Examining the Distinct Characteristics of *f-alpha* Flexible Q-Fuzzy Subgroups and *f-alpha* Flexible Normal Q-Fuzzy Subgroups

Geethalakshmi Manickam [1], Israr Ahmad[2], Gopal Thangavel[3], Kulandaivel Maruthamuthu Paramasivam[4], Balakrishnan Somasundaram[5]

[1]University of Technology and Applied Sciences - Nizwa, Sultanate of Oman.
geetha.lakshmi@utas.edu.om

[2] University of Technology and Applied Sciences - Nizwa, Sultanate of Oman.

israr.ahmed@utas.edu.om

[3] University of Technology and Applied Sciences - Nizwa, Sultanate of Oman.
gopal.thangavel@utas.edu.om

[4] University of Technology and Applied Sciences – Al Mussanah, Sultanate of Oman.
mpkoman@gmail.com (Corresponding Author)

[5] University of Technology and Applied Sciences - Nizwa, Sultanate of Oman.
bala.krish@utas.edu.om

Abstract

This paper introduces the concepts Q-fuzzy subsets and Q-fuzzy subgroups and establishes essential properties related to these two notions. This study aims to present and investigate new ideas of fuzzy subsets and fuzzy subsets that are $(f-\alpha)$-flexible. We define $(f-\alpha)$-Q-fuzzy subgroups, $(f-\alpha)$-flexible Q-fuzzy subgroups and $(f-\alpha)$-flexible normal Q-fuzzy subgroups based on these definitions. We investigate and analyze fundamental characteristics while incorporating new discoveries into the research on these topics.

Keywords: Fuzzy subset, $(f-\alpha)$-flexible fuzzy subset, $(f-\alpha)$-flexible Q-fuzzy subset, $(f-\alpha)$-flexible Q-fuzzy subgroup, $(f-\alpha)$-flexible Q-fuzzy normal subgroups.

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1. Introduction

The concept of fuzzy sets was first introduced by Zadeh in 1965. Following this, Rosenfeld in 1971 advanced the idea of fuzzy subgroups, marking the initial fuzzification of algebraic structures. This development inspired many mathematicians to explore various fuzzy algebraic structures, including fuzzy ideals in rings and semi-rings. Zadeh further expanded the theory in

1975 with the introduction of interval-valued fuzzy sets, where the values represent ranges rather than precise points. This concept was subsequently generalized by Anthony and Sherwood in 1979, leading to their definition of fuzzy normal subgroups. Additionally, Mukherjee and Bhattacharya (1986) investigated fuzzy cosets and normal fuzzy groupings.

Murali and Makamba (2006) introduced the concept of fuzzy subgroups in abelian groups by analyzing the number of fuzzy subgroups in an abelian group of order $p^n q$ where p, n, q are positive integers. In contrast, Solairaju and Nagarajan (2011) explored higher Q-fuzzy orders and Q-fuzzy subgroups. This study builds on findings from the works of Sarangapani and Muruganantham (2016) and Geethalakshmi Manickam and Solairaju. The aim of this work is to introduce the concepts of $(f-\alpha)$-Q-fuzzy subsets and $(f-\alpha)$-Q-fuzzy subgroups and to establish some basic algebraic properties associated with these concepts. We introduce the concepts of $(f-\alpha)$-flexible Q-fuzzy groups and $(f-\alpha)$-flexible normal Q-fuzzy subgroups and explore some of their essential properties through discussion and derivation.

2. Preliminaries and Definitions

The fundamental definitions and results are now outlined within the framework of a $(f-\alpha)$-flexible Q-fuzzy subgroups.

Definition 2.1: Let M_α be a set. A mapping μ_α: $M_\alpha \rightarrow [0,1]$ is referred to as a fuzzy subset of M_α.

Definition 2.2: Let M_α be any subgroup. A mapping $\Omega_\alpha^f : M_\alpha \rightarrow [0,1]$ is considered a

$(f-\alpha)$-fuzzy subgroup on M_α if it satisfies the following conditions:

i) $\Omega_\alpha^f(m_\alpha n_\alpha) \leq \max\{\Omega_\alpha^f(m_\alpha), \Omega_\alpha^f(n_\alpha)\}$

ii) $\Omega_\alpha^f(m_\alpha^{-1}) \leq \Omega_\alpha^f(m_\alpha)$ for all $m_\alpha, n_\alpha \in M_\alpha$ and for any α in M_α.

Definition 2.3: For any Q-fuzzy subset Z_α in M_α and k ∈ [0,1], the set

$U(Z_\alpha, \text{k}) = \{ m_\alpha \in M_\alpha \mid Z_\alpha(m_\alpha, q_{rl}) \geq \text{k}$ for all $q_{rl} \in \text{Q}\}$ is called the cut-set of Z_α at k.

Note 2.4: We use Q = $\{q_{rl}\}$ throughout this article

Definition 2.5: A $(f-\alpha)$-Q-fuzzy subset Z_α^f is termed as $(f-\alpha)$-Q-fuzzy subgroup of M_α if it satisfies the following conditions:

($(f-\alpha)$- QFG1): $Z_\alpha^f(m_\alpha\, n_\alpha, q_{rl}) \geq \min\{Z_\alpha^f(m_\alpha, q_{rl}), Z_\alpha^f(n_\alpha, q_{rl})\}$

($(f-\alpha)$-QFG2): $Z_\alpha^f(m_\alpha^{-1}, q_{rl}) = Z_\alpha^f(m_\alpha, q_{rl})$

($(f-\alpha)$-QFG3): $Z_\alpha^f(m_\alpha, q_{rl}) = 1$ for all $m_\alpha, n_\alpha \in M_\alpha$ and $q_{rl} \in$ Q.

