Left Generalized derivations on Prime ideals in Rings

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Abstract: In this work, we explore the commutativity of quotient rings, focusing on a prime ideal of the ring, and left generalized derivations acting on prime ideals satisfy certain algebraic identities.

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INTRODUCTION

Throughout the paper, P denotes a prime ideal of an associative ring R such that $r_1, r_2 \in R$, $r_1 R r_2 \subseteq P$ implies that $r_1 \in P$ or $r_2 \in P$. The commutator $[r_1, r_2]$ defined as $[r_1, r_2]$ $= r_1 r_2 - r_2 r_1$ for all $r_{1,} r_2$ in R. The anticommutator $(r_1 \text{o} r_2)$ is defined as $(r_1 \text{o} r_2) = r_1 r_2 + r_2 r_1$ for all r_1, r_2 in R. An additive mapping $\lambda(R)$ is called derivation if it is defined as $\lambda(r_1 r_2) = \lambda(r_1) r_2 + r_1 \lambda(r_2)$ for all r_1, r_2 in R. The inner derivation of R is the mapping $r_1 \to [r_1, a]$ for all r_1 in R and $a \in R$ is fixed. $\phi(R)$ is a Left generalized derivation connected with a derivation $\lambda(R)$ such that $\phi(r_1 r_2) = \lambda(r_1) r_2 + r_1 \phi(r_2)$ for all r_1, r_2 in R.

With great curiosity, many mathematicians investigated the commutativity of rings using various derivations on certain rings (see[3],[4],[5],[6] ,[7]). In [1], Raji et.al. verified the commutativity on near rings admitting the left generalized derivations with certain identities. Further, Jayasubbareddy and Mallikarjuna Rao investigated commutativity on prime rings admitting left generalized derivations with right ideals [2]. Daif and Bell [6] obtained the result that if A is a non-zero ideal of a semiprime ring R and d a derivation of R such that $d([x, y]) = [x, y]$ for every $x, y \in A$, then $A \subseteq Z(R)$ and also examined the Commutativity with algebraic identities involving prime ideals.

We continued in this line of research and established some commutativity theorems by studying the effect of left generalized derivations upon rings with certain algebraic conditions.

Preliminary Results

Fact 1.1. (Rehman & Alnoghashi, Hafedh M, 2021): Let P be a prime ideal of ring R and X an additive subgroup of R. Let ψ and ξ be additive functions defined from $X \to R$ such that $\psi(x)R\xi(x) \in P$ for all x in X .Then either $\psi(x) \in P$ for all $x \in P$ or $\xi(x) \in P$ for all $x \in X$.

Lemma 1.2. (Rehman & Alnoghashi, Hafedh M, 2021): Suppose P is a prime ideal of ring R. If (i) $[r_1, r_2] \in P$ (ii) r_1 or $r_2 \in P$ for all $r_1, r_2 \in R$, then R/P is a commutative integral domain.

Proposition 1.3**:** Suppose P is a prime ideal of ring R. Suppose $\phi : R \to R$ be a left generalized derivation of R associated with derivation λ in such a way that $[r_1, \phi(r_1)] \in P$ for all $r_1 \in P$, then either R/P is a commutative integral domain or $\lambda(R) \subseteq P$.

Proof: Consider $[r_1, \phi(r_1)] \in P$ for all $r_1 \in R$. (1.1)

Applying the method of linearization with regard to r_1, we obtain

$$[r_1, \phi(r_2)] + [r_2, \phi(r_1)] \in P \text{ for every } r_1, r_2 \in R. \quad (1.2)$$

When we substitute $r_2 r_1$ for r_1 in (1.2) and using (1.1) and (1.2), we get

$$[r_2, \lambda(r_2)]r_1 + \lambda(r_2)[r_2, r_1] \in P \text{ for all } r_1, r_2 \in R. \quad (1.3)$$

Right multiplying (1.3) by r where $r \in R$, we get

$$[r_2, \lambda(r_2)]r_1 r + \lambda(r_2)[r_2, r_1]r \in P \text{ for all } r_1, r_2, r \in R. \quad (1.4)$$

Substitute $r_1 r$ for r_1 in (1.3) and using it, we arrived

$$[r_2, \lambda(r_2)]r_1 r + \lambda(r_2)[r_2, r_1]r + \lambda(r_2)r_1[r_2, r] \in P \text{ for all } r_1, r_2, r \in R. \quad (1.5)$$

Subtracting (1.4) from (1.5), we have $\lambda(r_2)r_1[r_2, r] \in P$.

Then by fact 1.1, we have $\lambda(r_2)R[r_2, r] \subseteq P$. Further, we have either $\lambda(r_2) \in P$ or $[r_2, r] \in P$ for all $r_2, r \in R$. From the first one, we have $\lambda(R) \subseteq P$. By using the above lemma, $[r_2, r] \in P$ implies the required result.

