

Fractional Models in Laser Physics: The Role of the Hilfer-Prabhakar Operator and Khalouta Integral Transform

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Abstract. In this work, we seek to implement the new exponential type Kernel integral transformation called Khalouta transform to the Prabhakar integral (PFI), Prabhakar fractional derivatives (PFD), regularized Prabhakar fractional derivative (RPFDF), Hilfer-Prabhakar fractional derivatives (HPFD) and its regularized form (RHPFD). Next, we present these results in the form of functions containing the Mittag-Leffler function and apply them to the solution of the Generalized Fractional Free-Electron Laser (FEL) equation involving the HPFD and its regularized form.

Keywords: Prabhakar integral, Hilfer-Prabhakar fractional operator, Khalouta integral transform.

1 Introduction

Fractional derivative operators extend the concept of differentiation beyond integer orders. Traditionally, derivatives are defined for integer orders (like first, second, third derivatives, etc.), but fractional calculus allows for differentiation of non-integer orders. These operators are particularly useful in fields such as applied mathematics, physics, biology, chemistry, control theory, economics, electronics and finance, due to their ability to capture complex dynamics and anomalous behaviors [1, 8, 22, 24, 25, 26, 27, 30, 32].

Indeed, fractional integrals and derivatives have been studied extensively in literature, leading to the development of various definitions and approaches. These different definitions offer flexibility and applicability in different contexts and applications. Some of the commonly used definitions include: Riemann-Liouville Fractional Integral (RLFI) and Derivative (RLFD), Caputo Fractional Integral (CFI) and Derivative (CFD), Atangana–Baleanu Fractional Derivative (ABFD), etc. (see [9, 7, 22]).

The PFI is defined as an extension of the RLFI by replacing its kernel with a three-parameter Mittag-Leffler function [31]. In 2004, Kilbas et al. [21] defined the PFD as the inverse factor of the PFI. The HPFD and its regularized form were initially introduced in [13]. This operator has received great attention from many researchers in this field due to their unique characteristics and versatility in accommodating several integral transforms, including Fourier, Laplace, Sumudu, Elzaki, Shehu, Sawi, and Jafari. Numerous studies have leveraged these transforms alongside the PFD and HPFD to address fractional differential equations, often presenting solutions in terms of the Mittag-Leffler function.

In [13], researchers delve into the Laplace transformation of HPFD alongside their regularized version. Their investigation extends to classical equations within mathematical physics, including those governing heat dynamics, free-electron laser behavior, and homogeneous Poisson processes. Similarly, in [29], the authors utilize the Sumudu integral transform to address non-homogeneous Cauchy-type

problems, offering a novel approach to problem-solving in this domain. Additionally, in [14], the authors present an analytical solution to the generalized space-time fractional advection–dispersion equation by integrating the Sumudu and Fourier integral transforms involving the HPFD. This amalgamation of transforms showcases their efficacy in tackling complex mathematical models.

In a recent study by Sing et al. [33], the Elzaki transform of HPFD and its regularized version are employed to address integro-differential equations pertinent to free electron lasers. Their research provides valuable insights into the applicability of this transform in resolving intricate physical phenomena. Furthermore, in [4], researchers utilize the Shehu transform of HPFD along with its regular version to solve Cauchy problems, highlighting the versatility of this approach in handling differential equations. Moreover, in a study referenced as [19], the Sawi transform is applied to HPFD and its regular versions, enabling the solution of fractional differential equations. This novel approach opens avenues for addressing a wide range of mathematical challenges.

Additionally, the Jafari transformation was applied in [18, 5] on HPFD and its regular versions to address Cauchy-type fractional differential equations, further demonstrating the utility of integral transforms in this field.

In 2023, Khalouta [20] introduced a new generalization of famous integral transforms such as the Laplace, Elzaki, Sumudu, and Shehu transforms. Recently, the Khalouta transformation was used by Kumawat et al. [23], where they applied this transformation to the RLFD and CFD operators, and presented these results in applications for solving some differential equations.