Definition 2.6: If Ω_{α}^{f} is a $(f-\alpha)$-fuzzy subgroup of a subgroup M_{α} with identity e_{α}, then

i) Ω_{α}^{f} $(m_{\alpha}^{-1}, q_{rl}) = \Omega_{\alpha}^{f}$ (m_{α}, q_{rl})
ii) Ω_{α}^{f} $(e_{\alpha}, q_{rl}) \leq \Omega_{\alpha}^{f}(m_{\alpha}, q_{rl})$ for all $m_{\alpha} \in M_{\alpha}$.

Definition 2.7: Let Ω_{α}^{f} be a $(f-\alpha)$-Q-fuzzy subgroup of M_{α} . Then Ω_{α}^{f} is called a $(f-\alpha)$-normal Q-fuzzy subgroup of M_{α} if Ω_{α}^{f} $(m_{\alpha}\ n_{\alpha}, q_{rl}) = \Omega_{\alpha}^{f}$ $(n_{\alpha}\ m_{\alpha}, q_{rl})$ for all $m_{\alpha}, n_{\alpha} \in M_{\alpha}$.

Definition 2.8: A $T-nrm$ is a function that maps pairs of numbers from the unit interval to the unit interval, defined as $T-nrm$:[0,1] × [0,1] → [0,1]. The following four conditions are satisfied for all $\gamma_{1,}\gamma_{2,}\gamma_{3,}$and $\gamma_4 \in [0,1]$.

i. $T-nrm\ (\gamma_{1,}, \gamma_2) = T-nrm\ (\gamma_2, \gamma_{1,})$
ii. $T-nrm\ (\gamma_{1,}, T-nrm\ (\gamma_2, \gamma_3)) = T-nrm\ (T-nrm\ (\gamma_{1,}, \gamma_2), \gamma_3)$
iii. $T-nrm\ (\gamma_{1,}, 1) = T-nrm\ (1, \gamma_1) = 1$
iv. If $\gamma_{1,} \leq \gamma_3$and $\gamma_2 \leq \gamma_4$,then $T-nrm\ (\gamma_{1,}, \gamma_2) \leq T-nrm\ (\gamma_3, \gamma_4)$.

Note 2.9: The $T-nrm$ is a minimum-based norm.

Definition 2.10: Let M_{α} be a set. A mapping Ω_{α}^{f}: M_{α}×Q→Q^{sub*}([0,1]) is referred to as a $(f-\alpha)$ -flexible Q-fuzzy subset of M_{α}, where Q^{sub*}([0,1]) denotes the collection of all non-empty subsets of the interval [0,1].

Definition 2.11: Let M_{α} be a non-empty set and let Ω_{α}^{f} and δ_{α}^{f} be two $(f-\alpha)$ flexible Q-fuzzy subsets of M_{α}.

The intersection of Ω_{α}^{f} and δ_{α}^{f} denoted by $\Omega_{\alpha}^{f} \cap \delta_{\alpha}^{f}$ and is defined by

$\Omega_{\alpha}^{f} \cap \delta_{\alpha}^{f}$ = {min $\{m_{\alpha}, n_{\alpha}\}/m_{\alpha} \in \Omega_{\alpha}^{f}$ (M_{α}), $n_{\alpha} \in \delta_{\alpha}^{f}$ (M_{α})} for all $m_{\alpha} \in M_{\alpha}$.

Similarly, the union of Ω_{α}^{f} and δ_{α}^{f} is denoted by $(\Omega_{\alpha}^{f} \cup \delta_{\alpha}^{f})$ is defined by

$\Omega_{\alpha}^{f} \cup \delta_{\alpha}^{f}$= {max $\{m_{\alpha}, n_{\alpha}\}/m_{\alpha} \in \Omega_{\alpha}^{f}$ (M_{α}), $n_{\alpha} \in \delta_{\alpha}^{f}$ (M_{α})} for all $m_{\alpha} \in M_{\alpha}$.

Definition 2.12: Let M_{α} be a non-empty set. Let Ω_{α}^{f} and δ_{α}^{f} be two Q-fuzzy subsets ofM_{α}.Then for all $m_{\alpha} \in M_{\alpha}$ and $q_{rl} \in$ Q, the following statements hold true:

i) $\Omega_{\alpha}^{f} \subseteq \delta_{\alpha}^{f} \Leftrightarrow \Omega_{\alpha}^{f}(m_{\alpha}, q_{rl}) \leq \delta_{\alpha}^{f}(m_{\alpha}, q_{rl})$

ii) $\Omega_{\alpha}^{f} = \delta_{\alpha}^{f} \Leftrightarrow \Omega_{\alpha}^{f}(m_{\alpha}, q_{rl}) = \delta_{\alpha}^{f}(m_{\alpha}, q_{rl})$

Definition 2.13:

If Ω_{α} is a $(f-\alpha)$- Q fuzzy subgroup of G_{r}, then Com(Ω_{α}^{f}) represents the complement of a $(f-\alpha)$-Q-fuzzy subgroup of Ω_{α}^{f} and is defined as Com{$\Omega_{\alpha}^{f}(m_{\alpha}, q_{rl})$}=1-{$\Omega_{\alpha}^{f}(m_{\alpha}, q_{rl})$}, for all $m_{\alpha} \in M_{\alpha}$ and $q_{rl} \in$ Q.

