
 &= (\Gamma_1 + \Gamma_2)^* \left( m \left| \Gamma_1^k - \eta \right|^{\frac{2}{k}} + m \left| \Gamma_2^k - \eta \right|^{\frac{2}{k}} - \left| \Gamma_1^* - \bar{\eta} \right|^2 - \left| \Gamma_2^* - \bar{\eta} \right|^2 \right) (\Gamma_1 + \Gamma_2) \\
 &= \Gamma_1^* \left( m \left| \Gamma_1^k - \eta \right|^{\frac{2}{k}} - \left| \Gamma_1^* - \bar{\eta} \right|^2 \right) \Gamma_1 + \Gamma_2^* \left( m \left| \Gamma_2^k - \eta \right|^{\frac{2}{k}} - \left| \Gamma_2^* - \bar{\eta} \right|^2 \right) \Gamma_2 \\
 &\geq 0.
 \end{aligned}$$

Then  $\Gamma_1 + \Gamma_2$  is Quasi Totally m-Class  $A_k^*$  operator.

**Proposition 2.3.** Let  $\Gamma_1, \Gamma_2 \in B(H)$ . If  $\Gamma_2$  is a Quasi Totally m- Class  $A_k^*$  operator and  $\Gamma_1$  is unitary equivalent to  $\Gamma_2$ , then  $\Gamma_1$  is Quasi Totally m-Class  $A_k^*$  operator.

*Proof.*  $\Gamma_2$  is Quasi Totally m- Class  $A_k^*$  operator then

$$\begin{aligned}
 \Gamma_2^* \left( m \left| \Gamma_2^k - \eta \right|^{\frac{2}{k}} - \left| (\Gamma_2 - \eta)^* \right|^2 \right) \Gamma_2 \geq 0 &\quad \Rightarrow (U\Gamma_1U^*)^* \left( m \left| (U\Gamma_1U^*)^k - \eta \right|^{\frac{2}{k}} - \left| (U\Gamma_1U^* - \eta)^* \right|^2 \right) U\Gamma_1U^* \geq 0 \\
 &\Rightarrow U\Gamma_1U^* \left( m \left| U(\Gamma_1)^k U^* - \eta U U^* \right|^{\frac{2}{k}} - \left| (U\Gamma_1U^* - \eta U U^*)^* \right|^2 \right) U\Gamma_1U^* \geq 0 \\
 &\Rightarrow U\Gamma_1U^* \left( m U \left| (\Gamma_1)^k - \eta \right|^{\frac{2}{k}} U^* - U \left| (\Gamma_1 - \eta)^* \right|^2 U^* \right) U\Gamma_1U^* \geq 0 \\
 &\Rightarrow U\Gamma_1 \left( m \left| (\Gamma_1)^k - \eta \right|^{\frac{2}{k}} - \left| (\Gamma_1 - \eta)^* \right|^2 \right) \Gamma_1 U^* \geq 0 \\
 &\Rightarrow \Gamma_1 \left( m \left| (\Gamma_1)^k - \eta \right|^{\frac{2}{k}} - \left| (\Gamma_1 - \eta)^* \right|^2 \right) \Gamma_1 \geq 0; \quad \forall \eta \in \mathbb{C}
 \end{aligned}$$

Then  $\Gamma_1$  is a Quasi Totally m- Class  $A_k^*$  operator.

The following example shows that  $\Gamma_1$  and  $\Gamma_2$  are Quasi m- Class  $A_k^*$  operator but the product  $\Gamma_1\Gamma_2$  isn't Quasi m- Class  $A_k^*$  operator.

**Example 2.4.**

$$\text{Let } \Gamma_1 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \in B(\square^3)$$

So

$$\Gamma_1^* \left( m |\Gamma_1^k|^{\frac{2}{k}} - |\Gamma_1^*|^2 \right) \Gamma_1 = \begin{pmatrix} 8m-8 & 0 & -8m+8 \\ 0 & 0 & 0 \\ -8m+8 & 0 & 8m-8 \end{pmatrix}$$

for all  $m \geq 1$ . Then

$$\Gamma_1^* \left( m |\Gamma_1^k|^{\frac{2}{k}} - |\Gamma_1^*|^2 \right) \Gamma_1 = \begin{pmatrix} 8m-8 & 0 & -8m+8 \\ 0 & 0 & 0 \\ -8m+8 & 0 & 8m-8 \end{pmatrix} \geq 0$$

Hence  $\Gamma_1$  is Quasi  $m$ -Class  $A_k^*$  operator.

$$\text{and let } \Gamma_1 = \begin{pmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in B(\square^3)$$

So

$$\Gamma_2^* \left( m |\Gamma_2^k|^{\frac{2}{k}} - |\Gamma_2^*|^2 \right) \Gamma_2 = \begin{pmatrix} \frac{m2^{2/k}-16}{2} & 0 & \frac{m2^{2/k}-16}{2} \\ 2 & 0 & 2 \\ 0 & 0 & 0 \\ \frac{m2^{2/k}-16}{2} & 0 & \frac{m2^{2/k}-16}{2} \end{pmatrix}$$

for  $m2^{2/k}-16 \geq 0$ .

