Performance analysis of a limited capacity Queueing model incorporating the effects of two Environmental changing states and Catastrophes

Darvinder Kumar*
Department of Statistics, PGDAV College (University of Delhi)

Abstract: In this paper, performance analysis of a limited capacity queueing system incorporating the effects of two environmental changing states and catastrophes is studied. The effect of environmental change is taken to be a function of the number present in the system. We undertake the transient analysis of a limited capacity queueing system with two environmental states in the presence of catastrophes. Transient solution of the queueing model is obtained by using the probability generating function technique. Some interesting particular cases of the queueing model with and without catastrophes are obtained. Measures of effectiveness and steady state solutions of the model are also discussed.

Keywords: Transient analysis, Catastrophes, Environmental states, Finite capacity, Probability generating function.

1. Introduction:

In this paper, we consider a performance analysis of a limited capacity queueing model incorporates the effects of two environmental changing states and catastrophes. This involves studying the system's behavior under different environmental conditions and in the presence of disruptive events like server failures or network outages, called catastrophes. The analysis often focuses on transient and steady-state characteristics, including probabilities of various system states, queue length, and waiting time. The system's environment can transition between two changing states, potentially impacting arrival or service rates. Disruptive events like server breakdowns or network outages can occur, causing a sudden reset of the system or removal of customers in the system. In a queueing system with catastrophes, an unexpected event (like a server crash, network outage, or a virus infection) occurs, instantly clearing the queue and potentially disrupting the server. When a catastrophe occurs, all customers in the queue are removed, and the service may be temporarily unavailable until the system recovers.

In this connection, a special reference was made to our previous paper by Kumar, Darvinder [8,9] and Crescenzo et al. [2]. Kumar, Darvinder [2023] investigated a limited capacity queueing model working in two environmental changing states and possibilities of catastrophes. He have obtained transient state as well as the steady state solution of the model. A single server Markovian queuing system with catastrophe was studied by Kumar and Arivudainambi [3] in 2000 and later 2003 by Crescenzo et al. [2], where the authors deduced transient probabilities in the M/M/1 queue model with catastrophe. Jain and Kanethia [10] studied the transient analysis of a queue with environmental and catastrophic effects.
The proposed model may be useful for dealing with real-world queueing situations such as manufacturing firms, banks etc. Imagine a bank with a queue for a teller. A

* E-mail: darvinder.kumar@pgdav.du.ac.in

catastrophe could be a power outage, causing all customers in line to be temporarily removed and the teller's service to stop until the power is restored.

In this paper, we undertake the analysis of a queueing system in the presence of catastrophes and two environmental changing states in order to obtain some analytical results. In section 2, we have made the assumptions and definitions of the model. The detailed analysis of the main model is done in section 3 and some particular cases in section 4. In section 5 & 6, we have obtained the steady-state result and mean queue length. Application of the model is discussed in section 7.

2. Assumptions and Definitions:

(i) The customers arrive in the system one by one in accordance with a Poisson process at a single service station. The arrival pattern is non-homogeneous, i.e. there may exists two arrival rates, namely λ_1 and 0 of which only one is operative at any instant.

(ii) The customers are served one by one at the single channel. The service time is exponentially distributed. Further, corresponding to arrival rate λ_1 the Poisson service rate is μ_1 and the service rate corresponding to the arrival rate 0 is μ_2. The state of the system when operating with arrival rate λ_1 and service rate μ_1 is designated as E whereas the other with arrival rate 0 and service rate μ_2 is designated as F.

(iii) The Poisson rate d_n at which the system goes from environmental state E to F tends to decrease or increase whereas at the same time the Poisson rate b_n at which the system moves from environmental state F to E tends to increase or decrease according as the numbers in the queue (say n) increase or decrease from some fixed number (say N). We therefore define,

$$d_n = \beta[1+\varepsilon'(N-n)] \text{ with } n \le N + \frac{1}{\varepsilon'}$$

$$\text{and } \quad 0 \le n \le N + \frac{1}{\varepsilon'} \le M$$

Also

$$b_n = \alpha[1+\varepsilon(n-N)] \text{ with } n \ge N - \frac{1}{\varepsilon}$$

$$\text{and } \quad 0 \le N - \frac{1}{\varepsilon} \le n \le M$$

Where M denotes the size of the waiting space and ε, ε' are positive numbers such that $\varepsilon \ge \frac{1}{N}$ and $\varepsilon' \ge \frac{1}{M-N}$. These restrictions on M also are necessary to avoid the negative values of d_n and b_n. When n=N or ε=0, b_n gives the normal rate as α and when n=N or $\varepsilon = \varepsilon'$=0, d_n and b_n gives the normal rates as β and α.

(iv) When the system is not empty, catastrophes occur according to a Poisson process with rate ξ. The effect of each catastrophe is to make the queue instantly empty. Simultaneously, the system becomes ready to accept the new customers.

(v) The queue discipline is first- come-first-served.

(vi) The capacity of the system is limited to M.

Define,

$P_n(t)$ = Joint probability that at time t the system is in state E and n units are in the queue, including the one in service.

$Q_n(t)$ = Joint probability that at time t the system is in state F and n units are in the queue, including the one in service.

$R_n(t)$ = The probability that at time t there are n units in the queue, including the one in service.

Obviously,

$$R_n(t) = P_n(t) + Q_n(t)$$

Let us reckon time t from an instant when there are zero customers in the queue and the system is in the environmental state E so that the initial conditions associated with $P_n(t)$ and $Q_n(t)$ becomes,

$$P_n(0) = \begin{cases} 1 & ; \quad n = 0 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

$Q_n(0) = 0$; for all n.