Definition 2.14: Let M_{α} be a groupoid, meaning it is a set closed under a binary operation (multiplication). A mapping is termed a $(f-\alpha)$-Q-fuzzy groupoid if it satisfies the following conditions:

(i) inf $\Omega_{\alpha}^{f}(m_{\alpha}\ n_{\alpha}, q_{rl}) \geq T-nrm$ {(inf $\Omega^{f}(m_{\alpha}, q_{rl})$, inf $\Omega^{f}(n_{\alpha}, q_{rl})$),$\alpha$}

(ii) sup $\Omega_{\alpha}^{f}(m_{\alpha}\ n_{\alpha}, q_{rl}) \geq T-nrm$ {(sup $\Omega^{f}(m_{\alpha}, q_{rl})$, sup $\Omega^{f}(n_{\alpha}, q_{rl})$), α}

for all $m_{\alpha}, n_{\alpha} \in M_{\alpha}$ and $q_{rl} \in$ Q.

Definition 2.15:

Let M_{α} be a group. A mapping $\Omega_{\alpha}^{f}: M_{\alpha} \times Q \rightarrow Q^{sub*}([0,1])$ is called a $(f-\alpha)$-flexible Q-fuzzy subgroup on M_{α} if it satisfies the following conditions:

(i) inf $\Omega_{\alpha}^{f}(m_{\alpha} n_{\alpha}, q_{rl}) \geq T-nrm$ { (inf $\Omega^{f}(m_{\alpha}, q_{rl})$, inf $\Omega^{f}(n_{\alpha}, q_{rl})$), α }

(ii) sup $\Omega_{\alpha}^{f}(m_{\alpha} n_{\alpha}, q_{rl}) \geq T-nrm$ { (sup $\Omega^{f}(m_{\alpha}, q_{rl})$, sup $\Omega_{\alpha}^{f}(n_{\alpha}, q_{rl})$), α}

(iii) inf $\Omega_{\alpha}^{f}(m_{\alpha}^{-1}, q_{rl}) \geq$ inf $\Omega_{\alpha}^{f}(m_{\alpha}, q_{rl})$

(iv) sup $\Omega_{\alpha}^{f}(m_{\alpha}^{-1}, q_{rl}) \geq$ sup $\Omega_{\alpha}^{f}(m_{\alpha}, q_{rl})$ for all $m_{\alpha}, n_{\alpha} \in M_{\alpha}$ and $q_{rl} \in$ Q.

Example 2.16: Let G_{r} = {e_{α},p_{α},z_{α},r_{α}} be Klein's four-group. The multiplication operation in the group G_{r} is defined as follows:

•	e_{α}	p_{α}	z_{α}	r_{α}
e_{α}	e_{α}	e_{α}	e_{α}	e_{α}
p_{α}	p_{α}	p_{α}	p_{α}	p_{α}
z_{α}	e_{α}	e_{α}	e_{α}	z_{α}
r_{α}	p_{α}	p_{α}	p_{α}	e_{α}

Then (G_{r} ,·) is a group. Define a flexible fuzzy subset μ $_{\alpha}$: G_{r}→ Q^{sub*}([0,1]) as follows:

μ $_{\alpha}$(e) = 0.75, μ $_{\alpha}$ (p) = 0.25, μ $_{\alpha}$ (z) = 0.0.25, μ $_{\alpha}$ (r) = 0.75. With these assignments, μ $_{\alpha}$ qualifies as a flexible fuzzy subgroup of G_{r}.

Note 2.17: In definition *, if Ω^{f}_{α}: $M_{\alpha}\times Q \rightarrow Q^{sub*}$([0,1]) , then Ω^{f}_{α} (m_{α}, q_{rl}) for all $m \in M_{\alpha}$ are real values in [0,1] and it holds that inf (Ω^{f}_{α} (m_{α}, q_{rl})) = sup(Ω^{f}_{α} (m_{α}, q_{rl}))= Ω^{f}_{α} (m_{α}, q_{rl}) for all $m \in M_{\alpha}$ and q_{rl}∈Q. Consequently, definition * reduces to Rosenfeld's definition of a fuzzy subgroup. Therefore, $(f-\alpha)$-flexible Q-fuzzy subgroup is a generalization of Rosenfeld's fuzzy group.

3. Properties of $(f-\alpha)$-Flexible Q-Fuzzy Subgroups

Definition 3.1: Let G_{r} and Q be two non-empty subsets and for any $\alpha \in [0,1]$.Then , a mapping Ω^{f}_{α}: $G_{\mathrm{r}} \times Q \rightarrow [0,1]$ is called a $(f-\alpha)$-flexible Q-fuzzy set of G_{r} , with respect to the fuzzy subset Ω^{f}_{α}, if Ω^{f}_{α} $(m_{\alpha}, q_{rl}) = T-nrm$ { Ω^{f} (m_{α}, q_{rl}),α } for m_{α} Є G_{r} and q_{rl}∈ Q.