Then

$$\Gamma_2^* \left( m |\Gamma_2^k|^{\frac{2}{k}} - |\Gamma_2^*|^2 \right) \Gamma_2 = \begin{pmatrix} \frac{m2^{2/k}-16}{2} & 0 & \frac{m2^{2/k}-16}{2} \\ 2 & 0 & 2 \\ 0 & 0 & 0 \\ \frac{m2^{2/k}-16}{2} & 0 & \frac{m2^{2/k}-16}{2} \end{pmatrix} \geq 0$$

Hence  $\Gamma_2$  is Quasi m-Class  $A_k^*$  operator.

$$\text{And } \Gamma_1\Gamma_2 = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \in B(\mathbb{C}^3)$$

$$(\Gamma_1\Gamma_2)^* \left( m \left| (\Gamma_1\Gamma_2)^k \right|^{\frac{2}{k}} - \left| (\Gamma_1\Gamma_2)^* \right|^2 \right) (\Gamma_1\Gamma_2) = \begin{pmatrix} -8 & 0 & -8 \\ 0 & 0 & 0 \\ -8 & 0 & -8 \end{pmatrix} \geq 0.$$

Then  $\Gamma_1\Gamma_2$  is not Quasi m-Class  $A_k^*$  operator.

We say that an operator  $\Gamma_2$  doubly commutes with  $\Gamma_1$  if  $\Gamma_2$  commutes with  $\Gamma_1$  and  $\Gamma_1^*$ .

**Proposition 2.5.** Let  $\Gamma_1, \Gamma_2 \in B(H)$  are doubly commutes. If  $\Gamma_2$  is normal operator and  $\Gamma_1$  is Quasi m-Class  $A_k^*$  operator, then  $\Gamma_1\Gamma_2$  is Quasi m-Class  $A_k^*$  operator.

*Proof.* Let  $\Gamma_1$  is a Quasi m-Class  $A_k^*$  operator and  $\Gamma_1$  is a normal operator such that  $\Gamma_1\Gamma_2 = \Gamma_2\Gamma_1$  and  $\Gamma_1\Gamma_2^* = \Gamma_2^*\Gamma_1$ . we have that

$$\Gamma_1^* \left( m \left| \Gamma_1^k \right|^{\frac{2}{k}} - \left| \Gamma_1^* \right|^2 \right) \Gamma_1 \geq 0 \quad \text{and} \quad \Gamma_2\Gamma_2^* = \Gamma_2^*\Gamma_2$$

and so

$$\begin{aligned} \langle (\Gamma_1\Gamma_2)^* \left( m \left| (\Gamma_1\Gamma_2)^k \right|^{\frac{2}{k}} - \left| (\Gamma_1\Gamma_2)^* \right|^2 \right) (\Gamma_1\Gamma_2) \mu; \mu \rangle &= \langle \Gamma_1^* \Gamma_2^* \left( m \left| \Gamma_1^k \Gamma_2^k \right|^{\frac{2}{k}} - \left| \Gamma_1^* \Gamma_2^* \right|^2 \right) (\Gamma_1\Gamma_2) \mu; \mu \rangle \\ &= \langle \Gamma_1^* \Gamma_2^* \left( m (\Gamma_2^*)^{\frac{1}{k}} \left| \Gamma_1^k \right|^{\frac{2}{k}} (\Gamma_2^k)^{\frac{1}{k}} - \Gamma_2^* \left| \Gamma_1^* \right|^2 \Gamma_2 \right) (\Gamma_1\Gamma_2) \mu; \mu \rangle \\ &= \langle \Gamma_1^* \Gamma_2^* \Gamma_2^* \left( m \left| \Gamma_1^k \right|^{\frac{2}{k}} - \left| \Gamma_1^* \right|^2 \right) \Gamma_2 \Gamma_1 \Gamma_2 \mu; \mu \rangle \\ &= \langle \Gamma_2^* \Gamma_1^* \left( m \left| \Gamma_1^k \right|^{\frac{2}{k}} - \left| \Gamma_1^* \right|^2 \right) \Gamma_1 \Gamma_2^2 \mu; \mu \rangle \\ &= \langle \Gamma_1^* \left( m \left| \Gamma_1^k \right|^{\frac{2}{k}} - \left| \Gamma_1^* \right|^2 \right) \Gamma_1 \Gamma_2^2 \mu; \Gamma_2^2 \mu \rangle \geq 0 \end{aligned}$$

then  $\Gamma_1\Gamma_2$  is Quasi m-Class  $A_k^*$  operator.