In this research, we will attempt to apply the Khalouta transformation to various derivatives such as PFD, HPFD, and their regularized versions. Furthermore, we intend to utilize the outcomes from this transform to address generalized fractional free electron laser (FEL) equations involving HPFD. These equations are presented in the form of usual Mittag-Leffler (M-L) function, promising an innovative approach to solving many problems in this field.

The manuscript structure is as follows: Section 2 is devoted to presenting the basic definitions of fractional calculus. Section 3 provides an overview of recent integral transforms, including the Khalouta transform, along with associated results. In Section 4, we apply the Khalouta transform on several operators, namely PFI, PFD, and HPFD, including their regularized versions. Section 5 focuses on practical applications, illustrating the use of these transforms in solving Cauchy problems, such as the FEL equation, by using the main results obtained.

2 Preliminaries

In this part of this paper, we present key definitions concerning fractional integrals and derivatives, that will be essential for our upcoming discussions. For a deeper grasp of the topic, readers are encouraged to refer to the following references [22, 9, 27, 2, 16].

Definition 2.1. The RLFI operator for a given function $\mathcal{W}(t)$ of order $q > 0$, is given as the form:

$${}^{\text{RL}}\mathcal{I}_t^q \mathcal{W}(t) = \int_0^t \frac{(t-\mathcal{X})^{q-1}}{\Gamma(q)} \mathcal{W}(\mathcal{X}) d\mathcal{X} = \left(\mathcal{W} \star \frac{t^{q-1}}{\Gamma(q)} \right) (t), t > 0. \quad (2.1)$$

Definition 2.2. The RLFD operator for a given function $\mathcal{W}(t)$ of order $q > 0$, is defined as:

$${}^{\text{RL}}D_t^q \mathcal{W}(t) = \frac{d^r}{dt^r} \left({}^{\text{RL}}\mathcal{I}_t^{r-q} \mathcal{W}(t) \right) = \frac{d^r}{dt^r} \int_0^t \frac{\mathcal{W}(x)}{\Gamma(r-q)(t-x)^{r-q-1}} dx, \quad r-1 < q \leq r, r \in \mathbb{N}. \quad (2.2)$$

Definition 2.3. The CFD operator for a function $\mathcal{W}(t)$ of order $q > 0$, is a widely used definition, defined as:

$${}_0^C D_t^q \mathcal{W}(t) = \int_0^t \frac{(t-x)^{r-q-1}}{\Gamma(r-q)} \mathcal{W}^{(r)}(x) dx, \quad r-1 < q \leq r, r \in \mathbb{N}. \quad (2.3)$$

Definition 2.4. The HFD operator of order $q_1 > 0$ and $q_2 > 0$ for a function $\mathcal{W}(t)$ is another important definition [17]:

$${}_0 D_t^{q_1, q_2} \mathcal{W}(t) = \left({}_0 \mathcal{I}_t^{q_2(1-q_1)} \frac{d}{dt} \left({}_0 \mathcal{I}_t^{(1-q_2)(1-q_1)} \mathcal{W} \right) \right) (t), \quad (2.4)$$

where $q_1 \in (0, 1)$, $q_2 \in [0, 1]$.

Remark 2.5. The HFD (2.4) coincides with the RLFD (2.2) for $q_2 = 0$ and with the CFD (2.3) for $q_2 = 1$.

In practical applications, we often use another generalization of the HFD by replacing the integral kernel $\frac{1}{t^{q-1}}$ occurring in the RLI by the function $e_{\sigma, \eta, \omega}^q(t)$, which is expressed as follows:

$$e_{\sigma, \eta, \omega}^q(t) = t^{\eta-1} E_{\sigma, \eta}^q(\omega t^\sigma), \quad t \in \mathbb{R}, \omega, \sigma, \eta, q \in \mathbb{C}, \Re(\sigma) > 0, \Re(\eta) > 0. \quad (2.5)$$

where $E_{\sigma, \eta}^q(\omega t^\sigma)$ is the generalized Mittag-Leffler (Prabhakar) function which is defined in the following definition:

