

\$(n, m)\$- Power \$-(D, A)\$-Hyponormal Operators in Semi-Hilbertian Space

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Abstract

In this paper we introduce and analyze a new class of operators on semi-Hilbertian space $(\mathcal{H}, \|\cdot\|_A)$ called (n, m) power- (D, A) -hyponormal denoted $[(n, m)DH]_A$ associated with a Drazin invertible operator using its Drazin inverse. After establishing the basic properties of such operators and some examples are also given. We show some results related to this class on semi-Hilbertian space. In addition, we characterize the direct sum and the tensor product of these operators.

An operator $T \in \mathcal{B}_A(\mathcal{H})$ is said to be (n, m) power- (D, A) -normal for some positive operator A and for some positive integers n and m if

$$(T^\#)^m (T^D)^n - (T^D)^n (T^\#)^m \geq_A 0.$$

Keywords: Semi-Hilbertian space, A -positive, A -selfadjoint, A -hyponormal, Drazin inverse.

1 Introduction

The study of hyponormal and quasi-hyponormal on Hilbert space have received considerable attention in current literature ([10], [11], [15], [16], [18], [23]). The concept of Hyponormal was first introduced by P. Halmos [9], it generalizes the concept of normal operator, an operator T is Hyponormal if $T^*T - TT^* \geq 0$, it is proved that T is Hyponormal if and only if $\|Tx\| \leq \|T^*x\|$.

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This class has been generalized in some sense to the larger sets of so-called quasihyponormal, P -hyponormal, n power hyponormal, k -hyponormal, (n, m) power hyponormal operators on Hilbert spaces ([6], [11], [25]), recently these classes of operators have been generalized to semi Hilbert spaces by many authors such operators appear in different contexts in ([4], [8], [14]) and other papers.

The authors Chellali Cherifa and Benali Abdelkader introduced the class of (n, m) -power $-D$ -hyponormal operators on Hilbert spaces [2]. A bounded linear operator T on a complex Hilbert space is (n, m) -power $-D$ -hyponormal if $T^{*m}(T^D)^n \geq (T^D)^n T^{*m}$.

The purpose of this paper is to study the class of (n, m) power- (D, A) -hyponormal in semi-Hilbertian spaces, it is generalized of previous class.

The contents of the paper are the following. In section two we give notation and results about the concept of A -adjoint operators that will be useful in the sequel and we give the definition of the Drazin inverse of an operator. In Section 3 we introduce a new concept of hyponormality of operators in semi-Hilbertian space $(\mathcal{H}, \langle \cdot | \cdot \rangle_A)$ called (n, m) power- (D, A) -hyponormal operator and we investigate various structural properties of this class of operators with some examples studied. Moreover, the product, direct sum, tensor product and the sum of finite numbers of this type are discussed.

2 Preliminaries Results

We start by introducing some notations. Let $\mathcal{B}(\mathcal{H})$ be C^* -algebra of all bounded linear operators on a complex Hilbert space \mathcal{H} with inner product $\langle \cdot | \cdot \rangle$ and the corresponding norm $\| \cdot \|$, $I = I_{\mathcal{H}}$ being the identity operator. $\mathcal{B}(\mathcal{H})^+$ is the cone of positive operators of $\mathcal{B}(\mathcal{H})$ defined as

$$\mathcal{B}(\mathcal{H})^+ = \{T \in \mathcal{B}(\mathcal{H}) : \langle Tx|x \rangle \geq 0, \forall x \in \mathcal{H}\}.$$

For every $T \in \mathcal{B}(\mathcal{H})$, the symbols $\mathcal{N}(T)$, $\mathcal{R}(T)$ and $\overline{\mathcal{R}(T)}$ stand for respectively the null space, range and the closure of the range of T and its adjoint operator by T^* . if \mathcal{M} is closed linear subspace of \mathcal{H} satisfying $T\mathcal{M} \subset \mathcal{M}$ then, \mathcal{M} is called invariant subspace of T , In addition if \mathcal{M} also is invariant subspace of T^* , then \mathcal{M} is called a reducing subspace of T . We denote the orthogonal projection onto a closed linear subspace \mathcal{M} by $P_{\mathcal{M}}$.

For any operator $T \in \mathcal{B}(\mathcal{H})$, $|T| = (T^*T)^{\frac{1}{2}}$ and $[T^*, T] = T^*T - TT^* = |T|^2 - |T^*|^2$

Throughout this article, the operator A is considered to be non-zero and positive. Any $A \in \mathcal{B}(\mathcal{H})^+$ induces a positive semidefinite sesquilinear form $\langle \cdot | \cdot \rangle_A: \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$, defined by

$$\langle u|v \rangle_A = \langle Au|v \rangle.$$

By $\| \cdot \|_A$ we denote the semi-norm induced by $\langle \cdot | \cdot \rangle_A$ i.e $\|u\|_A = \langle u|u \rangle_A^{\frac{1}{2}} = \langle Au|u \rangle_A^{\frac{1}{2}}$.

Observe that $\|u\|_A = 0$ if and only if $u \in \mathcal{N}(A)$, then $\| \cdot \|_A$ is a norm if and only if A is an injective operator and the semi-normed space $(\mathcal{H}, \| \cdot \|_A)$ is complete if and only if $\mathcal{R}(A)$ is closed.

The above semi-norm induces a semi-norm on the subspace $\mathcal{B}^A(\mathcal{H})$ of $\mathcal{B}(\mathcal{H})$.

$$\mathcal{B}^A(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) / \exists c > 0, \|Tu\|_A \leq c\|u\|_A, \forall u \in \mathcal{H}\}.$$

Indeed, if $T \in \mathcal{B}^A(\mathcal{H})$, then $\|T\|_A = \sup \left\{ \frac{\|Tu\|_A}{\|u\|_A}, u \in \overline{\mathcal{R}(A)} \text{ and } u \neq 0 \right\}$.

Moreover $\|T\|_A = \sup \{ |\langle Tu|v \rangle|_A : u, v \in \mathcal{H} : \|u\| \leq 1, \|v\| \leq 1 \}$.

Definition 2.1. ([1]) For $T \in \mathcal{B}(\mathcal{H})$, For an operator $S \in \mathcal{B}(\mathcal{H})$ is called an A -adjoint operator of T if for every $u, v \in \mathcal{H}$, we have

$$\langle Tu|v \rangle_A = \langle u|Sv \rangle_A \text{ i.e. } AS = T^*A,$$

if T is an A -adjoint of itself, then T is called an A -selfadjoint operator ($AT = T^*A$).