Theorem 1.4**:** Suppose P is a prime ideal of ring R. Suppose $\phi : R \to R$ a left generalized derivation ϕ with associated derivation λ in such a way that:

(i) $\phi(r_1 r_2) - \phi(r_1)\phi(r_2) \in P$ for all $r_1, r_2 \in R$,
(ii) $\phi(r_1 r_2) - \phi(r_2)\phi(r_1) \in P$ for all $r_1, r_2 \in R$,

then either R/P is a commutative integral domain or $\lambda(R) \subseteq P$.

Proof: (i) Let us start with $\phi(r_1 r_2) - \phi(r_1)\phi(r_2) \in P$ for all $r_1, r_2 \in R$. (1.6)

Substitute rr_1 for r_1 in (1.6) and using (1.6), we have

$$\lambda(r)r_1r_2-\lambda(r)r_1\phi(r_2)\in P \text{ for all } r_1,r_2,r\in R. \tag{1.7}$$

Substitute r_3r_2 for r_2 in (1.7), we get

$$\lambda(r)r_1r_3r_2-\lambda(r)r_1\lambda(r_3)r_2-\lambda(r)r_1r_3\phi(r_2)\in P \text{ for all } r_1,r_2,r_3\in R. \tag{1.8}$$

Again substitute r_1r_3 for r_1 in (1.7), it reduces to

$$\lambda(r)r_1r_3r_2-\lambda(r)r_1r_3\phi(r_2)\in P \text{ for all } r_1,r_2,r_3,r\in R. \tag{1.9}$$

Now subtracting (1.8) from (1.9) ,we get that $\lambda(r)r_1\lambda(r_3)r_2\in P$ for all $r_1,r_2,r_3,r\in R.$

Hence $\lambda(r)r_1\lambda(r_3)[r_2,s]\in P$ for all $r,s,r_1,r_2,r_3\in R$ that is $\lambda(r)R\lambda(r_3)[r_2,s]\subseteq P.$

Therefore, either $\lambda(r)\in P$ or $\lambda(r_3)[r_2,s]\in P.$

From the first one, we get $\lambda(r)\in P$ which gives $\lambda(R)\subseteq P.$

From the second one, we get $\lambda(r_3)[r_2,s]\in P$ for all $r_2,r_3,s\in R$.

It implies that $\lambda(r_3)R[r_2,s]\subseteq P$.

By the primeness of *P*, we obtain either $\lambda(r_3)\in P$ or $[r_2,s]\in P$ for all $r_2,r_3,s\in R.$

Here $\lambda(r_3)\in P$ implies $\lambda(R)\subseteq P.$ Next, $[r_2,s]\in P$ gives the required result with the above lemma.

(iii) Take $\phi(r_1r_2)-\phi(r_2)\phi(r_1)\in p$ for all $r_1,r_2\in R.$ $\quad(1.10)$

Replace r_1r_2 in place of r_2 in (1.10) and using it, we have

$$\lambda(r_1)r_1r_2-\lambda(r_1)r_2\phi(r_1)\in P \text{ for all } r_1,r_2\in R. \tag{1.11}$$

Writing r_2t instead of r_2 in (1.11), we get

$$\lambda(r_1)r_1r_2t-\lambda(r_1)r_2t\phi(r_1)\in p \text{ for all } r_1,r_2,t\in R. \tag{1.12}$$

Right multiplying (1.11) by *t*, we get

$$\lambda(r_1)r_1r_2t-\lambda(r_1)r_2\phi(r_1)t\in P \text{ for all } r_1,r_2,t\in R. \tag{1.13}$$

$(1.13)-(1.12)$ gives $\lambda(r_1)r_2[\phi(r_1),t]\in P$. Thus $\lambda(r_1)R[\phi(r_1),t]\subseteq P.$

From the above fact, we have either $\left[\phi(r_1),t\right]\in P$ for all $r_1,t\in R$ or $\lambda(r_1)\subseteq P$ since *P* is prime ideal.

From the above lemma, the required results are obtained. This completes the proof.

Theorem 1.5**:** Suppose P be a prime ideal of ring R. Suppose $\phi : R \to R$ be a left generalized derivation R associated with derivation λ in such a way that:

(i) $\phi(r_1 r_2) - r_1 r_2 \in P$ for all $r_1, r_2 \in R$,

(ii) $\phi(r_1)\phi(r_2) - r_1 r_2 \in P$ for all $r_1, r_2 \in R$,

(iii) $\phi(r_1)\phi(r_2) - r_2 r_1 \in P$ for all $r_1, r_2 \in R$, then either R/P is a commutative integral domain or $\lambda(R) \subseteq P$.