Definition 2.6. The 1-parameter Mittag-Leffler function is stated as [12]

$$E_\sigma(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\sigma j + 1)}, \quad z, \sigma \in \mathbb{C}, \Re(\sigma) > 0. \quad (2.6)$$

The 2-parameter M-L function is given as [34]

$$E_{\sigma, \eta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\sigma j + \eta)}, \quad z, \sigma, \eta \in \mathbb{C}, \Re(\sigma) > 0, \quad (2.7)$$

such that $E_{\sigma, 1}(z) = E_\sigma(z)$.

A variant of the M-L function is given by the 3-parameter, also called Prabhakar function and given by [31, 6]

$$E_{\sigma, \eta}^q(z) = \frac{1}{\Gamma(q)} \sum_{j=0}^{\infty} \frac{\Gamma(q+j)}{\Gamma(\sigma j + \eta)} \frac{z^j}{j!}, \quad z, \sigma, \eta, q \in \mathbb{C}, \Re(\sigma) > 0. \quad (2.8)$$

Definition 2.7. Let $\mathcal{W} \in L^1([0, b])$, $0 < t < b \leq \infty$, where the parameters $q, q_1, q_2, q_3 \in \mathbb{C}$, with $\Re(q_1), \Re(q_2) > 0$. The PFI can be written in the form [31]:

$$\mathbf{E}_{q_1, q_2, q_3, 0^+}^q \mathcal{W}(t) = \int_0^t (t-\tau)^{q_2-1} E_{q_1, q_2}^q [q_3(t-\tau)^{q_1}] \mathcal{W}(\tau) d\tau = (\mathcal{W} * e_{q_1, q_2, q_3}^q)(t). \quad (2.9)$$

The inverse operation, the PFD, can now be presented in the following form.

Definition 2.8. The PFD operator is defined by [31]:

$$\mathbf{D}_{q_1, q_2, q_3, 0^+}^q \mathcal{W}(t) = \frac{d^r}{dt^r} \mathbf{E}_{q_1, m-q_2, q_3, 0^+}^{-q} \mathcal{W}(t), \quad (2.10)$$

where $\mathcal{W} * e_{q_1, r-q_2, q_3}^{-q} \in W^{r,1}[0, b] = \left\{ \mathcal{W} \in L^1([0, b]) : \frac{d^r}{dt^r} \mathcal{W} \in L^1([0, b]) \right\}$, $r = [q_2]$, $q, q_1, q_2, q_3 \in \mathbb{C}$, with $\Re(q_1), \Re(q_2) > 0$.

In the provided equations, the relationship between the RLFD and the PFI is established. Equation (2.11) expresses the RLFD in terms of PFI as:

$${}^{RL} \mathcal{I}_t^{r-(q_2+q_4)} \mathcal{W}(t) = \mathbf{E}_{q_1, r-(q_2+q_4), q_3, 0^+}^0 \mathcal{W}(t), \quad (2.11)$$

From which, we derive:

$$\mathbf{D}_{q_1, q_2, q_3, 0^+}^q \mathcal{W}(t) = \frac{d^r}{dt^r} \mathbf{E}_{q_1, r-q_2, q_3, 0^+}^{-q} \mathcal{W}(t) = D_{0^+}^{q_2+q_4} \mathbf{E}_{q_1, q_4, q_3, 0^+}^{-q} \mathcal{W}(t),$$

where $q_4 \in \mathbb{C}$, $\operatorname{Re}(q_4) > 0$ and $D_{0^+}^{q_2+q_4}$ is the RLFD operator.