**Proposition 2.6.** If  $\Gamma$  is Quasi m-Class  $A_k^*$  operator and  $\Gamma$  is normal operator, then  $\Gamma^*$  is Quasi m-Class  $A_k^*$  operator.

*Proof.*  $\Gamma$  is Quasi m-Class  $A_k^*$  operator and  $\Gamma$  is normal operator then

$$\begin{aligned} \Gamma \left( m \left( \Gamma^k \Gamma^{*k} \right)^{\frac{1}{k}} - \left( \Gamma^* \Gamma \right)^2 \right) \Gamma^{*} &= \Gamma \left( m \left( \Gamma^{*k} \Gamma^k \right)^{\frac{1}{k}} - \left( \Gamma \Gamma^* \right)^2 \right) \Gamma^{*} \\ &= \Gamma^{*} \left( m \left( \Gamma^{*k} \Gamma^k \right)^{\frac{1}{k}} - \left( \Gamma \Gamma^* \right)^2 \right) \Gamma \\ &\geq 0 \end{aligned}$$

Then  $\Gamma^*$  is Quasi m-Class  $A_k^*$  operator.

**Lemma.2.7:** [5] (Holder-mcCarthy's inequality) Let  $\Gamma \geq 0$ . Then

1.  $\langle \Gamma^r x, x \rangle \leq \langle \Gamma x, x \rangle^r \|x\|^{2(1-r)}$ , for  $r > 1$  and all  $x \in H$
2.  $\langle \Gamma^r x, x \rangle \leq \langle \Gamma x, x \rangle^r \|x\|^{2(1-r)}$ , for  $0 \leq r \leq 1$  and all  $x \in H$

**Proposition 2.8.** if  $\Gamma$  is Totally m- class  $A_k^*$  operator .then

$$m \left\| \left( \Gamma^k - \eta \right) \mu \right\|^{\frac{1}{k}} \left\| \mu \right\|^{\frac{1}{k}} \geq \left\| \left( \Gamma - \eta \right)^* \mu \right\|; \quad \forall \eta \in \square, \forall \mu \in H.$$

*Proof.* Suppose that  $\Gamma$  is Totally m-Class  $A_k^*$  . we have

$$m \left( \left( \Gamma^k - \eta \right)^* \left( \Gamma^k - \eta \right) \right)^{\frac{1}{k}} - \left( \Gamma - \eta \right) \left( \Gamma - \eta \right)^* \geq 0; \quad \forall \eta \in \square.$$

Let  $\mu \in H$  . then

$$\begin{aligned} & m \left\| \left( \Gamma^k - \eta \right) \mu \right\|^{\frac{2}{k}} \left\| \mu \right\|^{2-\frac{2}{k}} - \left\| \left( \Gamma - \eta \right)^* \mu \right\|^2 \\ &= m \langle \left( \Gamma^k - \eta \right) \mu, \left( \Gamma^k - \eta \right) \mu \rangle^{\frac{2}{k}} \left\| \mu \right\|^{2-\frac{2}{k}} - \langle \left( \Gamma - \eta \right)^* \mu, \left( \Gamma - \eta \right)^* \mu \rangle \\ &= m \langle \left( \Gamma^k - \eta \right)^* \left( \Gamma^k - \eta \right) \mu, \mu \rangle^{\frac{1}{k}} \left\| \mu \right\|^{2-\frac{2}{k}} - \langle \left( \Gamma - \eta \right) \left( \Gamma - \eta \right)^* \mu, \mu \rangle \\ &\geq \langle m \left[ \left( \Gamma^k - \eta \right)^* \left( \Gamma^k - \eta \right) \right]^{\frac{1}{k}} \mu, \mu \rangle - \langle \left( \Gamma - \eta \right) \left( \Gamma - \eta \right)^* \mu, \mu \rangle \\ &= \left\langle \left( m \left[ \left( \Gamma^k - \eta \right)^* \left( \Gamma^k - \eta \right) \right]^{\frac{1}{k}} - \left( \Gamma - \eta \right) \left( \Gamma - \eta \right)^* \right) \mu, \mu \right\rangle \\ &\geq 0 \end{aligned}$$

Therefore

$$m' \left\| (\Gamma^k - \eta) \mu \right\|^{\frac{1}{k}} \left\| \mu \right\|^{1-\frac{1}{k}} \geq \left\| (\Gamma - \eta)^* \mu \right\|; \quad \forall \eta \in \square, \forall \mu \in \square.$$