Remark 3.2: Obviously if $\alpha = 1$, then Ω^{f}_{α} $(m_{\alpha}, q_{rl}) = (\Omega^{f})^{\alpha}(m_{\alpha}, q_{rl}) = (\Omega^{f})^{1}(m_{\alpha}, q_{rl}) = \Omega^{f}$ (m_{α}, q_{rl}) and if $\alpha = 0$, then Ω^{f}_{α} $(m_{\alpha}, q_{rl}) = (\Omega^{f})^{\alpha}(m_{\alpha}, q_{rl}) = (\Omega^{f})^{0}(m_{\alpha}, q_{rl}) = 0$.

Theorem 3.3: Every $(f-\alpha)$-Q- fuzzy subgroup of a group M_{α} is $(f-\alpha)$-Q fuzzy subgroup of M_{α}.

Proof: Suppose Ω^{f}_{α} be a $(f-\alpha)$- Q- fuzzy subgroup of a group M_{α}.

Let m_{α}, n_{α} be any two elements in a subgroup M_{α}.

Then, $\Omega_{\alpha}^{f}\ (m_{\alpha}, q_{rl}) = T - nrm\ \{\ \Omega^{f}\ (m_{\alpha} n_{\alpha}, q_{rl}),\alpha\ \}$
$\geq T - nrm\ \{\ T - nrm\{\Omega^{f}(m_{\alpha}, q_{rl}), \Omega^{f}(n_{\alpha}, q_{rl})), \alpha\ \}$

$= T - nrm\ \{\ T - nrm\{\Omega^{f}(m_{\alpha}, q_{rl}),\alpha\ \}, T - nrm\ \{\Omega^{f}(m_{\alpha}, q_{rl})), \alpha\ \}\}$

$= T - nrm\{(\Omega_{\alpha}^{f}(m_{\alpha}, q_{rl}), \Omega_{\alpha}^{f}(m_{\alpha}, q_{rl}))^{\alpha}\}, \alpha\ \}$

$\Omega_{\alpha}^{f}(m_{\alpha}, q_{rl}) \geq\ T - nrm\{(\Omega_{\alpha}^{f}(m_{\alpha}, q_{rl}),\Omega_{\alpha}^{f}(n_{\alpha}, q_{rl})), \alpha\ \}$and

$\Omega_{\alpha}^{f}\ (m_{\alpha}^{-1}, q_{rl}) = T - nrm\{\ \Omega^{f}\ (m_{\alpha}^{-1}, q_{rl}),\alpha\ \}$

$\geq T - nrm\ \{\ \Omega^{f}\ (m_{\alpha}, q_{rl}),\alpha\ \}$

$=\Omega_{\alpha}^{f}\ (m_{\alpha}, q_{rl})$

Therefore $\Omega_{\alpha}^{f}(m_{\alpha}^{-1}, q_{rl}) \geq\ \Omega_{\alpha}^{f}\ (m_{\alpha}, q_{rl})$. Thus, Ω_{α}^{f} is a $(f - \alpha)$- Q fuzzy subgroup of M_{α}.

Remark 3.4: A $(f - \alpha)$- Q- fuzzy subgroup of M_{α} is not necessarily a Q-fuzzy subgroup of a group M_{α}.

Example 3.5: Consider the four Klein's group $G_r = \{p_{\alpha}, k_{\alpha}, r_{\alpha}, e_{\alpha}\}$ where $p_{\alpha}k_{\alpha} = q_{\alpha}k_{\alpha} = r_{\alpha}$ and $p_{\alpha}^{2} = k_{\alpha}^{2} = r_{\alpha}^{2} = e_{\alpha}$ and a non-empty set $Q = \{q_{rl}\}$.

A Q-fuzzy subset Ω_{α}^{f} of a group G_r is defined as

$$(m_{\alpha}, q_{rl}) = \begin{cases} 0.05 & if\ m_{\alpha} = e_{\alpha} \\ 0.08 & if\ m_{\alpha} = p_{\alpha}\ or\ m_{\alpha} = k_{\alpha} \\ 0.06 & if\ m_{\alpha} = r_{\alpha} \end{cases}$$

It is clear that the Q- fuzzy subset Ω_{α}^{f} is not a Q-fuzzy subgroup of a group G_r since first part of definition fails to hold as $\Omega_{\alpha}^{f}\ (r_{\alpha}, q_{rl}) <\ T - nrm\{(\Omega^{f}(p_{\alpha}, q_{rl}), \Omega^{f}(k_{\alpha}, q_{rl})), \alpha\ \}$

However, it can be demonstrated that $\Omega_{\alpha}^{f}$is a $(f - \alpha)$ - Q-fuzzy subgroups of a group G_r. If we set $\alpha = 0.03$, we find that $\Omega_{\alpha}^{f}\ (r_{\alpha}, q_{rl}) > \alpha$ for all elements in the group G_r and for all q_{rl} in Q. This indicates that , $\Omega_{\alpha}^{f}\ (m_{\alpha},\ q_{rl}) = T - nrm\ \{\ \Omega^{f}\ (m_{\alpha}, q_{rl}),\alpha\}=\alpha$, for all $m_{\alpha} \in G_r$ and $q_{rl} \in$ Q. Which fulfills the first condition of the definition of $(f - \alpha)$-Q-fuzzy subgroups:

$\Omega_{\alpha}^{f}\ (m_{\alpha} n_{\alpha}, q_{rl}) \geq T - nrm\ \{(\Omega^{f}\ (m_{\alpha}, q_{rl}),\ \Omega^{f}(n_{\alpha}, q_{rl})), \alpha\}$

Furthermore, for the second condition of the definition, since $p_{\alpha}^{-1} = p_{\alpha}, k_{\alpha}^{-1} = k_{\alpha}$ *and* $r_{\alpha}^{-1} = r_{\alpha}$ it follows that $\Omega_{\alpha}^{f}(m_{\alpha}^{-1}, q_{rl}) \geq \Omega_{\alpha}^{f}(m_{\alpha}, q_{rl})$.