It is possible that an operator T does not have an A -adjoint, and if S is an adjoint of T we may find many A -adjoint. In fact, if $AR = 0$ for some $R \in \mathcal{B}(\mathcal{H})$, then $S + R$ is an A -adjoint of T . The set of all A -bounded operators which admit an A -adjoint is denoted by $\mathcal{B}_A(\mathcal{H})$. By Douglas Theorem.

We have that

$$\mathcal{B}_A(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) / \mathcal{R}(T^*A) \subset \mathcal{R}(A)\}.$$

If $T \in \mathcal{B}_A(\mathcal{H})$, then there exists a distinguished A -adjoint operator of T , namely the reduced solution of equation $AX = T^*A$, This operator is denoted by $T^\#$.

Therefore, $T^\# = A^\dagger T^*A$ and $AT^\# = T^*A$, $\mathcal{R}(T^\#) \subset \overline{\mathcal{R}(A)}$ and $\mathcal{N}(T^\#) = \mathcal{N}(T^*A)$.

Note that in which A^\dagger is the Moore-Penrose inverse of A . For more details see ([1], [3]).

In the next proposition we add some important properties of $T^\#$ needed in the sequel (for the proof; we refer the reader to ([1] and [3]).

Proposition 2.1 Let $T \in \mathcal{B}_A(\mathcal{H})$. Then the following assertions hold

1. $T^\# \in \mathcal{B}_A(\mathcal{H})$, $(T^\#)^\# = P_{\overline{\mathcal{R}(A)}} T P_{\overline{\mathcal{R}(A)}}$ and $((T^\#)^\#)^\# = T^\#$.
2. If $S \in \mathcal{B}_A(\mathcal{H})$ then $TS \in \mathcal{B}_A(\mathcal{H})$ and $(TS)^\# = S^\# T^\#$.
3. $TT^\#$ and $T^\#T$ are A -selfadjoint.
4. If $AT = TA$, then $T^\# = P_{\overline{\mathcal{R}(A)}} T^*$.
5. $\|T\|_A = \|T^\#\|_A = \|T^\#T\|_A^{\frac{1}{2}} = \|TT^\#\|_A^{\frac{1}{2}}$.
6. $\|S\|_A = \|T^\#\|_A$ for every $S \in \mathcal{B}_A(\mathcal{H})$ which is an A -adjoint of T .
7. If $S \in \mathcal{B}_A(\mathcal{H})$ then $\|TS\|_A = \|ST\|_A$.
8. $I^\# = P_{\overline{\mathcal{R}(A)}}$ and $(T^n)^\# = (T^\#)^n$ for all $n \in \mathbb{N}^*$.

We recall that [12] the Drazin inverse of the operator $T \in \mathcal{B}_A(\mathcal{H})$ is the unique $T^D \in \mathcal{B}(\mathcal{H})$, provide it exists, satisfying the following conditions

$$[T^D, T] = T^D T - T T^D = 0, (T^D)^2 T = T^D \text{ and } T^{p+1} T^D = T^p \text{ for some integer } p \geq 0.$$

The smallest natural number p satisfying the previous system of equations is known as the index of the operator T and is denoted by $ind(T)$. We note that if T is nilpotent, then it is Drazin invertible $T^D = 0$ and $ind(T) = n$ where n is the power of nilpotency of T .

For $T \in \mathcal{B}(\mathcal{H})$ it was observed that the Drazin inverse T^D of T satisfies $(T^*)^D = (T^D)^*$ and $(T^k)^D = (T^D)^k$ for positive integer k . The Drazin invertibility of an operator in $\mathcal{B}(\mathcal{H})$ is similarly invariant, i.e. if T is Drazin invertible and $S \in \mathcal{B}(\mathcal{H})$ is an invertible operator then, $S^{-1}TS$ is Drazin invertible and

$$(S^{-1}TS)^D = S^{-1}T^D S.$$

We denote by $\mathcal{B}(\mathcal{H})^D$ the set of all Drazin invertible elements of $\mathcal{B}(\mathcal{H})$.

Lemma 2.2. ([17], [21]). Let $T, S \in \mathcal{B}(\mathcal{H})^D$. Then the following properties hold.

1. TS is Drazin invertible if and only if ST is Drazin invertible. Moreover

$$(TS)^D = T[(TS)^D]^2 S \quad \text{and} \quad ind(TS) \leq ind(ST) + 1.$$

2. If T is idempotent, then $T^D = T^\# = T$.

3. If $TS = ST$, then $(TS)^D = S^D T^D = T^D S^D$, $T^D S = S T^D$ and $TS^D = S^D T$.

4. If $TS = ST = 0$, then $(T + S)^D = T^D + S^D$.

The classes of normal, quasinormal, isometries, hyponormal, quasihyponormal and m -isometries on Hilbert spaces have been generalized to semi-Hilbert spaces by many authors in ([1], [3], [5], [19], [24]) and other papers.

Now we give the definition of some classes of operators in semi Hilbertian space.

Definition 3.1. Any operator $T \in \mathcal{B}_A(\mathcal{H})$, is called

1. A -normal if $TT^\# = T^\#T$.
2. (D, A) -normal if $T^D T^\# = T^\# T^D$.
3. A -isometry if $T^\# T = P_{\overline{\mathcal{R}(A)}}$.
4. A -Unitary if $T^\# T = T T^\# = P_{\overline{\mathcal{R}(A)}}$.
5. A -hyponormal if $T^\# T \geq_A T T^\#$.
6. A -quasi-hyponormal if $T^\#(T^\# T - T T^\#)T \geq_A$.
7. (A, n) -hyponormal if $T^\# T^n \geq_A T^n T^\#$.

3 (n, m) -Power- (D, A) -Hyponormal Operators in semi-Hilbertian space

In this section, we introduce the concept of (n, m) power- (D, A) -hyponormal operators in semi-Hilbertian Space. The following definition and results are useful for our study.

Definition 3.1. We say that $T \in \mathcal{B}(\mathcal{H})$ is A -positive if $AT \in \mathcal{B}(\mathcal{H})^+$ or equivalently

$$\langle Tu|u \rangle_A \geq 0, \forall u \in \mathcal{H}.$$

We note $T \geq_A 0$.