**Proposition 2.9.** if  $\Gamma$  is Quasi Totally  $m$ - class  $A_k^*$  operator .then

$$m \left\| (\Gamma^k - \eta) \Gamma \mu \right\|^{\frac{1}{k}} \left\| \Gamma \mu \right\|^{1-\frac{1}{k}} \geq \left\| (\Gamma - \eta)^* \Gamma \mu \right\|; \quad \forall \eta \in \square, \forall \mu \in H$$

*Proof.* Suppose that  $\Gamma$  is Quasi Totally  $m$ -Class  $A_k^*$  . we have

$$\Gamma^* \left( m \left[ (\Gamma^k - \eta)^* (\Gamma^k - \eta) \right]^{\frac{1}{k}} \right) \Gamma \geq \Gamma^* (\Gamma - \eta) (\Gamma - \eta)^* \Gamma$$

Let  $\mu \in H$  . Then

$$\begin{aligned} m \left\| (\Gamma^k - \eta) \Gamma \mu \right\|^{\frac{2}{k}} \left\| \Gamma \mu \right\|^{2-\frac{2}{k}} &= m \langle (\Gamma^k - \eta) \Gamma \mu, (\Gamma^k - \eta) \Gamma \mu \rangle^{\frac{1}{k}} \left\| \Gamma \mu \right\|^{2-\frac{2}{k}} \\ &= m \langle (\Gamma^k - \eta)^* (\Gamma^k - \eta) \Gamma \mu, \Gamma \mu \rangle^{\frac{1}{k}} \left\| \Gamma \mu \right\|^{2-\frac{2}{k}} \\ &\geq m \left\langle \left[ (\Gamma^k - \eta)^* (\Gamma^k - \eta) \right]^{\frac{1}{k}} \Gamma \mu, \Gamma \mu \right\rangle \\ &\geq \langle \Gamma^* m \left[ (\Gamma^k - \eta)^* (\Gamma^k - \eta) \right]^{\frac{1}{k}} \Gamma \mu, \mu \rangle \\ &\geq \langle \Gamma^* (\Gamma^k - \eta) (\Gamma^k - \eta)^* \Gamma \mu, \mu \rangle \\ &= \left\| (\Gamma - \eta)^* \Gamma \mu \right\|^2. \end{aligned}$$

Therefore

$$m' \left\| (\Gamma^k - \eta) \Gamma \mu \right\|^{\frac{1}{k}} \left\| \Gamma \mu \right\|^{1-\frac{1}{k}} \geq \left\| (\Gamma - \eta)^* \Gamma \mu \right\|; \quad \forall \eta \in \square, \forall \mu \in H$$

**Proposition 2.10.** Let  $\Gamma$  is a Quasi Totally  $m$ -Class  $A_k^*$  . then  $N(\Gamma - \alpha) \subset N\left((\Gamma - \alpha^k)^*\right)$  for each  $\alpha \neq 0$ .

*Proof.* Suppose  $\Gamma$  is Quasi Totally  $m$ -Class  $A_k^*$  .it follows from proposition 2.9 that

$$m \left\| (\Gamma^k - \eta) \Gamma \mu \right\|^{\frac{1}{k}} \left\| \Gamma \mu \right\|^{1-\frac{1}{k}} \geq \left\| (\Gamma - \eta)^* \Gamma \mu \right\|; \quad \forall \eta \in \square, \forall \mu \in H$$

and we have  $\mu \in N(\Gamma - \alpha)$  then  $\Gamma \mu = \alpha \mu$  . In particular,

$$m\left\|\left(\Gamma^k - \alpha^k\right) \Gamma \mu\right\|^{\frac{1}{k}}\left\|\Gamma \mu\right\|^{1-\frac{1}{k}} \geq\left\|\left(\Gamma - \alpha^k\right)^* \Gamma \mu\right\|$$

Since

$$0 \geq\left\|\left(\Gamma - \alpha^k\right)^* \Gamma \mu\right\|.$$

and we have  $\alpha \neq 0$ , then  $\left\|\left(\Gamma - \alpha^k\right)^* \mu\right\|=0$ .

Therefore  $\mu \in N\left(\left(\Gamma - \alpha^k\right)^*\right)$ .

Hence  $N(\Gamma - \alpha) \subset N\left(\Gamma - \alpha^k\right)^*$  for each  $\alpha \neq 0$ .

**Proposition 2.11.** Suppose  $\Gamma$  is Quasi Totally m-Class  $A_k^*$  and it has dense range. then  $\Gamma$  is Totally m-Class  $A_k^*$ .

**Proof.** Let  $\Gamma$  is Quasi Totally m-Class  $A_k^*$  then

$$m\left\|\left(\Gamma^k - \eta\right) \Gamma \mu\right\|^{\frac{1}{k}}\left\|\Gamma \mu\right\|^{1-\frac{1}{k}} \geq\left\|\left(\Gamma - \eta\right)^* \Gamma \mu\right\| ; \quad \forall \eta \in \square, \forall \mu \in H$$

Since  $\Gamma$  has dense range, then  $\overline{\Gamma(H)}=H$ .