Proof: (i) Let us start with $\phi(r_1 r_2) - r_1 r_2 \in P$ for all $r_1, r_2 \in R.$ (1.14)

If $\phi = 0$, $r_1 r_2 \in P$.

Here $P \neq R$ and P is prime. Hence R/P is a commutative integral domain.

Let us take $\phi \neq 0.$

Substitute $r r_1$ for r_1 in (1.14) and using it, we have $\lambda(r) r_1 r_2 \in P$,

i.e., $\lambda(r)[r_{1,}, r_3] r_2 \in P$ for all $r_1, r_2, r_3, r \in R$.

Here $P \neq R$ and P is prime, we have $\lambda(R) \subseteq P$ or $[r_{1,}, r_3] \in P$ for all $r_1, r_3 \in R$.

If $[r_1, r_3] \in P$ for all $r_1, r_3 \in R$.

From the above lemma, the required result is obtained. This completes the proof.

(ii) Let us start with $\phi(r_1)\phi(r_2) - r_1 r_2 \in P$ for all $r_1, r_2 \in R.$ (1.15)

If $\phi = 0$, then $r_1 r_2 \in P$ and hence $[r_1, r] r_2 \in P$ for all $r_1, r_2, r \in R.$

Since P is prime, we get the required result by using the above lemma.

Let us take $\phi \neq 0.$

Substitute $r r_1$ for r_1 in (1.15) and using it, where $r \in R$, we have $\lambda(r) r_1 \phi(r_2) \in P$ that is $\lambda(r) R \phi(r_2) \subseteq P$ for all $r_1, r_2, r \in R$. Then $\lambda(r) \in P$ or $\phi(r_2) \in P$.

First one gives $\lambda(R) \subseteq P$. Second one implies $\phi(R) \subseteq P$. By using (1.15), we get $r_1 r_2 \in P$.

i.e., $\left[r_1, r\right] r_2 \in P$ for all $r_1, r_2, r \in R$.

Since $P \neq R$ and P is prime, we get $\left[r_1, r\right] \in P$ for all $r_1, r_2 \in R.$

From the above lemma, the required result is obtained. This completes the proof.

(iii). Let us consider $\phi(r_1)\phi(r_2) - r_2 r_1 \in P$ for all $r_1, r_2 \in R.$ (1.16)

If $\phi = 0$, then $r_2 r_1 \in P$. It deduces that $\left[r_2, r\right] r_1 \in P$ for all $r_1, r_2, r \in R$.

As P is prime, Thus, we get the required result by using the above lemma.

Let us take that $\phi \neq 0$. Substitute $r_2 r_1$ for r_1 in (1.16), we obtain $\lambda(r_2)r_1\phi(r_2) \in P$ for all $r_1, r_2 \in R$.

i.e., $\lambda(r_2)R\phi(r_2) \subseteq P$.

By using the fact and P is prime, we have either $\phi(R) \subseteq P$ or $\lambda(R) \subseteq P$.

If $\varphi(R) \subseteq P$ by using (1.16), we obtain $r_2 r_1 \in P$.

By using the same method with the appropriate alternates as used in the above proof, we can get the required conclusion.

Theorem 1.6: Suppose $\phi : R \to R$ be a left generalized derivation of ring R associated with derivation λ in such a way that:

(i) $\phi(r_1 r_2) - [r_1, r_2] \in P$ for all $r_1, r_2 \in R,$

(ii) $\phi(r_1 r_2) - (r_1 o r_2) \in P$ for all $r_1, r_2 \in R,$

(iii) $\phi(r_1)\phi(r_2) - [r_1, r_2] \in P$ for all $r_1, r_2 \in R,$

(iv) $\phi(r_1)\phi(r_2) - (r_1 o r_2) \in P$ for all $r_1, r_2 \in R$ where P is prime ideal of R, then R/P is a commutative integral domain.

Proof: (i) Let us start with $\phi(r_1 r_2) - [r_1, r_2] \in P$ for all $r_1, r_2 \in R.$ (1.17)

If $\phi = 0$, then $[r_1, r_2] \in P$.

It gives the required result by using the above lemma.

Now take $\phi \neq 0$. Now, substitute rr_1 for r_1 in (1.17) and using (1.17), we get

$$\lambda(r)r_1 r_2 - [r, r_2]r_1 \in P \quad \text{for all} \quad r, r_1, r_2 \in R. \tag{1.18}$$

Replace $r_1 t$ in place of r_1 where $t \in R$, we get

$$\lambda(r)r_1 t r_2 - [r, r_2]r_1 t \in P \text{ for all } r, r_1, r_2, t \in R. \tag{1.19}$$

Right multiplying equation (1.18) by t where $t \in R$, we obtain

$$\lambda(r)r_1 r_2 t - [r, r_2]r_1 t \in P \text{ for all } r, r_1, r_2, t \in R. \tag{1.20}$$

$(1.20) - (1.19)$ gives $\lambda(r)r_1[r_2, t] \in P$. That is

$\lambda(r)R[r_2, t] \subseteq P$. Then, either $\lambda(r) \in P$ or $[r_2, t] \in P$.