Therefore , Ω_{α}^{f} is an $(f-\alpha)$-Q-fuzzy subgroup of the group G_{r}, denoted as Ω_{α}^{f}.

Theorem 3.6: A $(f-\alpha)$-flexible Q-fuzzy subset Ω_{α}^{f} of a group M_{α} is considered a $(f-\alpha)$-flexible Q-fuzzy subgroup if and only if the following conditions are met.

(i) $\inf \Omega_{\alpha}^{f}(m_{\alpha} n_{\alpha}^{-1}, q_{rl}) \geq T-nrm\{(\inf \Omega^{f}(m_{\alpha}, q_{rl}), \inf \Omega^{f}(n_{\alpha}, q_{rl})), \alpha\}$

(ii) $\sup \Omega_{\alpha}^{f}(m_{\alpha} n_{\alpha}^{-1}, q_{rl}) \geq T-nrm\{(\sup \Omega^{f}(m_{\alpha}, q_{rl}), \sup \Omega^{f}(n_{\alpha}, q_{rl})), \alpha\}$

for all $m_{\alpha}, n_{\alpha} \in M_{\alpha}$ and $q_{rl} \in Q$.

Proof: Initially, let Ω_{α}^{f} be a $(f-\alpha)$-flexible Q-fuzzy subgroup of M_{α}, and let m_{α} and n_{α} be elements of M_{α}. Then

$$\inf \Omega_{\alpha}^{f}(m_{\alpha} n_{\alpha}^{-1}, q_{rl}) \geq T-nrm\{(\inf \Omega^{f}(m_{\alpha}, q_{rl}), \inf \Omega^{f}(n_{\alpha}^{-1}, q_{rl})), \alpha\}$$
$$= T-nrm\{(\inf \Omega^{f}(m_{\alpha}, q_{rl}), \inf \Omega^{f}(n_{\alpha}, q_{rl})), \alpha\} \text{ and}$$
$$\sup \Omega_{\alpha}^{f}(m_{\alpha} n_{\alpha}^{-1}, q_{rl}) \geq T-nrm\{(\sup \Omega^{f}(m_{\alpha}, q_{rl}), \sup \Omega^{f}(n_{\alpha}, q_{rl})), \alpha\}$$
$$= T-nrm\{(\sup \Omega^{f}(m_{\alpha}, q_{rl}) \sup \Omega^{f}(n_{\alpha}, q_{rl})), \alpha\}.$$

Conversely, if Ω_{α}^{f} is a $(f-\alpha)$-flexible Q-fuzzy subset of M_{α} and the given conditions are satisfied, then it follows that

$$\inf \Omega_{\alpha}^{f}(e_{\alpha}, q_{rl}) = \inf \Omega_{\alpha}^{f}(e_{\alpha} m_{\alpha}^{-1}, q_{rl}) \geq T-nrm\{(\inf \Omega^{f}(e_{\alpha}, q_{rl}), \inf \Omega^{f}(m_{\alpha}, q_{rl})), \alpha\}$$
$$= \inf \Omega^{f}(m_{\alpha}, q_{rl}) \text{---------- (1)}$$

$$\sup \Omega_{\alpha}^{f}(e_{\alpha}, q_{rl}) = \sup \Omega_{\alpha}^{f}(e_{\alpha} m_{\alpha}^{-1}, q_{rl}) \geq T-nrm\{(\sup \Omega^{f}(e_{\alpha}, q_{rl}), \sup \Omega^{f}(m_{\alpha}, q_{rl})), \alpha\}$$
$$= \sup \Omega^{f}(m_{\alpha}, q_{rl}) \text{----------- (2)} \quad \text{for all } m_{\alpha} \in M_{\alpha}.$$

This implies that

$$\inf \Omega_{\alpha}^{f}(m_{\alpha}^{-1}, q_{rl}) = \inf \Omega_{\alpha}^{f}(e_{\alpha} m_{\alpha}^{-1}, q_{rl}) \geq T-nrm\{(\inf \Omega^{f}(e_{\alpha}, q_{rl}),$$

$\inf \Omega^f\ (m_\alpha, q_{rl})), \alpha\}$ by using (1)

and $\sup \Omega^f_\alpha\ (m_\alpha^{-1}, q_{rl}) = \sup \Omega^f_\alpha\ (e_\alpha m_\alpha^{-1}, q_{rl}) \geq T - nrm\ \{(\sup \Omega^f\ (e_\alpha, q_{rl}),$

$\sup \Omega^f\ (m_\alpha, q_{rl})), \alpha\}$ by using (2).