Let  $\mu \in H$ . Then there exists a sequence  $\left(x_n\right)$  in  $H$  such that  $\Gamma\left(x_n\right) \rightarrow \mu$  as  $n \rightarrow \infty$ .

In particular,

$$m\left\|\left(\Gamma^k - \eta\right) \Gamma x_n\right\|^{\frac{1}{k}}\left\|\Gamma x_n\right\|^{1-\frac{1}{k}} \geq\left\|\left(\Gamma - \eta\right)^* \Gamma x_n\right\|$$

Therefore

$$\begin{aligned} m\left\|\left(\Gamma^k - \eta\right) \mu\right\|^{\frac{1}{k}}\left\|\mu\right\|^{1-\frac{1}{k}} &= m\left\|\lim _{n \rightarrow \infty}\left(\Gamma^k - \eta\right) \Gamma x_n\right\|^{\frac{1}{k}}\left\|\lim _{n \rightarrow \infty} \Gamma x_n\right\|^{1-\frac{1}{k}} \\ &= \lim _{n \rightarrow \infty} m\left\|\left(\Gamma^k - \eta\right) \Gamma x_n\right\|^{\frac{1}{k}}\left\|\Gamma x_n\right\|^{1-\frac{1}{k}} \\ &\geq \lim _{n \rightarrow \infty}\left\|\left(\Gamma - \eta\right)^* \Gamma x_n\right\| \\ &= \left\|\lim _{n \rightarrow \infty}\left(\Gamma - \eta\right)^* \Gamma x_n\right\| \\ &= \left\|\left(\Gamma - \eta\right)^* \mu\right\| \end{aligned}$$

Hence  $\Gamma$  is Totally m-Class  $A_k^*$

**Theorem 2.12.** Let  $\Gamma \in B(H)$  such that  $H \neq \overline{R(\Gamma)}$ , if  $\Gamma$  is a Quasi Totally  $m$ -Class  $A_k^*$  operator then  $\Gamma_1 = \Gamma_{\overline{R(\Gamma)}}$  is a Totally  $m$ -Class  $A_k^*$  operator and  $\Gamma_3 = 0$ .

**Proof.** Let  $\Gamma \in B(H)$  such that  $H \neq \overline{R(\Gamma)}$ .

the matrix representation of  $\Gamma$  such that  $\Gamma = \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ 0 & \Gamma_3 \end{pmatrix}$  on  $H = \overline{R(\Gamma)} \oplus N(\Gamma^*)$  see[3]

and let  $P_{\overline{R(\Gamma)}}$  be the projection onto  $\overline{R(\Gamma)}$ .

$$\text{Then } \begin{pmatrix} \Gamma_1 & 0 \\ 0 & 0 \end{pmatrix} = \Gamma P_{\overline{R(\Gamma)}} = P_{\overline{R(\Gamma)}} \Gamma P_{\overline{R(\Gamma)}}.$$

Since  $\Gamma$  is an Quasi Totally  $m$ -Class  $A_k^*$  operator, we have

$$\Gamma^* \left( m \left[ (\Gamma^k - \eta)^* (\Gamma^k - \eta) \right]^{\frac{1}{k}} \right) \Gamma \geq \Gamma^* ((\Gamma - \eta)(\Gamma - \eta)^*) \Gamma$$

then

$$P_{\overline{R(\Gamma)}} \left( m \left[ (\Gamma^k - \eta)^* (\Gamma^k - \eta) \right]^{\frac{1}{k}} \right) P_{\overline{R(\Gamma)}} \geq P_{\overline{R(\Gamma)}} ((\Gamma - \eta)(\Gamma - \eta)^*) P_{\overline{R(\Gamma)}}$$

Therefore

$$\begin{aligned} P_{\overline{R(\Gamma)}} \left( m \left[ (\Gamma^k - \eta)^* (\Gamma^k - \eta) \right]^{\frac{1}{k}} \right) P_{\overline{R(\Gamma)}} &\leq m \left[ P_{\overline{R(\Gamma)}} (\Gamma^k - \eta)^* (\Gamma^k - \eta) P_{\overline{R(\Gamma)}} \right]^{\frac{1}{k}} \\ &= m \begin{pmatrix} \left[ (\Gamma^k - \eta)^* (\Gamma^k - \eta) \right]^{\frac{1}{k}} & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

On other hand

$$P_{\overline{R(\Gamma)}} ((\Gamma - \eta)(\Gamma - \eta)^*) P_{\overline{R(\Gamma)}} = \begin{pmatrix} (\Gamma_1 - \eta)(\Gamma_1 - \eta)^* + \Gamma_2 \Gamma_2^* & 0 \\ 0 & 0 \end{pmatrix}$$