3. Formulation of Model and Analysis (Time Dependent Solution):

The differential-difference equations governing the system are:

$$\frac{d}{dt}P_0(t) = -(\lambda_1 + d_0 + \xi)P_0(t) + \mu_1 P_1(t) + b_0 Q_0(t) + \xi\sum_{n=0}^{M} P_n(t)\,;\quad n = 0 \tag{1}$$

$$\frac{d}{dt}P_n(t) = -(\lambda_1 + \mu_1 + d_n + \xi)P_n(t) + \mu_1 P_{n+1}(t) + \lambda_1 P_{n-1}(t) + b_n Q_n(t)\,;\quad 0 < n < M \tag{2}$$

$$\frac{d}{dt}P_M(t) = -(\mu_1 + d_M + \xi)P_M(t) + \lambda_1 P_{M-1}(t) + b_M Q_M(t)\,;\quad n = M \tag{3}$$

$$\frac{d}{dt}Q_0(t) = -(b_0 + \xi)Q_0(t) + \mu_2 Q_1(t) + d_0 P_0(t) + \xi\sum_{n=0}^{M} Q_n(t)\,;\quad n = 0 \tag{4}$$

$$\frac{d}{dt}Q_n(t) = -(\mu_2 + b_n + \xi)\,Q_n(t) + \mu_2\,Q_{n+1}(t) + d_n P_n(t)\,;\quad 0 < n < M \tag{5}$$

$$\frac{d}{dt}Q_M(t) = -(\mu_2 + b_M + \xi)Q_M(t) + d_M P_M(t)\,;\quad n = M \tag{6}$$

Define, the Laplace Transform as

$$\text{L.T. } [f(t)] = \int_0^\infty e^{-st} f(t)\,dt = \bar{f}(s) \tag{7}$$

Now, taking the Laplace transforms of equations (1)–(6) and using the initial conditions, we get

$$(s + \lambda_1 + d_0 + \xi)\overline{P}_0(s) - 1 = \mu_1\,\overline{P}_1(s) + b_0\,\overline{Q}_0(s) + \xi\sum_{n=0}^{M}\overline{P}_n(s) \tag{8}$$

$$(s + \lambda_1 + \mu_1 + d_n + \xi)\overline{P}_n(s) = \mu_1\,\overline{P}_{n+1}(s) + \lambda_1\,\overline{P}_{n-1}(s) + b_n\,\overline{Q}_n(t) \tag{9}$$

$$(s + \mu_1 + d_M + \xi)\overline{P}_M(s) = \lambda_1\,\overline{P}_{M-1}(s) + b_M\,\overline{Q}_M(s) \tag{10}$$

$$\left(s+b_0+\xi\right)\overline{Q}_0(s)=\mu_2\,\overline{Q}_1(s)+d_0\,\overline{P}_0(s)+\xi\sum_{n=0}^{M}\overline{Q}_n(s) \tag{11}$$

$$\left(s+\mu_2+b_n+\xi\right)\overline{Q}_n(s)=\mu_2\,\overline{Q}_{n+1}(s)+d_n\,\overline{P}_n(s) \tag{12}$$

$$\left(s+\mu_2+b_M+\xi\right)\overline{Q}_M(s)=d_M\,\overline{P}_M(s) \tag{13}$$

Define, the probability generating functions

$$P(z,s)=\sum_{n=0}^{M}\overline{P}_n(s)z^n \tag{14}$$

$$Q(z,s)=\sum_{n=0}^{M}\overline{Q}_n(s)z^n \tag{15}$$

$$R(z,s)=\sum_{n=0}^{M}\overline{R}_n(s)z^n \tag{16}$$

where

$$\overline{R}_n(s)=\overline{P}_n(s)+\overline{Q}_n(s)$$

Multiplying equations (8)–(10) by the suitable powers of z, summing over all n and using equations (14)–(16), we have.

$$\beta\varepsilon' z^2P'(z,s)+\alpha\varepsilon z^2Q'(z,s)+\left[\lambda_1z^2-z\left\{s+\lambda_1+\mu_1+\xi+\beta(1+\varepsilon'N)\right\}+\mu_1\right]P(z,s)$$

$$+\alpha(1-\varepsilon N)zQ(z,s)=\lambda_1z^{M+1}(z-1)\overline{P}_M(s)+\mu_1(1-z)\overline{P}_0(s)-z-\xi z\sum_{n=0}^{M}\overline{P}_n(s) \tag{17}$$

Similarly, from equations (11)–(13) and using (14)–(16), we have

$$\beta\varepsilon' z^2P'(z,s)+\alpha\varepsilon z^2Q'(z,s)-\beta(1+\varepsilon' N)zP(z,s)+\left[z\left\{s+\mu_2+\xi+\alpha(1-\varepsilon N)\right\}-\mu_2\right]Q(z,s)$$

$$=\mu_2(z-1)\overline{Q}_0(s)+\xi z\sum_{n=0}^{M}\overline{Q}_n(s) \tag{18}$$

Subtracting equation (18) from (17), we have.

$$\left[\lambda_1z^2-z(s+\lambda_1+\mu_1+\xi)+\mu_1\right]P(z,s)+\left[\mu_2-z(s+\mu_2+\xi)\right]Q(z,s)=\lambda_1z^{M+1}(z-1)\overline{P}_M(s)$$

$$+\mu_1(1-z)\overline{P}_0(s)-\mu_2(z-1)\overline{Q}_0(s)-z-\xi z\sum_{n=0}^{M}\overline{P}_n(s)-\xi z\sum_{n=0}^{M}\overline{Q}_n(s) \tag{19}$$

Differentiating equation (19) with respect to z, we have

$$\left[\lambda_1z^2-z(s+\lambda_1+\mu_1+\xi)+\mu_1\right]P'(z,s)+\left[2\lambda_1z-(s+\lambda_1+\mu_1+\xi)\right]P(z,s)$$

$$+\left[\mu_2-z(s+\mu_2+\xi)\right]Q'(z,s)-(s+\mu_2+\xi)Q(z,s)=\lambda_1z^M\left[(M+2)z-(M+1)\right]\overline{P}_M(s)$$

$$-\mu_1\overline{P}_0(s)-\mu_2\overline{Q}_0(s)-1-\xi\sum_{n=0}^{M}\overline{P}_n(s)-\xi\sum_{n=0}^{M}\overline{Q}_n(s) \tag{20}$$