In case $\lambda(r) \subseteq P$. By using last relation in (1.18) , we get $[r, r_2]r_1 \in P$ and since $P \neq R$, then $[r, r_2] \in P$ .Thus, we get the required result by using the above lemma.

(ii) Consider $\phi(r_1 r_2) - (r_1 \text{ o} r_2) \in P$ for all $r_1, r_2 \in R.$ (1.21)

If $\phi = 0$ then $(r_1 \text{ o} r_2) \in P$.

From this, we get the result by the above lemma.

Now take $\phi \neq 0$. Now, substitute rr_1 for r_1 in (1.21) and using (1.21), it reduces to

$$\lambda(r)r_1 r_2 - [r, r_2]r_1 \in P \text{ for all } r, r_1, r_2 \in R. \tag{1.22}$$

Replace $r_1 t$ in place of r_1 where $r \in R$, we get

$$\lambda(r)r_1 t r_2 - [r, r_2]r_1 t \in P \text{ for all } r, r_1, r_2, t \in R. \tag{1.23}$$

Right multiplying equation (1.22) by t where $t \in R$, we obtain

$\lambda(r)r_1r_2t-[r,r_2]\,r_1t\in P$ for all $r,r_1,r_2,t\in R.$ (1.24)

Subtracting (1.23) from (1.24) gives $\lambda(r)r_1[r_2,t]\in P$. That is $\lambda(r)R[r_2,t]\subseteq P$. Thus, every $\lambda(r)\in P$ or $[r_2,t]\in P$.

In case $\lambda(r)\subseteq P$.

By using last relation in (1.22), we have $[r,r_2]r_1\in P$ and for $P\neq R$, then $[r,r_2]\in P$.

From this, the above lemma gives the proof.

If $[r_2,t]\in P$, it also gives the same result with the help of lemma.

(iii) Consider $\phi(r_1)\phi(r_2)-[r_1,r_2]\in P$ for all $r_1,r_2\in R.$ (1.25)

If $\phi=0$, then $[r_1,r_2]\in P$.

From this, the above lemma gives the proof.

Now consider $\phi\neq 0$. Then, substitute rr_1 for r_1 in (1.25) and using (1.25), we obtain

$$\lambda(r)r_1\phi(r_2)-[r,r_2]r_1\in P \text{ for all } r,r_1,r_2\in R. \tag{1.26}$$

Taking $r=r_2$ in (1.26), we get $\lambda(r_2)r_1\phi(r_2)\in P.$

That is $\lambda(r_2)R\phi(r_2)\subseteq P$. By the fact above, we have either $\lambda(r_2)\in P$ or $\phi(r_2)\in P$ for all $r_2\in P.$

In the first case, $\lambda(r_2)\in P$ gives $\lambda(r_2)\subseteq P$ and using the last relation in (1.26), then $[r,r_2]r_1\in P$ and since $P\neq R$, then $[r,r_2]\in P$ for all $r,r_2\in R.$

From this, the above lemma yields the required result.

Further, $\phi(r_2)\in P$ for all $r_2\in R$. It implies $\phi(R)\subseteq P$. By using the last relation (1.25) we get $[r_1,r_2]\in P$ for all $r,r_2\in R.$

From this, the above lemma yields the required result

(iv) Let us start with $\varphi(r_1)\varphi(r_2)-(r_1\,\mathrm{o}\,r_2)\in P$ for all $r,r_2\in R.$ (1.27)

If $\phi=0$ then $r_1\,\mathrm{o}\,r_2\in P$ .Thus, we get the required result by using the above lemma.

Now Let us take $\phi\neq 0$. Now, substitute rr_1 for r_1 in (1.27) and using (1.27), it reduces to

$$\lambda(r)r_1\phi(r_2)+[r,r_2]r_1\in P \text{ for all } r,r_1,r_2\in R. \tag{1.28}$$

Putting $r=r_2$ in (1.26), we get $\lambda(r_2)r_1\phi(r_2)\in P.$

That is $\lambda(r_2)R\phi(r_2)\subseteq P$. Applying the fact, we get either $\lambda(r_2)\in P$ or $\phi(r_2)\in P$ for all $r_2\in P.$

In the first case $\lambda(r_2)\in P$ which implies $\lambda(r_2)\subseteq P$.

As follows discussed in the proof of the above theorems, $\phi(r_2)\in P$ implies that R/P is a commutative integral domain.

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