Remark 3.1. We can define an order relation by $T \geq_A S \Leftrightarrow T - S \geq_A 0$.

Lemma 3.1. ([3]) Let $T, S \in \mathcal{B}(\mathcal{H})$ such that $T \geq_A S$ and $R \in \mathcal{B}(\mathcal{H})$, then the following properties hold.

1. $R^\#TR \geq_A R^\#SR$.
2. $RTR^\# \geq_A RSR^\#$.
3. If R is A -selfadjoint, then $RTR \geq_A RSR$.

Remark 3.2. From lemma (3.1) we deduce the following results

1. $R^{\#n}TR^n \geq_A R^{\#n}SR^n$ and $R^nTR^{\#n} \geq_A R^nSR^{\#n}$ for all positive integer n .
2. If $m \geq n$ and $TR^{\#k} \geq_A SR^{\#k}$, then $R^nTR^{\#m} \geq_A R^nSR^{\#m}$.
3. If $m \geq n$ and $R^{\#k}T \geq_A R^{\#k}$, then $R^{\#m}TR^n \geq_A R^{\#m}SR^n$.

Note that every positive operator has a square root as it is shown in the following theorem.

Theorem 3.1. ([13] theorem 2.4.4) Let $A \in \mathcal{B}(\mathcal{H})^+$. Then there exists a unique $B \in \mathcal{B}(\mathcal{H})^+$ such that $B^2 = A$. Moreover, B is a norm limit of polynomials in A . So, in particular

$$[C, A] = 0 \Rightarrow [B, A] = 0 \text{ for all } C \in \mathcal{B}(\mathcal{H})^+ . \quad (3.1)$$

We will write $B = A^{\frac{1}{2}}$, B is called the square root of A .

Now, we introduce the following useful lemma.

Lemma 3.3. Let $T, S \in \mathcal{B}(\mathcal{H})$ are A -positive operators, if T commutes with S then TS is A -positive.

Proof. Assume that $T, S \in \mathcal{B}(\mathcal{H})$ are an A -positive and $TS = ST$. Since T is A -positive then exists only one operator R is A -positive such that $R^{\frac{1}{2}} = T$ and commutes with S (Theorem (3.2)). Then for all $u \in \mathcal{H}$, we have

$$\langle TSu|u \rangle_A = \langle R^2Su|u \rangle_A$$

$$\begin{aligned}
 &= \langle RSu|Ru \rangle_A \\
 &= \langle S(Ru)|Ru \rangle_A \geq_A 0.
 \end{aligned}$$

Definition 2.1. Let $T \in \mathcal{B}_A(\mathcal{H})$ be Draizin invertible operator. We said that T is (n, m) power- (D, A) -hyponormal operator for some positive integers n, m if

$$(T^\#)^m (T^D)^n - (T^D)^n (T^\#)^m \geq_A 0,$$

or equivalently

$$\langle ((T^\#)^m (T^D)^n - (T^D)^n (T^\#)^m)u|u \rangle \geq_A 0, \quad \text{for all } u \in \mathcal{H}.$$

We denote the set of all (n, m) power- (D, A) -hyponormal operators by $[(n, m)DH]_A$.

Remark 2.1. We make the following observations:

1. if $n = m = 1$ then $(1,1)$ power- (D, A) -hyponormal is precisely (D, A) -hyponormal.
2. From [7] every (n, m) power- (D, A) -normal is (n, m) power- (D, A) -hyponormal i.e. $[(n, m)DN]_A \subset [(n, m)DH]_A$.

In the following examples we give some operators which are (n, m) power- (D, A) -hyponormal for

Example 3.1. Let

$$T = \begin{pmatrix} -3 & -2 \\ 0 & 3 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix},$$

be operators acting on two-dimensional Hilbert space space \mathbb{C}^2 . A simple calculation shows that

$$A \geq 0, \quad T^D = \begin{pmatrix} \frac{-1}{3} & \frac{-2}{9} \\ 0 & \frac{1}{3} \end{pmatrix} = \frac{1}{9} \begin{pmatrix} -3 & -2 \\ 0 & 3 \end{pmatrix} \quad \text{and} \quad T^\# = \begin{pmatrix} \frac{-3}{3} & 0 \\ \frac{-2}{3} & 3 \end{pmatrix}.$$

It is easy to check that $(T^\#)^3 (T^D)^2 \geq_A (T^D)^2 (T^\#)^3$ then T is of class $[(2,3)DH]_A$.

Example 3.2. Let

$$S = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \in \mathcal{B}(\mathbb{R}^3).$$

A simple computation shows that

$$S^D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad S^\# = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{2} & \frac{-1}{2} & 0 \end{pmatrix}.$$

It is easily to see that T is (n, m) power- (D, A) -hyponormal operator for all positive integers n and m .

In the following theorem we give the characterization of (n, m) power $-(D, A)$ -hyponormal.

Proposition 3.4. Let $T \in \mathcal{B}_A(\mathcal{H})^D$, $X = (T^D)^n + T^{\#m}$, $Y = (T^D)^n - T^{\#m}$ and $Z = (T^D)^n \cdot T^{\#m}$.

Then the following statements hold.

1. T is (n, m) power- (D, A) -hyponormal operator if and only if $[X, Y] \geq_A 0$.
2. If T is of class $[(n, m)DH]$, such that $(T^D)^n \geq_A 0$ and $(T^D)^n$ commutes with $((T^{\#})^m(T^D)^n - (T^D)^n(T^{\#})^m)$, then $[Z, X + Y]$ is A -positive.
3. If T is of class $[(n, m)DH]$, such that $(T^D)^n \geq_A 0$, $T^{\#m} \geq_A 0$ and $((T^{\#})^m(T^D)^n - (T^D)^n(T^{\#})^m)$ commutes with booth $(T^D)^n$ and $T^{\#m}$, then $[Z, Y]$ is A -positive.
4. T is of class $[(n, m)DH]_A$ if and only if $[X, (T^D)^n] \geq_A 0$ and $[(T^D)^n, Y] \geq_A 0$.
5. T is of class $[(n, m)DH]_A$ if and only if $[T^{\#m}, X] \geq_A 0$ and $[Y, T^{\#m}] \geq_A 0$.