Definition 2.9. Let $\mathcal{W} \in AC[0, b]$, $0 < t < b < \infty$, and $r = [q_2]$. The RPFDF is expressed as [13]:

$${}^C \mathbf{D}_{q_1, q_2, \omega, 0^+}^{q_3} \mathcal{W}(t) = \mathbf{E}_{q_1, r-q_2, \omega, 0^+}^{-q_3} \frac{d^r}{dx^r} \mathcal{W}(t), \quad (2.12)$$

where $q_1, q_2, q_3, \omega \in \mathbb{C}$, $\Re(q_1), \Re(q_2) > 0$.

Definition 2.10. Let $q_2 \in (0, 1)$, $q_4 \in [0, 1]$, $\mathcal{W} \in L^1[0, b]$, $0 < t < b \leq \infty$, $\mathcal{W} * e_{q_1, (1-q_4)(1-q_2), \omega}^{-q_3(1-q_4)}(\cdot) \in AC^1[0, b]$. The HPFD is expressed as [13, 28]:

$$D_{q_1, \omega, 0^+}^{q_3, q_2, q_4} \mathcal{W}(t) = \left(\mathbf{E}_{q_1, q_4(1-q_2), \omega, 0^+}^{-q_3 q_4} \frac{d}{dt} \left(\mathbf{E}_{q_1, (1-q_4)(1-q_2), \omega, 0^+}^{-q_3(1-q_4)} \mathcal{W} \right) \right) (t), \quad (2.13)$$

where $q_3, \omega \in \mathbb{R}$, $q_1 > 0$, and $\mathbf{E}_{q_1, 0, \omega, 0^+}^0 \mathcal{W} = \mathcal{W}$.

Definition 2.11. Let $\mathcal{W} \in AC[0, b]$, $q_2 \in (0, 1)$, $q_4 \in [0, 1]$, $\mathcal{W} \in L^1[0, b]$, $0 < t < b < \infty$. The regularized Hilfer-Prabhakar fractional derivatives (RHPD) of $\mathcal{W}(t)$ is defined as the form:

$${}^C \mathbf{D}_{q_1, \omega, 0^+}^{q_3, q_2, q_4} \mathcal{W}(t) = \left(\mathbf{E}_{q_1, q_4(1-q_2), \omega, 0^+}^{-q_3 q_4} \mathbf{E}_{q_1, (1-q_4)(1-q_2), \omega, 0^+}^{-q_3(1-q_4)} \frac{d}{dt} \mathcal{W} \right) (t) = \left(\mathbf{E}_{q_1, 1-q_2, \omega, 0^+}^{-q_3} \frac{d}{dt} \mathcal{W} \right) (t). \quad (2.14)$$

where $q_3, \omega \in \mathbb{R}$, $q_1 > 0$.

3 Khalouta transform

In this paragraph, we present the state of the art of results obtained so far for this new transformation (denoted by $\mathbb{K}\mathbb{H}$ -Transform), which we discuss in this paper.

Consider an integrable mapping $\mathcal{W}(t)$ defined on a set \mathcal{A}

$$\mathcal{A} = \{ \mathcal{W}(t) : \exists N > 0, \eta > 0, |\mathcal{W}(t)| < Ne(\eta t), \text{ if } t \in [0, \infty) \} \quad (3.1)$$

Definition 3.1. ([20]) Suppose the function $\mathcal{W} : \mathbb{R}^+ \rightarrow \mathbb{R}$ of exponential order $\eta > 0$, then Khalouta transform is defined as follows:

$$\mathbb{KH}[\mathcal{W}(t)](s, u, v) = \mathcal{K}(s, u, v) = \frac{s}{uv} \int_0^{\infty} e\left(-\frac{s}{uv}t\right) \mathcal{W}(t) dt, \quad t \geq 0, \quad (3.2)$$

where s, u and v are the \mathbb{KH} -Transform variables. Provided that the new integral transform (3.2) exists for $\frac{s}{uv} - \eta > 0$. This integral transform, similar to other integral transforms, is a linear operator and also has several properties. More details about this can be found in [20, 23].