Eliminating Q'(z,s) and Q(z,s) from equations (18), (19) and (20), we arrive at a computationally convenient equation.

$$P'(z,s)+\frac{\eta_1(z)}{\eta_2(z)}P(z,s)=\frac{1}{\eta_2(z)}\left[z_1+z_2\overline{Q}_0(s)+z_3\overline{P}_0(s)+z_4\overline{P}_M(s)+z_5\sum_{n=0}^{M}\overline{P}_n(s)+z_6\sum_{n=0}^{M}\overline{Q}_n(s)\right] \tag{21}$$

where

$$\eta_1(z) = a_1 z^4 + a_2 z^3 + a_3 z^2 + a_4 z + a_5$$

$$\eta_2(z) = z^2 \left[\mu_2 - z(s + \mu_2 + \xi)\right]\left[a_6 z^2 + a_7 z + a_8\right]$$

$$\frac{\eta_1(z)}{\eta_2(z)} = \frac{A}{\mu_2 - z(s+\mu_2+\xi)} + (B + C/z)\frac{1}{z} + \frac{D(2a_6 z + a_7)}{a_6(a_6 z^2 + a_7 z + a_8)}$$

$$+ \frac{(E - a_7 D/2a_6)}{a_6\left[\left(z + \frac{a_7}{2a_6}\right)^2 - \left\{\left(\frac{1}{2}a_7/a_6\right)^2 - \left(\frac{a_8}{a_6}\right)\right\}\right]}$$

$$C = \frac{a_5}{\mu_2 a_8}$$

$$B = \left[a_4 - \frac{a_5}{\mu_2 a_8}\{\mu_2 a_7 - (s+\mu_2+\xi)a_8\}\right]\frac{1}{\mu_2 a_8}$$

$$A = \begin{vmatrix} (a_3 - b_1 B - b_2 c) & 0 & \mu_2 \\ (a_2 - b_2 B - b_3 c) & \mu_2 & b_4 \\ (a_1 - b_3 B) & b_4 & 0 \end{vmatrix} \frac{1}{\Delta}$$

$$D = \begin{vmatrix} a_8 & \mu_2 & (a_3 - b_1 B - b_2 C) \\ a_7 & b_4 & (a_2 - b_2 B - b_3 C) \\ a_6 & 0 & (a_1 - b_3 B) \end{vmatrix} \frac{1}{\Delta}$$

$$E = \begin{vmatrix} a_8 & 0 & (a_3 - b_1 B - b_2 C) \\ a_7 & \mu_2 & (a_2 - b_2 B - b_3 C) \\ a_6 & b_4 & (a_1 - b_3 B) \end{vmatrix} \frac{1}{\Delta}$$

$$\Delta = \begin{vmatrix} a_8 & 0 & \mu_2 \\ a_7 & \mu_2 & b_4 \\ a_6 & b_4 & 0 \end{vmatrix}$$

$$b_1 = \mu_2\, a_7 - a_8\,(s+\mu_2+\xi)$$

$$b_2 = \mu_2\, a_6 - a_7\,(s+\mu_2+\xi)$$

$$b_3 = -\, a_6\,(s+\mu_2+\xi)$$

$$b_4 = -\,(s+\mu_2+\xi)$$

$$a_1 = \lambda_1(s+\mu_2+\xi)\left[\alpha\varepsilon + \{s + \mu_2 + \xi + \alpha(1 - \varepsilon N)\}\right]$$

$$a_2 = -\left[\ (s+\mu_2+\xi)\{\beta(1+\varepsilon' N)(s+\mu_2+\xi) + \lambda_1\mu_2\} + \{s+\mu_2+\xi+\alpha(1-\varepsilon N)\}.\right.$$
$$\left.\{\lambda_1\mu_2 + (s+\mu_2+\xi)(s+\lambda_1+\mu_1+\xi)\} + 2\lambda_1\varepsilon\,\mu_2\,\alpha\ \right]$$

$$a_3 = \left[\ (s+\mu_2+\xi)\{2\beta\mu_2(1+\varepsilon' N) - \alpha\varepsilon\mu_1\} + \mu_2(s+\lambda_1+\mu_1+\xi)\right.$$
$$\{\alpha\varepsilon + (s+\mu_2+\xi)\} + \{s+\mu_2+\xi+\alpha(1-\varepsilon N)\}$$

$$\{\mu_1(s+\mu_2+\xi)+\mu_2(s+\lambda_1+\mu_1+\xi)\}+(\mu_2 z)^2\lambda_1 \Big]$$

$$a_4=-\Big[\mu_2^2\{\beta(1+\varepsilon' N)+(s+\lambda_1+\mu_1+\xi)\}+\mu_1\mu_2\{s+\mu_2+\xi+\alpha(1-\varepsilon N)+(s+\mu_2+\xi)\}\Big]$$

$$a_5=\mu_1\mu_2^2$$

$$a_6=-\alpha\varepsilon\lambda_1^2$$

$$a_7=\alpha\varepsilon(s+\lambda_1+\mu_1+\xi)-\beta\varepsilon'(s+\mu_2+\xi)$$

$$a_8=(\beta\varepsilon'\mu_2-\alpha\varepsilon\mu_1)$$

$$z_1=\Big[-z^3(s+\mu_2+\xi)[\alpha\varepsilon+\{s+\mu_2+\xi+\alpha(1-\varepsilon N)\}]$$
$$+z^2\mu_2[2(s+\mu_2+\xi)+\alpha(1-\varepsilon N)+\alpha\varepsilon]-z\mu_2^2$$

