On an inequality of Hadamard product and the weighted version of arithmetic and harmonic

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Abstract

In this paper we give inequalities involving the Hadamard product and arithmetic-harmonic means of matrices. Moreover, we prove the trace inequality of the product of the arithmetic and harmonic means.

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1 Introduction

Let M_n denote the set of all $n \times n$ complex matrices. The Hadamard product of two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ is their element-wise product, which is given by

$$A \circ B = [a_{ij}b_{ij}].$$

If $A \in M_n$ and all eigenvalues for A are real, $\lambda_i(A)$ denotes the i th largest eigenvalue of A.

Let A and B be positive definite, and $t \in (0, 1)$. Then the weighted versions of arithmetic, geometric, and harmonic means are defined [8], respectively, by

$$A \nabla_t B = tA + (1-t)B$$
$$A \sharp_t B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^t A^{\frac{-1}{2}}$$
$$A !_t B = [tA^{-1} + (1-t)B^{-1}]^{-1}.$$

For $t = 1/2$, we simply put $A\sharp B$ for $A\sharp_{1/2}B$. The usual arithmetic, geometric, and harmonic means correspond to $t = 1/2$. The definitions extend to positive semidefinite matrices by continuity. For the rest of this paper, we assume that A and B are positive definite. It is well known that the following inequalities are valid:

$$A!_t\, B \leq A\sharp_t B \leq A \triangledown_t B,$$

$$(A \circ B) \geq (A\sharp B) \circ (A\sharp B)$$

$$(A!_t\, B)\sharp(A \triangledown_{1-t} B) = A\sharp B.$$

Bapat and Sunder [2] proved that

$$\prod_{i=k}^{n} \lambda_i(A \circ B) \geq \prod_{i=k}^{n} \lambda_i(A)\lambda_i(B), \qquad k = 1, \dots, n. \tag{1}$$

The Horn theorem [6] says that

$$\prod_{i=k}^{n} \lambda_i(AB) \geq \prod_{i=k}^{n} \lambda_i(A)\lambda_i(B), \qquad k = 1, \dots, n. \tag{2}$$

Bapat and Johnson [5] proved that

$$\prod_{i=k}^{n} \lambda_i(A \circ B) \geq \prod_{i=k}^{n} \lambda_i(AB), \qquad k = 1, \dots, n. \tag{3}$$

In 2017, Hiai and Lin [4] gave a weighted extension of the results of Ando [1]

$$\prod_{i=k}^{n} \lambda_i(A \circ B) \geq \prod_{i=k}^{n} \lambda_i((A\sharp_{1-t}B)(A\sharp_t B)) \geq \prod_{i=k}^{n} \lambda_i(AB), \qquad k = 1, \dots, n.$$

In this paper, we have to prove the inequalities

$$\prod_{i=k}^{n} \lambda_i(A \circ B) \geq \prod_{i=k}^{n} \lambda_i((A \triangledown_{1-t} B)(A!_t\, B)) \geq \prod_{i=k}^{n} \lambda_i(AB), \quad k = 1, \dots, n$$

for $0 \leq t \leq 1$. We also prove some trace inequalities of the product of the arithmetic and harmonic means.

2 Main Results

The following lemmas are well-known, we will apply them to prove the next theorem.

Lemma 1. *[4] Let X be a self-adjoint matrix. For any positive semidefinite A, B, the matrix* $\begin{bmatrix} A & X \\ X & B \end{bmatrix}$ *is positive semidefinite if and only if* $XA^{-1}X \le B$.

Lemma 2. *[6] Let A, B be positive semidefinite. Then $A\#B$ is the maximum of all sself-adjoint matrix X for which* $\begin{bmatrix} A & X \\ X & B \end{bmatrix}$ *is positive semidefinite.*

We have to prove some lemmas before proving the main theorem.

Lemma 3. *Let A, B be positive definite matrices. Then for $0 \le t \le 1$,* $\begin{bmatrix} A!_t\, B & A\#B \\ A\#B & A \bigtriangledown_{1-t} B \end{bmatrix}$ *is positive semidefinite.*

Proof. We have to show $A\sharp B(A!_t\, B)^{-1}A\sharp B \le A \bigtriangledown_{1-t} B$. Since we know that

$$\begin{aligned} A\sharp B(A!_t\, B)^{-1}A\sharp B &= tA^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}A^{-1}A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2} \\ &\quad +(1-t)A^{1/2}(A^{-1/2}BA^{-1/2})^{-1/2}A^{1/2}B^{-1}A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2} \\ &= tA^{1/2}(A^{-1/2}BA^{-1/2})A^{1/2} + (1-t)A \\ &= A \bigtriangledown_{1-t} B, \end{aligned}$$

by lemma 2 we have $\begin{bmatrix} A!_t\, B & A\sharp B \\ A\sharp B & A \bigtriangledown_{1-t} B \end{bmatrix}$ is positive semidefinite. □

Lemma 4. *Let A, B be positive definite matrices. For all $0 \le t \le 1$,*

$$A \circ B \ge (A!_t\, B) \circ (A \bigtriangledown_{1-t} B))$$

Proof. To prove this lemma, we will use the equation obtained from the definition of the geometric mean $(AB^{-1}A)\sharp B = A$ and the inequality $A \circ B \ge (A\sharp B) \circ (A\sharp B)$.