Proof. 1

$$\begin{aligned}
 [X, Y] = XY - YX \geq_A 0 &\Leftrightarrow ((T^D)^n + T^{\#m})((T^D)^n - T^{\#m}) - ((T^D)^n - T^{\#m})((T^D)^n + T^{\#m}) \geq_A 0 \\
 &\Leftrightarrow (T^D)^{2n} - (T^D)^n T^{\#m} + T^{\#m} (T^D)^n - T^{\#2m} - ((T^D)^{2n} \\
 &\quad + (T^D)^n T^{\#m} - T^{\#m} (T^D)^n - T^{\#2m} \geq_A 0 \\
 &\Leftrightarrow -2 \cdot (T^D)^n T^{\#m} + 2 \cdot T^{\#m} (T^D)^n \geq_A 0 \\
 &\Leftrightarrow T^{\#m} (T^D)^n - (T^D)^n T^{\#m} \geq_A 0 \\
 &\Leftrightarrow T \in [(n, m)DH]_A.
 \end{aligned}$$

2. Assume that $T \in [(n, m)DH]_A$, since $(T^D)^n \geq_A 0$ and $(T^D)^n$ commutes with

$((T^{\#})^m(T^D)^n - (T^D)^n(T^{\#})^m)$ by lemma (3,3) we have

$(T^D)^n((T^{\#})^m(T^D)^n - (T^D)^n(T^{\#})^m) \geq_A 0$ i.e. $(T^D)^n T^{\#m} (T^D)^n \geq_A (T^D)^{2n} T^{\#m}$. Then

$$\begin{aligned}
 [Z, X + Y] &= Z(X + Y) - (X + Y)Z \\
 &= (T^D)^n \cdot T^{\#m} \cdot 2(T^D)^n - 2(T^D)^n \cdot (T^D)^n \cdot T^{\#m} \\
 &= 2(T^D)^n \cdot T^{\#m} (T^D)^n - 2(T^D)^{2n} \cdot T^{\#m} \\
 &\geq_A (T^D)^{2n} \cdot T^{\#m} - (T^D)^{2n} \cdot T^{\#m} = 0.
 \end{aligned}$$

3. Under the assumption we have $(T^D)^n(T^{\#m} \cdot (T^D)^n - T^{\#m} \cdot (T^D)^n) \geq_A 0$ and

$(T^{\#m} \cdot (T^D)^n - T^{\#m} \cdot (T^D)^n)T^{\#m} \geq_A 0$, then

$$\begin{aligned} [Z, Y] &= ZY - YZ \\ &= (T^D)^n \cdot (T^{\#})^m ((T^D)^n - (T^{\#})^m) - ((T^D)^n - (T^{\#})^m) (T^D)^n \cdot (T^{\#})^m \\ &= (T^D)^n \cdot (T^{\#})^m \cdot (T^D)^n - (T^D)^n \cdot (T^{\#})^{2m} - (T^D)^{2n} \cdot (T^{\#})^m + (T^D)^n \cdot (T^{\#})^m \cdot (T^D)^n \\ &\geq_A (T^D)^{2n} \cdot (T^{\#})^m - (T^D)^n \cdot (T^{\#})^{2m} - (T^D)^{2n} \cdot (T^{\#})^m + (T^D)^n \cdot (T^{\#})^m \cdot (T^D)^n \\ &= (T^D)^n \cdot (T^{\#})^m \cdot (T^D)^n - (T^D)^n \cdot (T^{\#})^{2m} \\ &\geq_A (T^D)^n \cdot (T^{\#})^{2m} - (T^D)^n \cdot (T^{\#})^{2m} = 0. \end{aligned}$$

Hence $[Z, Y]$ is A -positive.

In the same we proved the Statements 4 and 5.

The following proposition discusses the relation between (n, m) power $-(D, A)$ -hyponormal operator and co- (n, m) power $-(D, A)$ -hyponormal operator.

Proposition 3.5. Let $T \in \mathcal{B}_A(\mathcal{H})^D$ such that $\mathcal{N}(A)$ is invariant subspace of T , if T is (n, m) power $-(D, A)$ -hyponormal operator, then $T^{\#}$ is co- (n, m) power $-(D, A)$ -hyponormal operator

Proof. Assume that T is (n, m) power $-(D, A)$ -hyponormal operator, it follows that

$$\begin{aligned} (T^{\#})^m (T^D)^n \geq_A (T^D)^n (T^{\#})^m &\Rightarrow ((T^{\#})^m (T^D)^n)^{\#} \geq_A ((T^D)^n (T^{\#})^m)^{\#} \\ &\Rightarrow ((T^D)^n)^{\#} ((T^{\#})^m)^{\#} \geq_A ((T^{\#})^m)^{\#} ((T^D)^n)^{\#} \\ &\Rightarrow ((T^{\#})^D)^n ((T^{\#})^{\#})^m \geq_A ((T^{\#})^{\#})^m ((T^{\#})^D)^n. \end{aligned}$$

Then $T^{\#}$ is co- (n, m) power $-(D, A)$ -hyponormal operator.

The following proposition shows the stability of the class of (n, m) power $-(D, A)$ -hyponormal operators under the A -unitary equivalent.

Proposition 3.6. Let $T \in \mathcal{B}_A(\mathcal{H})^D$ be an (n, m) power- (D, A) -hyponormal operator such that

$\mathcal{N}(A)$ is a reducing subspace of T . If $S = UTU^{\#}$ where U is A -unitary operator, then S is (n, m) power- (D, A) -hyponormal operator.

Proof. Let T be an (n, m) power- (D, A) -hyponormal operator, since S is A -unitary equivalent of T , then there exists A -unitary operator U such that $S = UTU^{\#}$, it is easily to check that $S^n = UT^nU^{\#}$, $S^D = UT^DU^{\#}$ and $S^{\#} = UT^{\#}U^{\#}$. We have

$$(S^D)^n (S^{\#})^m = (UT^DU^{\#})^n (UT^{\#}U^{\#})^m$$

$$\begin{aligned}
 &= U(T^D)^n U^\# \cdot UT^{\#m} U^\# \\
 &= U(T^D)^n P_{\overline{\mathcal{R}(A)}} T^{\#m} U^\# \\
 &= U(T^D)^n T^{\#m} U^\# \\
 &\leq_A UT^{\#m} (T^D)^n U^\# \quad (\text{Lemma ((3,1))}) \\
 &= (UT^{\#m} U^\#) (U(T^D)^n U^\#) \\
 &= (S^\#)^m (S^D)^n
 \end{aligned}$$

Hence $(S^D)^n (S^\#)^m \leq_A (S^\#)^m (S^D)^n$ then $S \in [(n, m)DH]_A$.