The inversion formula of \mathbb{KH} -Transform, is defined as:

$$\mathbb{KH}^{-1}[\mathcal{K}(s, u, v)](t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e\left(-\frac{st}{uv}\right) \mathcal{K}(s, u, v) ds, \text{ for } t \geq 0, a \in \mathbb{R}. \quad (3.3)$$

Where the integral 3.3 in the complex plane, is taken along $s = a$.

Theorem 3.2. [20] If $\mathcal{W}^{(r)}(t) \in \mathcal{A}$, then:

$$\mathbb{KH}[\mathcal{W}^{(r)}(t)] = \left[\frac{s}{uv}\right]^r \mathcal{K}(s, u, v) - \sum_{j=0}^{r-1} \left[\frac{s}{uv}\right]^{r-j} \mathcal{W}^{(j)}(0), \quad r \geq 1, \quad (3.4)$$

where $\mathcal{W}^{(r)}(t)$ is the r^{th} derivative of order $r \in \mathbb{N}^*$ of the function $\mathcal{W}(t)$.

Proposition 3.3. The \mathbb{KH} -Transform of r^{th} order partial derivative for a given the function $\mathcal{W}(\mathcal{X}, t)$ with respect to x , is defined as the form:

$$\mathbb{KH}\left[\frac{\partial^r \mathcal{W}(\mathcal{X}, t)}{\partial \mathcal{X}^r}\right] = \left[\frac{s}{uv}\right]^r \mathcal{K}(\mathcal{X}, s, u, v) - \sum_{j=0}^{r-1} \left[\frac{s}{uv}\right]^{r-j} \frac{\partial^j \mathcal{W}(x, 0)}{\partial \mathcal{X}^j} = \frac{d^r}{d\mathcal{X}^r} \mathcal{K}(\mathcal{X}, s, u, v), \quad (3.5)$$

where $\mathcal{K}(\mathcal{X}, s, u, v)$ the \mathbb{KH} -Transform of the function $\mathcal{W}(\mathcal{X}, t)$.

Theorem 3.4. (Convolution property) Let $\mathcal{W}_1(t)$ and $\mathcal{W}_2(t)$ be two functions given in \mathcal{A} with \mathbb{KH} -transforms $\mathcal{K}_1(s, u, v)$ and $\mathcal{K}_2(s, u, v)$, respectively. Then the \mathbb{KH} -transform of their convolution is given by the form:

$$\mathbb{KH}\{\mathcal{W}_1 * \mathcal{W}_2\} = \frac{uv}{s} \mathcal{K}_1(s, u, v) \cdot \mathcal{K}_2(s, u, v), \quad (3.6)$$

where (\cdot) denotes multiplication, and the convolution is defined as:

$$(\mathcal{W}_1 * \mathcal{W}_2)(t) = \int_0^t \mathcal{W}_1(\tau) \mathcal{W}_2(t - \tau) d\tau. \quad (3.7)$$

Based on this new integral transformation, many integral transformations can easily be deduced. This is a result of the generality of this integral transformation. In Table 1, we mention some of these transformations and their relationship to the Khaluta transformation.