Once again, we even prove that

$\inf \Omega^f_\alpha\ (m_\alpha n_\alpha, q_{rl}) \geq T - nrm\ \{(\inf \Omega^f\ (m_\alpha, q_{rl}), \inf \Omega^f\ (n_\alpha^{-1}, q_{rl})), \alpha\}$

$\geq T - nrm\ \{(\inf \Omega^f\ (m_\alpha, q_{rl}), \inf \Omega^f\ (n_\alpha, q_{rl})), \alpha\}.$

and $\sup \Omega^f_\alpha\ (m_\alpha n_\alpha, q_{rl}) \geq T - nrm\ \{(\sup \Omega^f\ (m_\alpha, q_{rl}), \sup \Omega^f\ (n_\alpha^{-1}, q_{rl})), \alpha\}$

$\geq T - nrm\ \{(\sup \Omega^f\ (m_\alpha, q_{rl}), \sup \Omega^f\ (n_\alpha, q_{rl})), \alpha\}.$

Therefore, Ω^f_α is a $(f - \alpha)$-flexible Q-fuzzy group of M_α .

Theorem 3.7: If Ω^f_α is a $(f - \alpha)$-flexible Q-fuzzy groupoid of an infinite group M_α, then Ω^f_α is a $(f - \alpha)$-flexible Q-fuzzy group of M_α.

Proof: Let $m \in M_\alpha$. Since M_α is finite, m_α has a finite order, say p. Therefore,$m_\alpha^p = e_\alpha$, where e_α is the identity element of M_α.

Thus, $m_\alpha^{-1}$is equivalent to m_α^{p-1} in the context of $(f - \alpha)$-flexible Q-fuzzy groupoids.

Thus, it follows that $\inf \Omega^f_\alpha\ (m_\alpha^{-1}, q_{rl}) = \inf \Omega^f_\alpha\ (m_\alpha^{p-1}, q_{rl}) = \inf \Omega^f_\alpha\ (m_\alpha^{p-2}, q_{rl})$

$\geq T - nrm\ \{(\inf \Omega^f\ (m_\alpha^{p-2}, q_{rl}), \Omega^f\ (m_\alpha, q_{rl})), \alpha\}.$

Once again, we even prove that

$\inf \Omega^f_\alpha\ (m_\alpha^{p-2}, q_{rl}) = \inf \Omega^f_\alpha\ (m_\alpha^{p-3}, q_{rl})$

$\geq T - nrm\ \{(\inf \Omega^f\ (m_\alpha^{p-3}, q_{rl}), \Omega^f\ (m_\alpha, q_{rl})), \alpha\}.$

Consequently, it can be deduced that

$\inf \Omega^f_\alpha\ (m_\alpha^{-1}, q_{rl}) \geq T - nrm\ \{(\inf \Omega^f\ (m_\alpha^{p-3}, q_{rl}), \Omega^f\ (m_\alpha, q_{rl})), \alpha\}.$

By repeatedly applying the definition of a $(f-\alpha)$-flexible Q-fuzzy groupoid, we obtain:

$$\inf \Omega_{\alpha}^{f}(m_{\alpha}^{-1}, q_{rl}) \leq \inf \Omega_{\alpha}^{f}(m_{\alpha}, q_{rl}).$$

Similarly, we have $\sup \Omega_{\alpha}^{f}(m_{\alpha}^{-1}, q_{rl}) \leq \sup \Omega_{\alpha}^{f}(m_{\alpha}, q_{rl})$.

Thus, Ω_{α}^{f} is a $(f-\alpha)$-flexible Q-fuzzy subgroup of M_{α}.

Theorem 3.8: The intersection of any two$(f-\alpha)$-flexible Q-fuzzy subgroups is also a $(f-\alpha)$-flexible Q-fuzzy subgroup of M_{α}.

Proof: Let Z_{α}^{f} and Y_{α}^{f} be two $(f-\alpha)$-flexible Q-fuzzy subgroups of M_{α}. Consider the elements m_{α} and n_{α} are of M_{α}. Then

$\inf (Z_{\alpha}^{f} \cap Y_{\alpha}^{f})(m_{\alpha} n_{\alpha}^{-1}, q_{rl}) = T-nrm\{(\inf Z^{f}(m_{\alpha} n_{\alpha}^{-1}, q_{rl}), \inf Y^{f}(m_{\alpha} n_{\alpha}^{-1}, q_{rl})), \alpha\}$

$\geq T-nrm\{T-nrm\{(\inf Z^{f}(m_{\alpha}, q_{rl}), \inf Z^{f}(m_{\alpha}, q_{rl})), \alpha\}, \{(\inf Y^{f}(n_{\alpha}, q_{rl}), \inf Y^{f}(n_{\alpha}, q_{rl})), \alpha\}\}\}$

$= T-nrm\{(T-nrm\{\inf Z^{f}(m_{\alpha}, q_{rl}), \inf Y^{f}(n_{\alpha}, q_{rl})), \alpha\}, T-nrm\{(\inf Z^{f}(m_{\alpha}, q_{rl}), \inf Y^{f}(n_{\alpha}, q_{rl})), \alpha\}\}\}$

$= T-nrm\{(\inf Z^{f} \cap Y^{f}(m_{\alpha}, q_{rl}), \inf Z^{f} \cap Y^{f}(n_{\alpha}, q_{rl})), \alpha\}$ ...... (1).