Hence

$$m \left[ (\Gamma^k - \eta)^* (\Gamma^k - \eta) \right]^{\frac{1}{k}} \geq (\Gamma_1 - \eta)(\Gamma_1 - \eta)^* + \Gamma_2 \Gamma_2^*$$

Then

$$m\left[(\Gamma^k - \eta)^*(\Gamma^k - \eta)\right]^{\frac{1}{k}} \geq (\Gamma_1 - \eta)(\Gamma_1 - \eta)^*$$

Hence  $\Gamma_1$  is a Totally  $m$ -Class  $A_k^*$ .

In addition, let  $\mu = \mu_1 + \mu_2 \in H = \overline{R(\Gamma)} \oplus N(\Gamma^*)$ . a simple computation shows that

$$\begin{aligned} \langle \Gamma_3 \mu_2; \mu_2 \rangle &= \langle \Gamma(I - P_{\overline{R(\Gamma)}})\mu; (I - P_{\overline{R(\Gamma)}})\mu \rangle \\ &= \langle (I - P_{\overline{R(\Gamma)}})\mu; \Gamma^*(I - P_{\overline{R(\Gamma)}})\mu \rangle \\ &= 0 \end{aligned}$$

So,  $\Gamma_3 = 0$ .

**Theorem 2.13.** Let  $\Gamma$  is Quasi Totally  $m$ -Class  $A_k^*$  such that  $H \neq \overline{R(\Gamma)}$ . Then

$$\Gamma = \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ 0 & 0 \end{pmatrix} \text{ on } H = \overline{\Gamma(H)} \oplus N(\Gamma^*)$$

where  $\Gamma_1 = \Gamma_{\overline{\Gamma(H)}}$  is a Totally  $m$ -Class  $A_k^*$  operator.

*Proof.* Let  $\Gamma$  is Quasi Totally  $m$ -Class  $A_k^*$  then

$$\Gamma^* \left( m\left[(\Gamma^k - \eta)^*(\Gamma^k - \eta)\right]^{\frac{1}{k}} - (\Gamma - \eta)(\Gamma - \eta)^* \right) \Gamma \geq 0 \quad \forall \eta \in \square$$

And  $\Gamma$  doesn't have dense range. we can represent  $\Gamma$  as the  $2 \times 2$  operator matrix as follows:

$$\Gamma = \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ 0 & 0 \end{pmatrix} \text{ on } H = \overline{\Gamma(H)} \oplus N(\Gamma^*)$$

Therefore

$$\Gamma^* \left( m\left[(\Gamma_1^k - \eta)^*(\Gamma_1^k - \eta)\right]^{\frac{1}{k}} - (\Gamma_1 - \eta)(\Gamma_1 - \eta)^* \right) \Gamma \geq 0 \quad \forall \eta \in \square$$

Hence

$$\left\langle \left( m\left[(\Gamma_1^k - \eta)^*(\Gamma_1^k - \eta)\right]^{\frac{1}{k}} - (\Gamma_1 - \eta)(\Gamma_1 - \eta)^* \right) \Gamma \mu; \Gamma \mu \right\rangle \geq 0; \quad \forall \eta \in \square, \forall \mu \in H$$

So

$$\left\langle \left( m \left[ (\Gamma_1^k - \eta)^* (\Gamma_1^k - \eta) \right]^{\frac{1}{k}} - (\Gamma_1 - \eta)(\Gamma_1 - \eta)^* \right) \mathbf{u}; \mathbf{u} \right\rangle \geq 0 ; \quad \forall \eta \in \square, \forall \mu \in H.$$

Then  $\Gamma_1$  is a Totally m-Class  $A_k^*$  operator.

### 3. TENSOR PRODUCT OF QUASI TOTALLY m-CLASS $A_k^*$ operators

The tensor product of operators is a mathematical operation that combines two operators into a single operator acting on a tensor product space. If you have two operators  $\Gamma_1$  and  $\Gamma_2$  acting on spaces  $H$  and  $K$  respectively, the tensor product operator  $\Gamma_1 \otimes \Gamma_2$  acts on the tensor product space  $H \otimes K$  see [4,6] .It is defined as:

$$(\Gamma_1 \otimes \Gamma_2)(\mu_1 \otimes \mu_2) = (\Gamma_1 \mu_1) \otimes (\Gamma_2 \mu_2)$$

These are some of the properties we need in the following theorem

1.  $(\Gamma_1 \otimes \Gamma_2)(\Gamma_1 \otimes \Gamma_2)^* = (\Gamma_1 \Gamma_1^*) \otimes (\Gamma_2 \Gamma_2^*)$
2.  $|\Gamma_1 \otimes \Gamma_2|^\rho = |\Gamma_1|^\rho \otimes |\Gamma_2|^\rho \quad \forall \rho \in \square^+.$