$$z_2=z(z-1)\Big[\mu_2\{2(s+\mu_2+\xi)+\alpha(1-\varepsilon N)\}-z(s+\mu_2+\xi)\cdot$$
$$\{s+\mu_2+\xi+\alpha(1-\varepsilon N)\}\Big]-\mu_2(z-1)\Big[\mu_2^2-\{\mu_2-z(s+\mu_2+\xi)\}^2\Big]$$
$$+\mu_2\varepsilon\alpha z^2\{\mu_2-z(s+\mu_2+\xi)\}$$

$$z_3=\Big[\mu_1\mu_2\{2(s+\mu_2+\xi)+\alpha(1-\varepsilon N)\}z(z-1)+\mu_1(s+\mu_2+\xi)z^2(1-z).$$
$$\{s+\mu_2+\xi+\alpha(1-\varepsilon N)\}+(1-z)\mu_1\mu_2^2+\mu_1 z^2\alpha\varepsilon\cdot\{\mu_2-(s+\mu_2+\xi)z\}\Big]$$

$$z_4=z^M\Big[z^3(z-1)\lambda_1(s+\mu_2+\xi)\{s+\mu_2+\xi+\alpha(1-\varepsilon N)\}$$
$$-z^2(z-1)\lambda_1\mu_2\{2(s+\mu_2+\xi)+\alpha(1-\varepsilon N)\}+$$
$$z(z-1)\lambda_1\mu_2^2-\alpha\varepsilon z^2\lambda_1\{z(M+2)-(M+1)\}\{\mu_2-z(s+\mu_2+\xi)\}\Big]$$

$$z_5=z^3\xi(s+\mu_2+\xi)[(s+\mu_2+\xi)-\{s+\mu_2+\xi+\alpha(1-\varepsilon N)\}]+$$
$$z^2\mu_2\xi[\alpha\varepsilon+\{s+\mu_2+\xi+\alpha(1-\varepsilon N)\}-(s+\mu_2+\xi)]$$

$$z_6=-z^3\xi(s+\mu_2+\xi)\{s+\mu_2+\xi+\alpha(1-\varepsilon N)\}+$$
$$z^2\mu_2\xi[\alpha\varepsilon+(s+\mu_2+\xi)+\{s+\mu_2+\xi+\alpha(1-\varepsilon N)\}]-z\mu_2{}^2\xi$$

On solving equation (21), we have

$$P(z,s)=\frac{L_1(z)+L_2(z)\overline{Q}_0(s)+L_3(z)\overline{P}_0(s)+L_4(z)\overline{P}_M(s)+L_5(z)\sum_{n=0}^{M}\overline{P}_n(s)+L_6(z)\sum_{n=0}^{M}\overline{Q}_n(s)}{L(z)} \tag{22}$$

where

$$L(z)=[\mu_2-z(s+\mu_2+\xi)]^{-A/(s+\mu_2+\xi)}\left[\frac{X(z)-a}{X(z)+a}\right]^{E'} z^B\cdot(a_6 z^2+a_7 z+a_8)^{D'}\exp^{(-C/2)}$$

$$D'=\frac{D}{2a_6}$$

$$E'=\left(E-\frac{a_7}{2a_6}D\right)\frac{1}{2a_6}$$

$$X(z)=z+\frac{a_7}{2a_6}$$

$$a=\left[\left(\frac{1}{2}\frac{a_7}{a_6}\right)^2-\left(\frac{a_8}{a_6}\right)\right]^{1/2}$$

$$L_j(z)=\int_0^Z \frac{z_j}{\eta_2(z)}L(z)\,dz\,; \qquad j=1,2,3,4,5,6.$$

Now, from equations (19) and (22), we have

$$Q(z,s)=\frac{L_7(z)+\overline{Q}_0(s)L_8(z)+\overline{P}_0(s)L_9(z)+\overline{P}_M(s)L_{10}(z)+\sum_{n=0}^{M}\overline{P}_n(s)L_{11}(z)+\sum_{n=0}^{M}\overline{Q}_n(s)L_{12}(z)}{B(z)L(z)} \tag{23}$$

where

$$L_7(z)=L_1(z)\,g(z)-zL(z)$$
$$L_8(z)=L_2(z)\,g(z)-\mu_2(z-1)\,L(z)$$
$$L_9(z)=L_3(z)\,g(z)-\mu_1(z-1)\,L(z)$$
$$L_{10}(z)=L_4(z)\,g(z)+\lambda_1 z^{M+1}(z-1)\,L(z)$$
$$L_{11}(z)=L_5(z)\,g(z)-\xi z\,L(z)$$
$$L_{12}(z)=L_6(z)\,g(z)-\xi z\,L(z)$$
$$g(z)=\left[z\,(s+\lambda_1+\mu_1+\xi)-\mu_1-\lambda_1 z^2\right]$$
$$B(z)=\left[\mu_2-z(s+\mu_2+\xi)\right]$$

Adding equations (22) and (23), we have

$$R(z,s)=\frac{C_1(z)+C_2(z)\overline{Q}_0(s)+C_3(z)\overline{P}_0(s)+C_4(z)\overline{P}_M(s)+C_5(z)\sum_{n=0}^{M}\overline{P}_n(s)+C_6(z)\sum_{n=0}^{M}\overline{Q}_n(s)}{B(z)\,L(z)} \tag{24}$$

where

$C_i(z) = B(z)\,L_i(z) + L_{i+6}(z)\,;\ i=1, 2, 3, 4, 5, 6.$

Since,

$$R(1,s)=\sum_{n=0}^{M}\overline{R}_n(s)=\frac{1}{s} \tag{25}$$

Thus equation (24) for z=1, gives

$$R(1,s)=\frac{1}{s}=\lim_{z\to 1}R(z,s) \tag{26}$$

$$P(0,s)=\overline{P}_0(s)=\lim_{z\to 0}P(z,s) \tag{27}$$

And $$Q(0,s)=\overline{Q}_0(s)=\lim_{z\to 0}Q(z,s) \tag{28}$$

The equations (26), (27) and (28) on solution gives the values of $\overline{P}_0(s)$, $\overline{Q}_0(s)$, $\overline{P}_M(s)$, $\sum_{n=0}^{M}\overline{P}_n(s)$ and $\sum_{n=0}^{M}\overline{Q}_n(s)$.