The following discusses the conditions for product and sum of two (n, m) power- (D, A) -hyponormal operators to be (n, m) power- (D, A) -hyponormal.

Theorem 2.3. Let $T, S \in \mathcal{B}_A(\mathcal{H})^D$ are commuting (n, m) power- (D, A) -normal operators such that $T^D S^\# = S^\# T^D$ and $S^D T^\# = T^\# S^D$, then the following statements hold.

1. TS is (n, m) power- (D, A) -hyponormal.
2. If $TS = 0$ then $T + S$ is (n, m) power- (D, A) -hyponormal.

Proof. Assume that $T, S \in [(n, m)DH]_A$ are commuting, $T^D S^\# = S^\# T^D$ and $S^D T^\# = T^\# S^D$, it

Follows that $(T^D)^n S^{\#m} = (T^D)^n S^{\#m}$ and $(S^D)^n T^{\#m} = T^{\#m} (S^D)^n$. So

1.
$$\begin{aligned}
 ((TS)^D)^n (TS)^{\#m} &= (T^D S^D)^n (T^\# S^\#)^m \\
 &= (T^D)^n (S^D)^n (S^\#)^m (T^\#)^m \\
 &\leq_A (T^D)^n (S^\#)^m (S^D)^n (T^\#)^m \\
 &= (S^\#)^m (T^D)^n (S^D)^n (T^\#)^m \\
 &= (S^\#)^m (T^D)^n (T^\#)^m (S^D)^n \\
 &\leq_A (S^\#)^m (T^\#)^m (T^D)^n (S^D)^n \\
 &= (TS)^{\#m} ((TS)^D)^n.
 \end{aligned}$$

2. Under the assumptions and from lemma (2,2) we get

$$\begin{aligned}
 ((T + S)^D)^n ((T + S)^\#)^m &= (T^D + S^D)^n (T^\# + S^\#)^m \\
 &= ((T^D)^n + (S^D)^n) (T^{\#m} + S^{\#m})
 \end{aligned}$$

$$\begin{aligned}
 &= (T^D)^n T^{\#m} + (T^D)^n S^{\#m} + (S^D)^n T^{\#m} + (S^D)^n S^{\#m} \\
 &\leq_A T^{\#m} (T^D)^n + S^{\#m} (T^D)^n + T^{\#m} (S^D)^n + S^{\#m} (S^D)^n \\
 &= (T^{\#m} + S^{\#m}) ((T^D)^n + (S^D)^n) \\
 &= ((T + S)^{\#})^m ((T + S)^D)^n.
 \end{aligned}$$

Hence $(T + S)$ is an (n, m) power- (D, A) -hyponormal.

Proposition 3.8. Let $T, S \in \mathcal{B}_A(\mathcal{H})^D$ are commuting $(n, 1)$ power- (D, A) -hyponormal such that $T^D S^{\#} = S^{\#} T^D, T^{\#} S^D = S^D T^{\#}$ and $(T + S)^{\#}$ commutes with $\sum_{1 \leq p \leq n-1} \binom{n}{p} (T^D)^p (S^D)^{n-p}$,

then $(T + S)$ is an $(n, 1)$ power- (D, A) -hyponormal.

Proof.

Assume the conditions hold, then

$$\begin{aligned}
 ((T + S)^D)^n (T + S)^{\#} &= (T^D + S^D)^n (T + S)^{\#} \\
 &= \left(\sum_{0 \leq p \leq n} \binom{n}{p} (T^D)^p (S^D)^{n-p} \right) (T + S)^{\#} \\
 &= \left((S^D)^n + (T^D)^n + \sum_{1 \leq p \leq n-1} \binom{n}{p} (T^D)^p (S^D)^{n-p} \right) (T + S)^{\#} \\
 &= ((S^D)^n + (T^D)^n) (T + S)^{\#} + \left(\sum_{1 \leq p \leq n-1} \binom{n}{p} (T^D)^p (S^D)^{n-p} \right) (T + S)^{\#} \\
 &= (S^D)^n T^{\#} + (S^D)^n S^{\#} + (T^D)^n T^{\#} + (T^D)^n S^{\#} \\
 &\quad + (T + S)^{\#} \sum_{1 \leq p \leq n-1} \binom{n}{p} (T^D)^p (S^D)^{n-p} \\
 &\leq_A T^{\#} (S^D)^n + S^{\#} (S^D)^n + T^{\#} (T^D)^n + S^{\#} (T^D)^n \\
 &\quad + (T + S)^{\#} \sum_{1 \leq p \leq n-1} \binom{n}{p} (T^D)^p (S^D)^{n-p} \\
 &= (T + S)^{\#} ((S^D)^n + (T^D)^n) + (T + S)^{\#} \sum_{1 \leq p \leq n-1} \binom{n}{p} (T^D)^p (S^D)^{n-p} \\
 &= (T + S)^{\#} \left(\sum_{0 \leq p \leq n} \binom{n}{p} (T^D)^p (S^D)^{n-p} \right) \\
 &= (T + S)^{\#} ((T + S)^D)^n.
 \end{aligned}$$

Therefore $(T + S)$ is an $(n, 1)$ power- (D, A) -hyponormal operator.

Proposition 3.9. Let $T \in \mathcal{B}_A(\mathcal{H})^D$ such that $T \in [(n, m)DN]_A \cap [(n + 1, m)DH]_A$. If T^D is A -positive and commutes with $((T^\#)^m(T^D)^{n+1} - (T^D)^{n+1}(T^\#)^m)$, then $T \in [(n + 2, m)DH]_A$ for some positive integers n and m .