We also establish that

$\sup (Z_{\alpha}^{f} \cap Y_{\alpha}^{f})(m_{\alpha} n_{\alpha}^{-1}, q_{rl}) = T-nrm\{(\sup Z^{f}(m_{\alpha} n_{\alpha}^{-1}, q_{rl}), \sup Y^{f}(m_{\alpha} n_{\alpha}^{-1}, q_{rl})), \alpha\}$ by definition

$\geq T-nrm\{T-nrm\{\{(\sup Z^{f}(m_{\alpha}, q_{rl}), \sup Z^{f}(m_{\alpha}, q_{rl})), \alpha\}, \{(\sup Y^{f}(m_{\alpha}, q_{rl}), \sup Y^{f}(n_{\alpha}, q_{rl})), \alpha\}\}\}$

$= T-nrm\{T-nrm\{(\sup Z^{f}(m_{\alpha}, q_{rl}), \sup Y^{f}(m_{\alpha}, q_{rl})), \alpha\}, T-nrm\{(\sup Z^{f}(m_{\alpha}, q_{rl}), \sup Y^{f}(n_{\alpha}, q_{rl}), \alpha\}\}\}$

$= T-nrm\{(\sup Z^{f} \cap Y^{f}(m_{\alpha}, q_{rl}), \sup Z^{f} \cap Y^{f}(n_{\alpha}, q_{rl})), \alpha\}$...... (2).

Based on results (1) and (2), it follows that the intersection $Z_{\alpha}^{f} \cap Y_{\alpha}^{f}$ is a $(f-\alpha)$-flexible Q-fuzzy subgroup of M_{α}.

Remark 3.9: If Ω_{α}^{f} and δ_{α}^{f} be two $(f-\alpha)$-Q fuzzy subgroup of the group G_{r}, then $\Omega_{\alpha}^{f} \cup \delta_{\alpha}^{f}$ need not to be a $(f-\alpha)$-flexible Q-fuzzy subgroup of a group G_{r}.

Theorem 3.10: The intersection of any arbitrary collection of $(f-\alpha)$-flexible Q-fuzzy subgroups is also a $(f-\alpha)$-flexible Q-fuzzy subgroups of M_{α}.

Proof: This is straightforward.

Theorem 3.11: If Z_{α}^{f} is a $(f-\alpha)$-flexible Q-fuzzy subgroup of a group M_{α} with identity element e_{α}, then

(i) $\inf Z_{\alpha}^{f}(m_{\alpha}^{-1}, q_{rl}) = \inf Z_{\alpha}^{f}(m_{\alpha}, q_{rl})$, and $\sup Z_{\alpha}^{f}(m_{\alpha}^{-1}, q_{rl}) = \sup Z_{\alpha}^{f}(m_{\alpha}, q_{rl})$

(ii) $\inf Z_{\alpha}^{f}(e_{\alpha}, q_{rl}) = \inf Z_{\alpha}^{f}(m_{\alpha}, q_{rl})$ and $\sup Z_{\alpha}^{f}(e_{\alpha}, q_{rl}) = \sup Z_{\alpha}^{f}(m_{\alpha}, q_{rl})$

for all $m \in M_{\alpha}$.

Proof: (i) Since Z_{α}^{f} is a $(f-\alpha)$ −flexible Q-fuzzy subgroup of the group M_{α},

it follows that $\inf Z_{\alpha}^{f}(m_{\alpha}^{-1}, q_{rl}) \leq \inf Z_{\alpha}^{f}(m_{\alpha}, q_{rl})$.

In the same way, it can be established that

$$\inf Z_{\alpha}^{f}(m_{\alpha}, q_{rl}) = \inf Z_{\alpha}^{f}((m_{\alpha}^{-1})^{-1}, q_{rl})$$

$$\leq \inf Z_{\alpha}^{f}(m_{\alpha}^{-1}, q_{rl}).$$

Thus, $\inf Z_{\alpha}^{f}(m_{\alpha}^{-1}, q_{rl}) = \inf Z_{\alpha}^{f}(m_{\alpha}, q_{rl})$.

Similarly, it can be shown that $\sup Z_{\alpha}^{f}(m_{\alpha}^{-1}, q_{rl}) = \sup Z_{\alpha}^{f}(m_{\alpha}, q_{rl})$

(ii) Given that Z_{α}^{f} is a $(f-\alpha)$-flexible Q-fuzzy subgroup of the group M_{α}, it follows that

$\inf Z_{\alpha}^{f}(e_{\alpha}, q_{rl}) = \inf Z_{\alpha}^{f}(m_{\alpha} n_{\alpha}^{-1}, q_{rl}) \geq T-nrm\,\{(\inf Z^{f}(m_{\alpha}, q_{rl}), \inf Z^{f}(m_{\alpha}^{-1}, q_{rl})), \alpha\}$ and

$\sup Z_{\alpha}^{f}(e_{\alpha}, q_{rl}) = \sup Z_{\alpha}^{f}(m_{\alpha} n_{\alpha}^{-1}, q_{rl}) \geq T-nrm\,\{(\sup Z^{f}(m_{\alpha}, q_{rl}), \sup Z^{f}(m_{\alpha}^{-1}, q_{rl})), \alpha\}$.

Theorem 3.12: Let Ω_{α}^{f} and δ_{α}^{f} be two $(f-\alpha)$-flexible Q-fuzzy subgroups of M_{α_1} and M_{α_2} respectively and let f be a homomorphism from M_{α_1} to M_{α_2} . Then:

- $h(\Omega_\alpha^f, q_{rl})$ is a $(f-\alpha)$-flexible Q-fuzzy group of M_{α_2}.
- $h(\delta_\alpha^f, q_{rl})$ is a $(f-\alpha)$-flexible Q-fuzzy group of M_{α_1}.

Proof: This is straightforward.