Again, we have from equations (22) and (23) on setting z=1 and $\overline{P}_0(s) = P_0$, $\overline{Q}_0(s) = Q_0$, $\overline{P}_M(s) = P_M$, $\sum_{n=0}^{M}\overline{P}_n(s) = P_n$ and $\sum_{n=0}^{M}\overline{Q}_n(s) = Q_n$

$$P(1,s)=\frac{L_1(1)+L_2(1)Q_0+L_3(1)P_0+L_4(1)P_M+L_5(1)\sum_{n=0}^{M}P_n+L_6(1)\sum_{n=0}^{M}Q_n}{L(1)} \tag{29}$$

$$Q(1,s)=\frac{L_7(1)+L_8(1)Q_0+L_9(1)P_0+L_{10}(1)P_M+L_{11}(1)\sum_{n=0}^{M}P_n+L_{12}(1)\sum_{n=0}^{M}Q_n}{B(1)\,L(1)} \tag{30}$$

These on inversions give the respective probabilities for the system to be in the environmental states E and F.

4. Particular Cases:

Case I Setting n=N or $\varepsilon=0$ in equations (22) and (23), (i.e., when the rate of change of environment from state F to E is constant), we have

$$P(z,s)=\frac{L_1'(z)+L_2'(z)\overline{Q}_0(s)+L_3'(z)\overline{P}_0(s)+L_4'(z)\overline{P}_M(s)+L_5'(z)\sum_{n=0}^{M}\overline{P}_n(s)+L_6'(z)\sum_{n=0}^{M}\overline{Q}_n(s)}{L'(z)}\,. \tag{31}$$

$$Q(z,s)=\frac{L_7'(z)+L_8'(z)\overline{Q}_0(s)+L_9'(z)\overline{P}_0(s)+L_{10}'(z)\overline{P}_M(s)+L_{11}'(z)\sum_{n=0}^{M}\overline{P}_n(s)+L_{12}'(z)\sum_{n=0}^{M}\overline{Q}_n(s)}{B'(z)\,L'(z)} \tag{32}$$

where

$$L_i'(z)=L_i(z)\big|_{\varepsilon=0} \quad ; \; i=1, 2, 3, \ldots . 12.$$

$$L'(z)=L(z)\big|_{\varepsilon=0}$$

$$B'(z)=B(z)\big|_{\varepsilon=0}$$

On adding equations (31) and (32), we have.

$$R(z,s)=\frac{C_1'(z)+C_2'(z)\overline{Q}_0(s)+C_3'(z)\overline{P}_0(s)+C_4'(z)\overline{P}_M(s)+C_5'(z)\sum_{n=0}^{M}\overline{P}_n(s)+C_6'(z)\sum_{n=0}^{M}\overline{Q}_n(s)}{B'(z)\,L'(z)} \tag{33}$$

where

$$C_i'(z)=B'(z)L_i'(z)+L_{i+6}'(z)\,; \qquad i=1, 2, 3, 4, 5, 6.$$

The unknown quantities $\overline{Q}_0(s)$, $\overline{P}_0(s)$, $\overline{P}_M(s)$, $\sum_{n=0}^{M}\overline{P}_n(s)$ and $\sum_{n=0}^{M}\overline{Q}_n(s)$ can be evaluated as before.

Case II: Setting $\varepsilon=\varepsilon'=0$ or n=N in equation (17) and (18), (i.e. when the rates of interchange of environmental states from E to F and F to E is constant), we have

$$X_1(z)P(z,s)+X_2(z)Q(z,s)+X_3(z)=0 \tag{34}$$

$$X_4(z)P(z,s)+X_5(z)Q(z,s)+X_6(z)=0 \tag{35}$$

where

$$X_1(z)=-\left[\lambda_1 z^2 - z\,(s+\lambda_1+\mu_1+\beta+\xi)+\mu_1\right]$$

$$X_2(z)=-\alpha z$$
$$X_3(z)=-\left[\mu_1(z-1)\,\overline{P}_0(s)+\lambda_1 z^{M+1}(1-z)\overline{P}_M(s)+z+\xi z\sum_{n=0}^{M}\overline{P}_n(s)\right]$$
$$X_4(z)=\beta z$$
$$X_5(z)=[\mu_2-z(s+\mu_2+\alpha+\xi)]$$
$$X_6(z)=\left[\mu_2(z-1)\overline{Q}_0(s)+\xi z\sum_{n=0}^{M}\overline{Q}_n(s)\right]$$

From equations (34) and (35), we have

$$P(z,s)=\frac{X_2(z)X_6(z)-X_3(z)X_5(z)}{X_1(z)X_5(z)-X_2(z)X_4(z)} \quad (36)$$

$$Q(z,s)=\frac{X_4(z)X_3(z)-X_1(z)X_6(z)}{X_1(z)X_5(z)-X_2(z)X_4(z)} \quad (37)$$

Thus, we have

$$R(z,s)=\frac{\begin{array}{l}\mu_2(z-1)[X_2(z)-X_1(z)]\overline{Q}_0(s)+\xi z\sum_{n=0}^{M}\overline{Q}_n(s)\,[X_2(z)-X_1(z)]+\mu_1(1-z)\\ [X_4(z)-X_5(z)]\overline{P}_0(s)+\lambda_1 z^{M+1}[X_5(z)-X_4(z)]\,(1-z)\overline{P}_M(s)+\\ z[X_5(z)-X_4(z)]+\xi z\sum_{n=0}^{M}\overline{P}_n(s)\,[X_5(z)-X_4(z)]\end{array}}{-z^2s^2+sX_7(z)+(1-z)X_8(z)-z^2\xi(\alpha+\beta+\xi)} \quad (38)$$

where

$$X_7(z)=\lambda_1 z^3-z^2(\lambda_1+\mu_1+\mu_2+\alpha+\beta+2\xi)+z(\mu_1+\mu_2)$$
$$X_8(z)=-z^2\lambda_1(\alpha+\mu_2+\xi)+z\,[\alpha\mu_1+\mu_2(\lambda_1+\mu_1+\beta+\xi)]-\mu_1\mu_2.$$