Proof. Assume $T \in [(n, m)DN]_A \cap [(n + 1, m)DH]_A$. Since $T^D \geq_A 0$ and commutes with $((T^\#)^m(T^D)^{n+1} - (T^D)^{n+1}(T^\#)^m)$ by lemma (3.3) we deduce that $(T^\#)^m(T^D)^{n+1}T^D \geq_A (T^D)^{n+1}(T^\#)^mT^D$ and $T^D(T^\#)^m(T^D)^{n+1} \geq_A T^D(T^D)^{n+1}(T^\#)^m$. So

$$\begin{aligned} (T^\#)^m(T^D)^{n+2} &= (T^\#)^m(T^D)^{n+1}T^D \\ &\geq_A (T^D)^{n+1}(T^\#)^mT^D \\ &= T^D(T^D)^n(T^\#)^mT^D \\ &= T^D(T^\#)^m(T^D)^nT^D \\ &= T^D(T^\#)^m(T^D)^{n+1} \\ &\geq_A T^D(T^D)^{n+1}(T^\#)^m \\ &= (T^D)^{n+2}(T^\#)^m. \end{aligned}$$

Hence $T \in [(n + 2, m)DH]_A$.

Proposition 3.10. Let $T \in \mathcal{B}_A(\mathcal{H})^D$ such that $T \in [(n, m)DN]_A \cap [(n, m + 1)DH]_A$. If $T^\# \geq_A 0$ and commutes with $((T^\#)^{m+1}(T^D)^n - (T^D)^n(T^\#)^{m+1})$, then $T \in [(n, m + 2)DH]_A$ for some positive integers n and m .

Proof. Assume that $T \in [(n, m)DN]_A \cap [(n, m + 1)DH]_A$. Since $T^\# \geq_A 0$ and commutes with $((T^\#)^{m+1}(T^D)^n - (T^D)^n(T^\#)^{m+1})$ from lemma (3.3) we get

$$((T^\#)^{m+1}(T^D)^n - (T^D)^n(T^\#)^{m+1})T^\# \geq_A 0 \text{ and } T^\#((T^\#)^{m+1}(T^D)^n - (T^D)^n(T^\#)^{m+1}) \geq_A 0.$$

Or equivalently

$$(T^\#)^{m+1}(T^D)^nT^\# \geq_A (T^D)^n(T^\#)^{m+1}T^\# \text{ and } T^\#(T^\#)^{m+1}(T^D)^n \geq_A T^\#(T^D)^n(T^\#)^{m+1}. \text{ Now}$$

$$\begin{aligned} (T^\#)^{m+2}(T^D)^n &= T^\#(T^\#)^{m+1}(T^D)^n \\ &\geq_A T^\#(T^D)^n(T^\#)^{m+1} \\ &= T^\#(T^D)^n(T^\#)^mT^\# \\ &= T^\#(T^\#)^m(T^D)^nT^\# \\ &= (T^\#)^{m+1}(T^D)^nT^\# \end{aligned}$$

$$\begin{aligned} &\geq_A (T^D)^n (T^\#)^{m+1} T^\# \\ &= (T^D)^n (T^\#)^{m+2}. \end{aligned}$$

We deduce that $T \in [(n, m + 2)DH]_A$.

Theorem 3.11. Let $T \in \mathcal{B}_A(\mathcal{H})^D$ be an (n, m) power- (D, A) -hyponormal operator for some positive integers n and m . The following statements hold

1. If $n \geq m$ and $(T^D)^m (T^\#)^m (T^D)^m = (T^D)^m$ such that $(T^D)^m \geq_A 0$ and commutes with $((T^\#)^m (T^D)^n - (T^D)^n (T^\#)^m)$ then $T \in [(n + m, m)DH]_A$.
2. If $m \geq n$ and $(T^\#)^n (T^D)^n (T^\#)^n = (T^\#)^n$ such that $(T^\#)^n \geq_A 0$ and commutes with $((T^\#)^m (T^D)^n - (T^D)^n (T^\#)^m)$ then $T \in [(n, n + m)DH]_A$.

Proof. We have $(T^D)^m (T^\#)^m (T^D)^m = (T^D)^m$. It follows that

$(T^D)^n (T^\#)^m (T^D)^m = (T^D)^n$ and $(T^D)^m (T^\#)^m (T^D)^n = (T^D)^n$ for $n \geq m$, which means that

$$(T^D)^n (T^\#)^m (T^D)^m = (T^D)^m (T^\#)^m (T^D)^n, \tag{3,2}$$

Since $T \in [(n, m)DH]_A$ and $(T^D)^m$ is A -positive commutes with $((T^\#)^m (T^D)^n - (T^D)^n (T^\#)^m)$. We get

$$(T^D)^n (T^\#)^m (T^D)^m \leq_A (T^\#)^m (T^D)^{n+m}, \tag{3,3}$$

and

$$(T^D)^{n+m} (T^\#)^m \leq_A (T^D)^m (T^\#)^m (T^D)^n. \tag{3,4}$$

By combining (3,2), (3,3) and (3,4) we get

$$(T^D)^{n+m} (T^\#)^m \leq_A (T^\#)^m (T^D)^{n+m}.$$

Then $T \in [(n + m, m)DH]_A$.

2. Since $(T^\#)^n (T^D)^n (T^\#)^n = (T^\#)^n$. It follows that

$(T^\#)^m (T^D)^n (T^\#)^n = (T^\#)^m$ and $(T^\#)^n (T^D)^n (T^\#)^m = (T^\#)^m$ for $m \geq n$, which means that

$$(T^\#)^m (T^D)^n (T^\#)^n = (T^\#)^n (T^D)^n (T^\#)^m. \tag{3,5}$$

Since $T \in [(n, m)DH]_A$ and $(T^\#)^n$ A -positive and commutes with $((T^\#)^m (T^D)^n - (T^D)^n (T^\#)^m)$,

we conclude that $(T^\#)^{n+m} (T^D)^n \geq_A (T^\#)^n (T^D)^n (T^\#)^m$, (3,6)

and

$$(T^\#)^m (T^D)^n (T^\#)^n \geq_A (T^D)^n (T^\#)^{n+m}. \tag{3,7}$$

By combining (3,5), (3,6) and (3,7) we get

$$(T^\#)^{n+m} (T^D)^n \geq_A (T^D)^n (T^\#)^{n+m}.$$

Then $T \in [(n, n + m)DH]_A$.