In the sequel, we present an important result related to the tensor product of elements of the considered class of operators

**Theorem 3.1.** Let  $\Gamma_1 \in B(H)$  and  $\Gamma_2 \in B(K)$  such that  $\Gamma_1 \otimes I_K = -I_H \otimes \Gamma_2$  if  $\Gamma_1$  and  $\Gamma_2$  are a Quasi Totally m-Class  $A_k^*$  operator then  $\Gamma_1 \otimes \Gamma_2$  is a Quasi Totally m-Class  $A_k^*$

*Proof.* Let  $\Gamma_1 \in B(H)$  and  $\Gamma_2 \in B(K)$  .

let  $\Gamma_1$  and  $\Gamma_2$  are a Quasi Totally m-Class  $A_k^*$  operator. we have

$$\langle \Gamma_1^* \left( m \left[ (\Gamma_1^k - \eta I_H) \right]^{\frac{2}{k}} - \left| (\Gamma_1 - \eta I_H) \right|^2 \right) \Gamma_1 \mu; \mu \rangle \geq 0; \quad \forall \eta \in \square, \forall \mu \in H$$

And

$$\langle \Gamma_2^* \left( m \left[ (\Gamma_2^k - \eta I_H) \right]^{\frac{2}{k}} - \left| (\Gamma_2 - \eta I_H) \right|^2 \right) \Gamma_2 \mu; \mu \rangle \geq 0; \quad \forall \eta \in \square, \forall \mu \in H$$

Then

$$\begin{aligned}
& (\Gamma_1 \otimes \Gamma_2)^* \left( m \left| (\Gamma_1 \otimes \Gamma_2)^k - \eta' \right|^{\frac{2}{k}} - \left| (\Gamma_1 \otimes \Gamma_2) - \eta' \right|^{*2} \right) (\Gamma_1 \otimes \Gamma_2) \\
&= (\Gamma_1 \otimes \Gamma_2)^* \left( m \left| \Gamma_1^k \otimes \Gamma_2^k - \eta \Gamma_1^k \otimes I_k - \eta I_H \otimes \Gamma_2^k + \eta^2 I_H \otimes I_K \right|^{\frac{2}{k}} \right. \\
&\quad \left. - \left| (\Gamma_1 \otimes \Gamma_2 - \eta \Gamma_1 \otimes I_k - \eta I_H \otimes \Gamma_2 + \eta^2 I_H \otimes I_K) \right|^{*2} \right) (\Gamma_1 \otimes \Gamma_2) \\
&= (\Gamma_1 \otimes \Gamma_2)^* \left( m \left| \Gamma_1^k \otimes (\Gamma_2^k - \eta I_K) - \eta I_H \otimes (\Gamma_2^k - \eta I_K) \right|^{\frac{2}{k}} - \left| \Gamma_1^* \otimes (\Gamma_2^* - \eta I_K) - \eta I_H \otimes (\Gamma_2^* - \eta I_K) \right|^2 \right) (\Gamma_1 \otimes \Gamma_2) \\
&= (\Gamma_1 \otimes \Gamma_2)^* \left( m' \left| (\Gamma_1^k - \eta I_H) \otimes (\Gamma_2^k - \eta I_K) \right|^{\frac{2}{k}} - \left| (\Gamma_1 - \eta I_H)^* \otimes (\Gamma_2 - \eta I_K) \right|^{*2} \right) (\Gamma_1 \otimes \Gamma_2) \\
& \\
&= \left( \Gamma_1^* \left[ m \left| (\Gamma_1^k - \eta I_H) \right|^{\frac{2}{k}} - \left| (\Gamma_1 - \eta I_H)^* \right|^2 \right] \otimes \Gamma_2^* \left| (\Gamma_2^k - \eta I_K) \right|^{\frac{2}{k}} \right. \\
&\quad \left. + \Gamma_1^* \left| (\Gamma_1 - \eta I_H)^* \right|^2 \otimes \Gamma_2^* \left[ m \left| (\Gamma_2^k - \eta I_K) \right|^{\frac{2}{k}} - \left| (\Gamma_2 - \eta I_K)^* \right|^2 \right] \right) (\Gamma_1 \otimes \Gamma_2) \\
&= \Gamma_1^* \left[ m \left| (\Gamma_1^k - \eta I_H) \right|^{\frac{2}{k}} - \left| (\Gamma_1 - \eta I_H)^* \right|^2 \right] \Gamma_1 \otimes \Gamma_2^* \left| (\Gamma_2^k - \eta I_K) \right|^{\frac{2}{k}} \Gamma_2 \\
&\quad + \Gamma_1^* \left| (\Gamma_1 - \eta I_H)^* \right|^2 \Gamma_1 \otimes \Gamma_2^* \left[ m \left| (\Gamma_2^k - \eta I_K) \right|^{\frac{2}{k}} - \left| (\Gamma_2 - \eta I_K)^* \right|^2 \right] \Gamma_2
\end{aligned}$$