Remark 3.13: If Ω_α^f is a $(f-\alpha)$-flexible Q-fuzzy group of M_α and S_{gr} is a subgroup of M_α , then the restriction of Ω_α^f to S_{gr} (denoted Ω_α^f/S_{gr}) is a$(f-\alpha)$-flexible Q-fuzzy subgroup of S_{gr}.

4. Normal $(f-\alpha)$-Flexible Q-Fuzzy Subgroups

Definition 4.1: If Ω_α^f is a $(f-\alpha)$-flexible Q-fuzzy group of a group M_α, then Ω_α^f is referred to as a normal $(f-\alpha)$-flexible Q-fuzzy subgroup of M_α if

inf Ω_α^f $(m_\alpha n_\alpha)$ = inf Ω_α^f $(n_\alpha m_\alpha)$ and

sup Ω_α^f $(m_\alpha n_\alpha)$ =sup Ω_α^f $(n_\alpha m_\alpha)$} for all $m_\alpha, n_\alpha \in M_\alpha$.

Theorem 4.2: A normal $(f-\alpha)$-flexible Q-fuzzy subgroup of a group M_α can be expressed as the intersection of any two normal $(f-\alpha)$-flexible Q-fuzzy subgroups of M_α.

Proof: Let Z_α^f and Y_α^f be two normal $(f-\alpha)$-flexible Q-fuzzy subgroups of a group M_α. We need to show that $Z_\alpha^f \cap Y_\alpha^f$ is also a normal $(f-\alpha)$-flexible Q-fuzzy group of M_α.

To do this, consider any elements $m_\alpha, n_\alpha \in M_\alpha$. By definition,

inf $(Z_\alpha^f \cap Y_\alpha^f)$ $(m_\alpha n_\alpha, q_{rl})$ = $T-nrm$ {(inf Z^f $(m_\alpha n_\alpha, q_{rl})$, inf Y^f $(m_\alpha n_\alpha, q_{rl})$), α}

= $T-nrm$ {(inf Z^f $(n_\alpha m_\alpha, q_{rl})$, inf Y^f $(n_\alpha m_\alpha, q_{rl})$), α}

= inf $Z_\alpha^f \cap Y_\alpha^f$ $(m_\alpha n_\alpha, q_{rl})$.

In a similar manner, sup $(Z_\alpha^f \cap Y_\alpha^f)$ $(m_\alpha n_\alpha, q_{rl})$ = sup $(Z_\alpha^f \cap Y_\alpha^f)$ $(n_\alpha m_\alpha, q_{rl})$.

This demonstrates that $Z_\alpha^f \cap Y_\alpha^f$ is a normal $(f-\alpha)$-flexible Q-fuzzy group of M_α.

Theorem 4.3: A normal $(f-\alpha)$-flexible Q-fuzzy subgroup of a group M_α can be expressed as the intersection of any arbitrary collection of normal $(f-\alpha)$-flexible Q-fuzzy subgroups of M_α.

Proof: Let $m_\alpha, n_\alpha \in M_\alpha$ and $\alpha \in M_\alpha$. Now it finds that

$\inf Z_\alpha^f (m_\alpha n_\alpha^{-1}, q_{rl}) = \inf Z_\alpha^f (\alpha^{-1} m_\alpha n_\alpha^{-1}\alpha, q_{rl})$ by definition

$= \inf Z_\alpha^f (\alpha^{-1} m_\alpha \alpha\alpha^{-1} n_\alpha^{-1}\alpha, q_{rl}) = \inf (Z_\alpha^f (\alpha^{-1} m_\alpha \alpha, q_{rl}), Z_\alpha^f ((\alpha^{-1} n_\alpha \alpha)^{-1}, q_{rl}))$

$\geq T-nrm \{(\inf (Z^f (\alpha^{-1} m_\alpha \alpha, q_{rl}), \inf Z^f (\alpha^{-1} n_\alpha \alpha, q_{rl})), \alpha\}$

$= T-nrm \{(\inf (Z^f (m_\alpha, q_{rl}), Z^f (n_\alpha, q_{rl})), \alpha\}$.

Again $\sup Z_\alpha^f (m_\alpha n_\alpha^{-1}, q_{rl}) = \sup Z_\alpha^f (\alpha^{-1} m_\alpha n_\alpha^{-1}\alpha, q_{rl})$ by definition

$= \sup (Z_\alpha^f (\alpha^{-1} m_\alpha \alpha\alpha^{-1} n_\alpha^{-1}\alpha, q_{rl})$

$= \sup (Z_\alpha^f (\alpha^{-1} m_\alpha \alpha, q_{rl}), Z_\alpha^f ((\alpha^{-1} n_\alpha \alpha)^{-1}, q_{rl}))$

$\geq T-nrm \{(\sup (Z^f (\alpha^{-1} m_\alpha \alpha, q_{rl}), \sup (Z^f (\alpha^{-1} n_\alpha \alpha), q_{rl})), \alpha\}$

$= T-nrm \{(\sup (Z^f (m_\alpha, q_{rl}), Z^f (m_\alpha, q_{rl})), \alpha\}$.

Consequently, it can be concluded that Z_α^f is a normal $(f-\alpha)$-flexible Q-fuzzy subgroup of the group M_α.

5. Conclusion

We proposed the notions of $(f-\alpha)$-flexible fuzzy sets and $(f-\alpha)$-flexible Q-fuzzy groups in this paper. Furthermore, we described and analyzed some of the fundamental features of $(f-\alpha)$-flexible Q-fuzzy normal groups. Many of these attributes have been demonstrated.

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