3.1 Tensor Product and Direct sum of (n, m) Power-(D, A)-Hyponormal Operators in Semi-Hilbertian Spaces

Let $T, S \in \mathcal{B}(\mathcal{H})$ and $T \otimes S \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ denote the tensor product defined on the Hilbert space $\mathcal{H} \otimes \mathcal{H}$ as follows

$$\langle T \otimes S(\xi_1 \otimes \eta_1) | \xi_2 \otimes \eta_2 \rangle = \langle T \xi_1 | \xi_2 \rangle \langle S \eta_1 | \eta_2 \rangle.$$

The operation of taking tensor products $T \otimes S$ preserves many properties of $T, S \in \mathcal{B}(\mathcal{H})$ but by no means all of them. Whereas $T \otimes S$ is normal if and only if T and S are normal, there exist paranormal operators T and S such that $T \otimes S$ is not paranormal. It is proved that for non-zero $T, S \in \mathcal{B}(\mathcal{H})$, $T \otimes S$ is p -hyponormal if and only if T and S are p -hyponormal. This result was extended to P -quasi-hyponormal operators, for more details see ([20], [22]).

Recall that $(T \otimes S)^*(T \otimes S) = T^*T \otimes S^*S$ and $(T \otimes S)^\#(T \otimes S) = T^\#T \otimes S^\#S$.

The following elementary results on tensor products of operators will be used often in the sequel.

Lemma 3.12. ([4]) Let $T_k, S_k \in \mathcal{B}(\mathcal{H})$, $k = 1, 2$ and let $A, B \in \mathcal{B}(\mathcal{H})^+$ such that $T_1 \geq_A T_2 \geq_A 0$ and $S_1 \geq_A S_2 \geq_A 0$, then $T_1 \otimes S_1 \geq_{A \otimes B} T_2 \otimes S_2 \geq_{A \otimes B} 0$.

Proposition 3.13. ([4]) Let $T_k, S_k \in \mathcal{B}(\mathcal{H})$, $k = 1, 2$ and let $A, B \in \mathcal{B}(\mathcal{H})^+$ such that

T_k is A -positive and S_k is B -positive. If $T_1 \neq 0$ and $S_1 \neq 0$. Then the following conditions are equivalents.

1. $T_2 \otimes S_2 \geq_{A \otimes B} T_1 \otimes S_1$.
2. There exist $d > 0$ such that $dT_2 \geq_A T_1$ and $d^{-1}S_2 \geq_A S_1$.

In the following theorem we will prove the stability of the class of (n, m) power-(D, A)-hyponormal operators under the direct sum and tensor product.

Theorem 3.14. Let T, S be (n, m) power-(D, A)-hyponormal operators in $\mathcal{B}_A(\mathcal{H})^D$ such that $T^\#_d(T^D_d)^n \geq_A 0$ for all $d \in \{1, 2, \dots, k\}$. Then

1. $(T_1 \oplus T_2 \oplus \dots \oplus T_k)$ is (n, m) power-(D, $A \oplus A \oplus \dots \oplus A$)-hyponormal.
2. $(T_1 \otimes T_2 \otimes \dots \otimes T_k)$ is (n, m) power-(D, $A \otimes A \otimes \dots \otimes A$)-hyponormal operator.

Proof. 1. Assume $T_1, T_2, T_3 \dots T_k$ are (n, m) power-(D, A)-hyponormal operators, then

$T_d^\#(T^D_d)^n \geq_A (T^D_d)^n T_d^\#$ for $d = 1, 2, \dots, k$ and we have

$$\begin{aligned} & (T_1 \oplus T_2 \oplus \dots \oplus T_k)^\# \left((T_1 \oplus T_2 \oplus \dots \oplus T_k)^D \right)^n \\ &= (T_1^\# \oplus T_2^\# \oplus \dots \oplus T_k^\#) \left((T^D_1)^n \oplus (T^D_2)^n \oplus \dots \oplus (T^D_k)^n \right) \end{aligned}$$

$$\begin{aligned}
 &= T_1 \#^m (T^D_1)^n \oplus T_2 \#^m (T^D_2)^n \oplus \dots \oplus T_k \#^m (T^D_k)^n \\
 &\geq_A (T^D_1)^n T_1 \#^m \oplus (T^D_2)^n T_2 \#^m \oplus \dots \oplus (T^D_k)^n T_k \#^m \\
 &= ((T^D_1)^n \oplus (T^D_2)^n \oplus \dots \oplus (T^D_k)^n) (T_1 \#^m \oplus T_2 \#^m \oplus \dots \oplus T_k \#^m) \\
 &= (T_1 \oplus T_2 \oplus \dots \oplus T_k) \#^m ((T_1 \oplus T_2 \oplus \dots \oplus T_k)^D)^n.
 \end{aligned}$$

Then $(T_1 \oplus T_2 \oplus \dots \oplus T_k)$ is (n, m) power- $(D, A \oplus A \oplus \dots \oplus A)$ -hyponormal.

2. Let $x_1, x_2, \dots, x_k \in \mathcal{H}$, then

$$\begin{aligned}
 &((T_1 \otimes T_2 \otimes \dots \otimes T_k)^D)^n (T_1 \otimes T_2 \otimes \dots \otimes T_k) \#^m (x_1 \otimes x_1 \otimes \dots \otimes x_1) \\
 &= ((T^D_1)^n \otimes (T^D_2)^n \otimes \dots \otimes (T^D_k)^n) (T_1 \#^m \otimes T_2 \#^m \otimes \dots \otimes T_k \#^m) (x_1 \otimes x_1 \otimes \dots \otimes x_1) \\
 &= (T^D_1)^n T_1 \#^m x_1 \otimes (T^D_2)^n T_2 \#^m x_2 \otimes \dots \otimes (T^D_k)^n T_k \#^m x_k \\
 &\leq_A T_1 \#^m (T^D_1)^n x_1 \otimes T_2 \#^m (T^D_2)^n x_2 \otimes \dots \otimes T_k \#^m (T^D_k)^n x_k \\
 &= (T_1 \#^m \otimes T_2 \#^m \otimes \dots \otimes T_k \#^m) ((T^D_1)^n \otimes (T^D_2)^n \otimes \dots \otimes (T^D_k)^n) (x_1 \otimes x_1 \otimes \dots \otimes x_1) \\
 &= (T_1 \otimes T_2 \otimes \dots \otimes T_k) \#^m ((T_1 \otimes T_2 \otimes \dots \otimes T_k)^D)^n (x_1 \otimes x_1 \otimes \dots \otimes x_1).
 \end{aligned}$$

Then $(T_1 \otimes T_2 \otimes \dots \otimes T_k)$ is (n, m) power- $(D, A \otimes A \otimes \dots \otimes A)$ -hyponormal operator

The following theorem gives a necessary and sufficient condition for $T \otimes S$ to be (n, m) power- $(D, A \otimes B)$ -hyponormal when T and S are non-zero operators.