$$\begin{aligned}
&= (\Gamma_1 \otimes \Gamma_2)^* \left( m' \left| (\Gamma_1^k - \eta I_H) \right|^{\frac{2}{k}} \otimes \left| (\Gamma_2^k - \eta I_K) \right|^{\frac{2}{k}} - \left| (\Gamma_1 - \eta I_H) \right|^{*2} \otimes \left| (\Gamma_2 - \eta I_K) \right|^{*2} \right) (\Gamma_1 \otimes \Gamma_2) \\
&= (\Gamma_1^* \otimes \Gamma_2^*) \left( m \left| (\Gamma_1^k - \eta I_H) \right|^{\frac{2}{k}} \otimes \left| (\Gamma_2^k - \eta I_K) \right|^{\frac{2}{k}} - \left| (\Gamma_1 - \eta I_H) \right|^{*2} \otimes \left| (\Gamma_2^k - \eta I_K) \right|^{\frac{2}{k}} \right. \\
&\quad \left. + \left| (\Gamma_1 - \eta I_H) \right|^{*2} \otimes \left| (\Gamma_2^k - \eta I_K) \right|^{\frac{2}{k}} - \left| (\Gamma_1 - \eta I_H) \right|^{*2} \otimes \left| (\Gamma_2 - \eta I_K) \right|^{*2} \right) (\Gamma_1 \otimes \Gamma_2) \\
&= (\Gamma_1^* \otimes \Gamma_2^*) \left( \left[ m \left| (\Gamma_1^k - \eta I_H) \right|^{\frac{2}{k}} - \left| (\Gamma_1 - \eta I_H) \right|^{*2} \right] \otimes \left| (\Gamma_2^k - \eta I_K) \right|^{\frac{2}{k}} \right. \\
&\quad \left. + \left| (\Gamma_1 - \eta I_H) \right|^{*2} \otimes \left[ m \left| (\Gamma_2^k - \eta I_K) \right|^{\frac{2}{k}} - \left| (\Gamma_2 - \eta I_K) \right|^{*2} \right] \right) (\Gamma_1 \otimes \Gamma_2)
\end{aligned}$$

Therefore for ever  $\mu \in H$  and  $u \in K$

$$\begin{aligned}
&\langle \Gamma_1^* \left[ m \left| (\Gamma_1^k - \eta I_H) \right|^{\frac{2}{k}} - \left| (\Gamma_1 - \eta I_H) \right|^{*2} \right] \Gamma_1 \mu; \mu \rangle \langle \Gamma_2^* \left| (\Gamma_2^k - \eta I_K) \right|^{\frac{2}{k}} \Gamma_2 u; u \rangle \\
&+ \langle \Gamma_1^* \left| (\Gamma_1 - \eta I_H) \right|^{*2} \Gamma_1 \mu; \mu \rangle \langle \Gamma_2^* \left[ m \left| (\Gamma_2^k - \eta I_K) \right|^{\frac{2}{k}} - \left| (\Gamma_2 - \eta I_K) \right|^{*2} \right] \Gamma_2 u; u \rangle \\
&= \langle \Gamma_1^* \left[ m \left| (\Gamma_1^k - \eta I_H) \right|^{\frac{2}{k}} - \left| (\Gamma_1 - \eta I_H) \right|^{*2} \right] \Gamma_1 \mu; \mu \rangle \left\| \left( \Gamma_2^k - \eta I_K \right)^{\frac{1}{k}} \Gamma_2 u \right\|^2 \\
&+ \left\| \left( \Gamma_1 - \eta I_H \right)^* \Gamma_1 \mu \right\|^2 \langle \Gamma_2^* \left[ m \left| (\Gamma_2^k - \eta I_K) \right|^{\frac{2}{k}} - \left| (\Gamma_2 - \eta I_K) \right|^{*2} \right] \Gamma_2 u; u \rangle \\
&\geq 0
\end{aligned}$$

Then

$$(\Gamma_1 \otimes \Gamma_2)^* \left( m \left| (\Gamma_1 \otimes \Gamma_2)^k - \eta' \right|^{\frac{2}{k}} - \left| (\Gamma_1 \otimes \Gamma_2 - \eta') \right|^{*2} \right) (\Gamma_1 \otimes \Gamma_2) \geq 0$$

Hence  $\Gamma_1 \otimes \Gamma_2$  is a Quasi Totally m-Class  $A_k^*$  operator.

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