And $P(1,s)=\sum_{n=0}^{M}\overline{P}_n(s)=\dfrac{s+\alpha+\xi}{s(s+\alpha+\beta+\xi)}$

$$Q(1,s)=\sum_{n=0}^{M}\overline{Q}_n(s)=\frac{\beta}{s(s+\alpha+\beta+\xi)}$$

Relation (38) is a polynomial in z and exists for all values of z, including the three zeros of the denominator. Hence $\overline{P}_0(s)$, $\overline{Q}_0(s)$ and $\overline{P}_M(s)$ are obtained by setting the numerator equal to zero and substituting the three zeros, α_1, α_2 and α_3 (say) of the denominator (at each of which the numerator must vanish).

Now letting $\alpha\to\infty$, $\beta\to 0$ and setting $\mu_1=\mu_2=\mu$ (say) in relation (38), we have

$$r(z,s)=\frac{(1-z)\,\mu\overline{R}_0(s)-(1-z)\lambda_1 z^{M+1}\overline{P}_M(s)-z-\xi z/s}{\lambda_1 z^2-z(s+\lambda_1+\mu+\xi)+\mu} \quad (39)$$

where

$$\overline{R}_0(s)=\overline{P}_0(s)+\overline{Q}_0(s)$$
$$r(z,s)=\lim_{\beta\to 0}\left[\lim_{\alpha\to\infty}R(z,s)\right]$$

Relation (39) is a polynomial in z and exists for all values of z, including the two zeros of the denominator. Hence, $\overline{R}_0(s)$ and $\overline{P}_M(s)$ can be evaluated as before.

Case III Putting $\varepsilon=\varepsilon'=1$, N=1 in equation (24), (i.e. when $d_n=\beta n$ and $b_n=\alpha n$), we have.

$$R(z,s)=\frac{C_1''(z)+\overline{Q}_0(s)C_2''(z)+\overline{P}_0(s)C_3''(z)+\overline{P}_M(s)C_4''(z)+\sum_{n=0}^{M}\overline{P}_n(s)C_5''(z)+\sum_{n=0}^{M}\overline{Q}_n(s)C_6''(z)}{B(z)L''(z)} \tag{40}$$

where

$$L''(z)=L(z)\big|_{\varepsilon=\varepsilon'=1,N=1}$$

$$C_i''(z)=C_i(z)\big|_{\varepsilon=\varepsilon'=1,N=1};\quad i=1,2,3,4,5,6$$

5. Steady State Results:

This can at once be obtained by the well-known property of the Laplace transform given below:

$$\lim_{t\to\infty} f(t)=\lim_{s\to 0} s\,\bar{f}(s),\quad \text{If the limit on the left hand side exists.}$$

Thus if $\quad R(z)=\sum_{n=0}^{M} R_n z^n$

Then $\quad R(z)=\lim_{s\to 0} s\,R(z,s)$

By using this property, we have from equation (24) for the steady state

$$R(z)=\frac{N_1(z)Q_0+N_2(z)P_0+N_3(z)P_M+N_4(z)\sum_{n=0}^{M}P_n+N_5(z)\sum_{n=0}^{M}Q_n+N}{B^*(z)L^*(z)} \tag{41}$$

where

$$R_n=\lim_{s\to 0} s\,\overline{R}_n(s)$$

$$N_i(z)=B(z)L_{i+1}^*(z)+L_{i+7}^*(z)\big|_{s=0};\ i=1,2,3,4,5$$

$$B^*(z)=B(z)\big|_{s=0}$$

$$L^*(z)=L(z)\big|_{s=0}$$

$$L_i^*(z)=\int\frac{z_i}{\eta_2(z)}L(z)\,dz\ ;\ i=1,2,3,4,5,6$$

$$L_8^*(z)=L_2^*(z)\,g(z)-\mu_2(z-1)L(z)$$

$$L_9^*(z)=L_3^*(z)\,g(z)-\mu_1(z-1)L(z)$$

$$L_{10}^*(z)=L_4^*(z)\,g(z)+\lambda_1 z^{M+1}(z-1)L(z)$$

$$L_{11}^*(z)=L_5^*(z)\,g(z)-\xi z\,L(z)$$

$$L_{12}^*(z)=L_6^*(z)\,g(z)-\xi z\,L(z)$$

and N = The constant of integration.

The unknown quantities P_0, Q_0, P_M, $\sum_{n=0}^{M}P_n$ and $\sum_{n=0}^{M}Q_n$ can be evaluated as before.

Particular cases:

Case I Relation (33), on applying the theory of Laplace transform, we have

$$R(z)=\frac{Q_0 N_1'(z)+N_2'(z)P_0+N_3'(z)P_M+N_4'(z)\sum_{n=0}^{M}P_n+N_5'(z)\sum_{n=0}^{M}Q_n+N'}{T_1(z)\,T_2(z)} \tag{42}$$

where,

$$N_i'(z)=B'(z)L_{i+1}^{**}(z)+L_{i+7}^{**}(z)\Big|_{s=0} \quad ; \; i=1, 2, 3, 4, 5.$$

$$T_1(z)=B'(z)\big|_{s=0}$$

$$L_j^{**}(z)=\int\left[\frac{z_j}{\eta_2(z)}L(z)\right]_{\in=0} dz \quad ; \; j=2, 3, 4, 5, 6.$$

$$L_8^{**}(z)=L_2^{**}(z)\;g(z)-\mu_2(z-1)L'(z)$$

$$L_9^{**}(z)=L_3^{**}(z)\;g(z)-\mu_1(z-1)L'(z)$$

$$L_{10}^{**}(z)=L_4^{**}(z)\;g(z)+\lambda_1 z^{M+1}(z-1)L'(z)$$

$$L_{11}^{**}(z)=L_5^{**}(z)\;g(z)-\xi z L'(z)$$

$$L_{12}^{**}(z)=L_6^{**}(z)\;g(z)-\xi z L'(z)$$

N' = the constant of integration.