Then we have that,

$$\begin{aligned}(A!_t B)\circ(A \triangledown_{1-t} B) &= ((tA^{-1}+(1-t)B^{-1})^{-1}\circ((1-t)A+tB)\\ &= t^{-1}[A-(1-t)A((1-t)A+tB)^{-1}A]\circ((1-t)A+tB)\\ &= [t^{-1}(1-t)A\circ A + A\circ B]\\ &\quad -t^{-1}(1-t)A((1-t)A+tB)^{-1}A\circ((1-t)A+tB)\\ &\leq [t^{-1}(1-t)A\circ A + A\circ B] - t^{-1}(1-t)A\circ A\\ &= A\circ B.\end{aligned}$$

□

Theorem 5. *Let A, B be positive definite matrices. Then for* $0 \leq t \leq 1$

$$\prod_{i=k}^{n} \lambda_i(A\circ B) \geq \prod_{i=k}^{n} \lambda_i((A \triangledown_{1-t} B)(A!_t B)) \geq \prod_{i=k}^{n} \lambda_i(AB), \quad k=1,\dots,n.$$

Proof. By Lemma 4 and the inequality (2), we get

$$\lambda_i(A\circ B) \geq \lambda_i((A\sharp_{1-t}B)\circ(A!_t B)), \quad i=1,\dots,n$$

and by the inequality (2) and (3) for any $k=1,\dots,n$, we have

$$\begin{aligned}&\prod_{i=k}^{n} \lambda_i(A\circ B) \geq \prod_{i=k}^{n} \lambda_i((A \triangledown_{1-t} B)\circ(A!_t B))\\ &\geq \prod_{i=k}^{n} \lambda_i((A \triangledown_{1-t} B)(A!_t B))\\ &\geq \prod_{i=k}^{n} \lambda_i(A \triangledown_{1-t} B)\lambda_i(A!_t B).\end{aligned} \tag{4}$$

We proceed to prove the second inequality of this theorem by using lemma 3 and theorem 2.5 of [4]. As for any $k=1,\dots,n$

$$\begin{aligned}\prod_{i=k}^{n} \lambda_i(A \triangledown_{1-t} B) &= \prod_{i=k}^{n} \lambda_i((A!_t B)^{-1})\lambda_i((A\sharp B)(A\sharp B))\\ &\geq \prod_{i=k}^{n} \lambda_i(A!_t B)^{-1}\lambda_i(AB).\end{aligned} \tag{5}$$

That is, $\prod_{i=k}^{n} \lambda_i(A \triangledown_{1-t} B)\lambda_i(A!_t B) \geq \prod_{i=k}^{n} \lambda_i(AB)$. □

Next we will prove some trace inequality of the product of the arithmetic and harmonic means.

Lemma 6 *Let A, B be positive definite matrices. Then*

$$\det((A!_t B)(A \triangledown_{1-t} B)) = \det(AB).$$

Proof. For any the positive definite matrices A and B, $\det(A\sharp B)$ can be directly calculated as follows

$$\begin{aligned}\det(A\sharp B) &= \det(A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}) \\ &= \det(A^{1/2})\det(B^{1/2}).\end{aligned} \quad (6)$$

Applying this result, it follows that

$$\det((A!_t B)\sharp(A \triangledown_{1-t} B)) = \det((A!_t B)^{1/2})\det((A \triangledown_{1-t} B)^{1/2}).$$

But we know that $(A!_t B)\sharp(A \triangledown_{1-t} B) = A\sharp B$. Thus

$$\det((A!_t B)(A \triangledown_{1-t} B)) = \det(AB).$$

□

Theorem 7. *Let A, B be positive definite matrices. Then*

$$\operatorname{tr}((A \triangledown_{1-t} B)(A!_t B)) \leq \operatorname{tr}(AB).$$

Proof. Apply Lemma 6 by using the relation between the determinant and eigenvalue, we get

$$\prod_{i=1}^{n} \lambda_i((A!_t B)(A \triangledown_{1-t} B)) = \prod_{i=1}^{n} \lambda_i(AB).$$

Thus, the second inequality of Theorem 5 is equivalent to

$$\prod_{i=1}^{k} \lambda_i((A!_t B)(A \triangledown_{1-t} B)) \leq \prod_{i=1}^{k} \lambda_i(AB), \qquad k = 1 \dots n.$$

Next, we use the example II.3.5(vii) of [3] implies

$$\operatorname{tr}((A \triangledown_{1-t} B)(A!_t B)) \leq \operatorname{tr}(AB).$$

□

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