Theorem 3.15. Let $A, B \in \mathcal{B}(\mathcal{H})^+$. If $T \in \mathcal{B}_A(\mathcal{H})$ and $S \in \mathcal{B}_B(\mathcal{H})$ are nonzero operators, then $(T \otimes S)$ is (n, m) power- $(D, A \otimes B)$ -hyponormal if and only if T is (n, m) power- (D, A) -hyponormal and S is (n, m) power- (D, B) -hyponormal.

Proof. Assume that $T \in [(n, m)DH]_A$ and $S \in [(n, m)DH]_B$. Then

$$\begin{aligned}
 (T \otimes S) \#^m ((T \otimes S)^D)^n &= (T \#^m (T^D)^n \otimes S \#^m (T^D)^n) \\
 &\geq_{A \otimes B} ((T^D)^n T \#^m \otimes (T^D)^n S \#^m) \\
 &= ((T^D)^n \otimes (S^D)^n) (T \#^m \otimes S \#^m) \\
 &= ((T \otimes S)^D)^n (T \otimes S) \#^m.
 \end{aligned}$$

Which implies that $(T \otimes S)$ is (n, m) power- $(D, A \otimes B)$ -hyponormal.

Conversely, assume that $(T \otimes S)$ is (n, m) power $(D, A \otimes B)$ -hyponormal operator. So

$$(T \otimes S)^{\#m} ((T \otimes S)^D)^n \geq_{A \otimes B} ((T \otimes S)^D)^n (T \otimes S)^{\#m} ,$$

i.e.
$$T^{\#m} (T^D)^n \otimes S^{\#m} (S^D)^n \geq_{A \otimes B} (T^D)^n T^{\#m} \otimes (S^D)^n S^{\#m}$$

From proposition (3.13) there exists $d > 0$ such that
$$\begin{cases} d T^{\#m} (T^D)^n \geq_A (T^D)^n T^{\#m} \\ \text{and} \\ d^{-1} S^{\#m} (S^D)^n \geq_B (S^D)^n S^{\#m}. \end{cases}$$

A simple computation shows that $d = d^{-1} = 1$ and hence

$$T^{\#m} (T^D)^n \geq_A (T^D)^n T^{\#m} \quad \text{and} \quad S^{\#m} (S^D)^n \geq_B (S^D)^n S^{\#m}.$$

Therefore, T is (n, m) power (D, A) -hyponormal and S is (n, m) power (D, A) -hyponormal.

Proposition 3.16. Let $T, S \in \mathcal{B}_A(\mathcal{H})^D$ are an (n, m) power- (D, A) -hyponormal such that $(T^D)^n T^{\#m} \geq_A 0$ and $(S^D)^n S^{\#m} \geq_A 0$, then the following statement hold.

1. If $(S^D)^n ((T^D)^n T^{\#m}) = ((T^D)^n T^{\#m}) (S^D)^n$ and $(T^D)^n (S^{\#m} (S^D)^n) = (S^{\#m} (S^D)^n) (T^D)^n$, then

$(TS \otimes T) \in \mathcal{B}_{A \otimes A}(\mathcal{H} \otimes \mathcal{H})$ is (n, m) power- $(D, A \otimes A)$ -hyponormal operators.

2. If $(T^D)^n ((S^D)^n S^{\#m}) = ((S^D)^n S^{\#m}) (T^D)^n$ and $(S^D)^n (T^{\#m} (T^D)^n) = (T^{\#m} (T^D)^n) (S^D)^n$, then

$(TS \otimes S) \in \mathcal{B}_{A \otimes A}(\mathcal{H} \otimes \mathcal{H})$ is (n, m) power- $(D, A \otimes A)$ -hyponormal operator.

Proof. 1 Assume that the conditions (1) are hold, T and S are (n, m) power- (D, A) -hyponormal, we have.

$$\begin{aligned} (TS \otimes T)^{\#m} . ((TS \otimes T)^D)^n &= ((TS)^{\#m} \otimes T^{\#m}) . (((TS)^D)^n \otimes (T^D)^n) \\ &= S^{\#m} T^{\#m} (T^D)^n (S^D)^n \otimes T^{\#m} (T^D)^n. \end{aligned}$$

Since $T^{\#m} (T^D)^n \geq_A (T^D)^n T^{\#m} \geq_A 0$ and $S^{\#m} (S^D)^n \geq_A (S^D)^n S^{\#m} \geq_A 0$, from lemma (3,1) and remark (3,2) we have

$$\begin{aligned} S^{\#m} T^{\#m} (T^D)^n (S^D)^n &\geq_A S^{\#m} (T^D)^n T^{\#m} (S^D)^n \\ &= S^{\#m} (S^D)^n (T^D)^n T^{\#m} \\ &= (T^D)^n S^{\#m} (S^D)^n T^{\#m} \\ &\geq_A (T^D)^n (S^D)^n S^{\#m} T^{\#m} \\ &= ((TS)^D)^n (TS)^{\#m}. \end{aligned}$$

Thus $S^{\#m}T^{\#m}(T^D)^n(S^D)^n \geq_A ((TS)^D)^n(TS)^{\#m}$.

And $T^{\#m}(T^D)^n \geq_A (T^D)^nT^{\#m} \geq_A 0$.

Lemma (3,1) in [20] implies that

$$S^{\#m}T^{\#m}(T^D)^n(S^D)^n \otimes T^{\#m}(T^D)^n \geq_A ((TS)^D)^n(TS)^{\#m} \otimes (T^D)^nT^{\#m}.$$

Then $(TS \otimes T)^{\#m} \cdot ((TS \otimes T)^D)^n \geq_A ((TS \otimes T)^D)^n \cdot (TS \otimes T)^{\#m}$.

Hence $(TS \otimes T)$ is (n, m) power- $(D, A \otimes A)$ -hyponormal.

In the same way, we way deduce the (n, m) power- $(D, A \otimes A)$ -hyponormality of $(TS \otimes S)$.

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