The unknown quantities Q_0, P_0, P_M, $\sum_{n=0}^{M}P_n$ and $\sum_{n=0}^{M}Q_n$ can be evaluated as before.

Case II Relation (38), on applying the theory of Laplace transforms gives

$$R(z)=\frac{\begin{array}{l}\mu_2(1-z)\{\alpha z+z(\lambda_1+\mu_1+\beta+\xi)-\lambda_1 z^2-\mu_1\}Q_0+\\ \mu_1(1-z)[\beta z-\{\mu_2-z(\mu_2+\alpha+\xi)\}]P_0+\lambda_1 z^{M+1}(1-z)\{\mu_2-z(\mu_2+\alpha+\xi)-\beta z\}P_M+\\ \xi z/(\alpha+\beta+\xi)\left[\beta\{\lambda_1 z^2-z(\lambda_1+\mu_1+\alpha+\beta+\xi)+\mu_1\}+(\alpha+\xi)\{\mu_2-z(\mu_2+\alpha+\beta+\xi)\}\right]\end{array}}{\begin{array}{l}z^3\lambda_1(\mu_2+\alpha+\xi)-z^2[\lambda_1(\mu_2+\alpha+\xi)+\{\alpha\mu_1+\mu_2(\lambda_1+\mu_1+\beta+\xi)\}+\xi(\alpha+\beta+\xi)]\\ +z[\{\alpha\mu_1+\mu_2(\lambda_1+\mu_1+\beta+\xi)\}+\mu_1\mu_2]-\mu_1\mu_2\end{array}} \tag{43}$$

or, we can write

$$R(z)=\frac{T(z)Q_0+N(z)P_0+L(z)P_M+M(z)}{K(z)} \tag{44}$$

Where T(z), N(z) and L(z) are the co-efficient of Q_0, P_0 and P_M respectively in the numerator of equation (43) and K(z) is the denominator of (43).

Equation (44) is a polynomial in z and exists for all values of z, including three zeros of the denominator. Hence Q_0, P_0 and P_M can be obtained by setting the numerator equal to zero. Substituting the three zeros b_1, b_2 and b_3 (say) of the denominator (at each of which the numerator must vanish).

Three equations determining the constants Q_0, P_0 and P_M are

$$T(b_1)Q_0 + N(b_1)P_0 + L(b_1)P_M = -M(b_1) \quad (45)$$

$$T(b_2)Q_0 + N(b_2)P_0 + L(b_2)P_M = -M(b_2) \quad (46)$$

$$T(b_3)Q_0 + N(b_3)P_0 + L(b_3)P_M = -M(b_3) \quad (47)$$

After solving these equations, we have

$$Q_0 = \frac{-M(b_1)A_{11} + M(b_2)A_{21} - M(b_3)A_{31}}{A}$$

$$P_0 = \frac{M(b_1)A_{12} - M(b_2)A_{22} + M(b_3)A_{32}}{A}$$

$$P_M = \frac{-M(b_1)A_{13} + M(b_2)A_{23} - M(b_3)A_{33}}{A}$$

where

$$A = \begin{vmatrix} T(b_1) & N(b_1) & L(b_1) \\ T(b_2) & N(b_2) & L(b_2) \\ T(b_3) & N(b_3) & L(b_3) \end{vmatrix}$$

A_{ij} is the co-factor of the $(i, j)^{th}$ element of A.

By putting the values of Q_0, P_0 and P_M in equation (44), we have

$$R(z) = \frac{\begin{array}{l} T(z)\left[-M(b_1)A_{11} + M(b_2)A_{21} - M(b_3)A_{31}\right] + N(z)\left[M(b_1)A_{12} - M(b_2)A_{22} + M(b_3)A_{32}\right] \\ + L(z)\left[-M(b_1)A_{13} + M(b_2)A_{23} - M(b_3)A_{33}\right] + A \cdot M(z) \end{array}}{A \cdot K(z)} \quad (48)$$

Case III Relation (40), on applying the theory of Laplace Transform gives

$$R(z) = \frac{H_1(z)Q_0 + H_2(z)P_0 + H_3(z)P_M + H_4(z)\sum_{n=0}^{M} P_n + H_5(z)\sum_{n=0}^{M} Q_n + H'}{B_1(z)\ H(z)} \quad (49)$$

where,

$$B_1(z) = B(z)\big|_{s=0}$$

$$H(z) = L''(z)\big|_{s=0}$$

$$H_i(z) = B(z)L''_{i+1}(z) + L''_{i+7}(z)\big|_{s=0} \qquad ; i=1, 2, 3, 4, 5.$$

$$L''_j(z) = \int \left[\frac{z_j}{\eta_2(z)} L(z)\right]_{\varepsilon=\varepsilon'=1,\, N=1} dz \qquad ; j= 2, 3, 4, 5, 6.$$

$$L''_k(z) = L_k(z)\big|_{\varepsilon=\varepsilon'=1,\, N=1} \qquad ; k= 8, 9, 10, 11, 12.$$

H' = the constant of Integration.

The unknown quantities of equation (49) can be evaluated as before.

6. Mean Queue Length:

Define,

L_q= Expected number of customers in the queue including the one in service.

Then

$$L_q = R'(z)\big|_{z=1}$$

Therefore, from equation (48), we have

$$L_q = \frac{\begin{array}{l} K(1)[T'(1)\{-M(b_1)A_{11} + M(b_2)A_{21} - M(b_3)A_{31}\} + N'(1)\{M(b_1)A_{12} - M(b_2)A_{22} + M(b_3)A_{32}\} \\ + L'(1)\{-M(b_1)A_{13} + M(b_2)A_{23} - M(b_3)A_{33}\} + A\,M'(1)] - [T(1)\{-M(b_1)A_{11} + M(b_2)A_{21} \\ - M(b_3)A_{31}\} + N(1)\{M(b_1)A_{12} - M(b_2)A_{22} + M(b_3)A_{32}\} + L(1)\{-M(b_1)A_{13} + M(b_2)A_{23} \\ - M(b_3)A_{33}\} + A \cdot M(1)]\, K'(1) \end{array}}{A \cdot [K(1)]^2} \tag{50}$$

where dashes denotes the first derivative w. r. t. z.

Relation (39), on applying the theory of Laplace transforms gives

$$r(z) = \frac{(1-z)\mu R_0 - (1-z)\lambda_1 z^{M+1} P_M - \xi z}{\lambda_1 z^2 - z(\lambda_1 + \mu + \xi) + \mu} \tag{51}$$

where

$$r(z) = \lim_{s\to 0} s\, r(z,s)$$

Equation (51) is a polynomial in z and exists for all values of z, including the two zeros of the denominator. Hence R_0 and P_M can be obtained by setting the numerator equal to zero. Substituting the two zeros a_1 and a_2 (say) of the denominator (at each of which the numerator must vanish).

If $\xi = 0$ (i.e., no catastrophe is allowed in the system), from equation (51), we have

$$r(z) = \frac{\mu R_0 - \lambda_1 z^{M+1} P_M}{\mu - \lambda_1 z} \tag{52}$$

The condition, $\lim\limits_{z\to 1} r(z) = 1$ gives

$$\mu R_0 - \lambda_1 P_M = \mu - \lambda_1 \tag{53}$$

As r(z) is analytic, the numerator and denominator of equation (52) must vanish simultaneously for $z=\mu/\lambda_1$, which is a zero of its denominator. Equating the numerator of equation (52) to zero for $z= \mu/\lambda_1$ we have

$$R_0 = \rho^{-M} P_M, \qquad \rho = \lambda_1/\mu < 1 \tag{54}$$

Relation (53) and (54) gives

$$R_0 = \frac{1-\rho}{1-\rho^{M+1}}, \quad P_M = \frac{(1-\rho)\rho^M}{1-\rho^{M+1}}$$

Now, from equation (52), we have

$$r(z) = \frac{1-\rho}{1-\rho^{M+1}} \cdot \left[\frac{1-(\rho z)^{M+1}}{1-\rho z}\right] \tag{55}$$

which is a well known result of the M/M/1 queue with finite waiting space M and The results agrees to one which are studied by Jain and Kanethia [10] and Goel, L.R. [7].

7. Conclusion and application of the model:

In this paper, we have established the limited capacity queueing system incorporating the effects of environmental changing states and catastrophes. We have obtained some particular cases and steady state solutions with (without) catastrophes. Some measures of effectiveness are also obtained. The direct applications of such queueing models are found in call centers, industries, banking and health sectors etc. Assuming a queueing system modeling a call center wherein environmental changing states could be the time of the day, with higher arrival rates during peak hours. Catastrophes might be a power outage or system failure. The model would analyze how these factors affect queue length, waiting time and the probability of a customer getting service.

References:

1. A. Di Crescenzo and A.G. Nobile, (1995) Diffusion approximation to a queueing system with time dependent arrival and service rates, Queueing Systems 19,41-62.
2. A. Di Crescenzo, V. Giorno, A.G. Nobile and L.M. Ricciardi, (2003) On the M/M/1 queue with catastrophes and its continuous approximation, Queueing Systems 43, 329-347.
3. B. Krishna Kumar and D. Arivudainambi, (2000) Transient solution of an M/M/1 queue with catastrophes, Comp. and Maths with Applications 40, 1233-1240.
4. D.V. Widder, (1946) The Laplace transforms, Princeton University Press.
5. E.G. Kyriakidis and A. Abakuks,(1989) Optimal pest control through catastrophes, Journal of Appl. Prob. 27, 873-879.
6. G. Arul Freeda Vinodhini and V. Vidhya, (2016) Computational Analysis of Queues with Catastrophes in a Multiphase Random Environment, Math. Problems in Engineering, Article ID 2917917, 7 pg.
7. Goel. L.R. (1979) Transient solution of a certain type of heterogeneous queues, Trabajos de Estadistica y de Investigacion Operativa, Vol. 30, 63-70.
8. Kumar, Darvinder (2023) A Queueing system with Catastrophe, State dependent Service and Environmental change, Bulletin of Mathematics and Statistics Research, Vol. 11(4), 9-20.
9. Kumar, Darvinder (2023) A Queueing System with Catastrophe, State dependent Input Parameter and Environmental change, International Journal of Applied Mathematics and Statistical Sciences, Vol. 12 (2), 21-36.
10. N.K. Jain and D.K. Kanethia, (2006) Transient Analysis of a Queue with Environmental and Catastrophic Effects, International Journal of Information and Management Sciences Vol. 17 No.1, 35-45.
11. P.J. Brockwell, (1985) The Extinction time of a birth, death and catastrophe process & related diffusion model, Adv. in Appl. Prob. 17, 42-52.
12. P.J. Brockwell, (1986) The Extinction time of a general birth and death process with catastrophes, Journal of Applications in Prob. 23, 851-858.
13. R.J. Swift, (2001) Transient probabilities for a simple birth-death-immigration process under the influence of total catastrophes, I .J. Math Math Sci. 25, 689-692.
14. X. Chao, (1995) A queueing network model with catastrophes and product form solution, O.R. letters 18,75-79.
15. Youxin Liu and Liwei Liu, (2023) An M/PH/1 Queue with Catastrophes, Research Square, DOI: 10.21203/rs.3.rs-2634820/v1.