Transform	Definition	\mathbb{KH} -Transform:
Laplace	$\mathcal{L}\{\mathcal{W}(t), s\} = \int_0^\infty \mathcal{W}(t)e(-st) dt$	$\frac{1}{s}\mathcal{K}(s, 1, 1)$
Laplace-Carson	$\mathcal{L}_c\{\mathcal{W}(t), s\} = s \int_0^\infty \mathcal{W}(t)e(-st) dt$	$\mathcal{K}(s, 1, 1)$
Sumudu	$\mathcal{S}\{\mathcal{W}(t), u\} = \frac{1}{u} \int_0^\infty \mathcal{W}(t)e\left(-\frac{t}{u}\right) dt$	$\mathcal{K}(1, u, 1)$
ZZ	$\mathcal{Z}\{\mathcal{W}(t), s, v\} = \frac{s}{v} \int_0^\infty \mathcal{W}(t)e\left(-\frac{st}{v}\right) dt$	$\mathcal{K}(s, 1, v)$
ZMA	$\mathcal{M}\{\mathcal{W}(t), u, v\} = \frac{1}{uv} \int_0^\infty \mathcal{W}(t)e\left(-\frac{t}{uv}\right) dt$	$\mathcal{K}(1, u, v)$
Elzaki	$E\{\mathcal{W}(t), u\} = u \int_0^\infty \mathcal{W}(t)e\left(-\frac{t}{u}\right) dt$	$u^2 \mathcal{K}(1, u, 1)$
Natural	$N\{\mathcal{W}(t), s, v\} = \frac{1}{v} \int_0^\infty \mathcal{W}(t)e\left(-\frac{st}{v}\right) dt$	$\frac{1}{s}\mathcal{K}(s, 1, v)$
Aboodh	$A\{\mathcal{W}(t), s\} = \frac{1}{s} \int_0^\infty \mathcal{W}(t)e(-st) dt$	$\frac{1}{s^2}\mathcal{K}(s, 1, 1)$
Mohand	$\mathcal{M}\{\mathcal{W}(t), s\} = s^2 \int_0^\infty \mathcal{W}(t)e(-st) dt$	$s\mathcal{K}(s, 1, 1)$
Sawi	$\mathcal{S}\mathcal{A}\{\mathcal{W}(t), u\} = \frac{1}{u^2} \int_0^\infty \mathcal{W}(t)e\left(-\frac{t}{u}\right) dt$	$\frac{1}{u}\mathcal{K}(1, u, 1)$
Kamel	$\mathcal{K}\{\mathcal{W}(t), v\} = \int_0^\infty \mathcal{W}(t)e\left(-\frac{t}{v}\right) dt$	$v\mathcal{K}(1, 1, v)$
Shehu	$\mathcal{S}\mathcal{H}\{\mathcal{W}(t), s, u\} = \int_0^\infty \mathcal{W}(t)e\left(-\frac{st}{u}\right) dt$	$\frac{u}{s}\mathcal{K}(s, u, 1)$

Table 1: Relationships for relating the \mathbb{KH} -Transform with other integral transforms.

4 Main Results

Here in this manuscript, we present the main results of our research. Let $\mathcal{W}(t) \in \mathcal{A}$ with \mathbb{KH} -transform $\mathcal{K}(s, s, v)$.

Property 4.1. The Khalouta transform of t^{x-1} is given by:

$$\mathbb{KH}[t^{x-1}] = \left(\frac{uv}{s}\right)^{x-1} \Gamma(x). \quad (4.1)$$

Proof. For $x > 0$, we know:

$$\Gamma(x) = \int_0^\infty \xi^{x-1} e(-\xi) d\xi.$$

From the formula (3.2), we obtain:

$$\mathbb{KH}[t^{x-1}] = \frac{s}{uv} \int_0^\infty t^{x-1} e\left(-\frac{s}{uv}t\right) dt.$$

If we set $\xi = \frac{s}{uv}t$, or $\left(t = \frac{\xi uv}{s}\right)$, we get:

$$\begin{aligned} \mathbb{KH}[t^{x-1}] &= \frac{s}{uv} (uv)^{x-1} \frac{1}{s^{x-1}} \int_0^\infty \xi^{x-1} e(-\xi) \frac{uv}{s} d\xi, \\ &= (uv)^{x-1} \frac{1}{s^{x-1}} \int_0^\infty \xi^{x-1} e(-\xi) d\xi, \\ &= \left(\frac{uv}{s}\right)^{x-1} \Gamma(x). \end{aligned}$$

□

Lemma 4.2. Let $0 < \sigma < 1$ and $\omega \in \mathbb{R}$ such that $\frac{s}{uv} < |\omega|^{\frac{1}{\sigma}}$, then:

$$\mathbb{K}\mathbb{H} [t^{\eta-1} E_{\sigma,\eta}^{q_3}(\omega t^\sigma)] = \left(\frac{uv}{s}\right)^{\eta-1} \left(1 - \omega \left(\frac{uv}{s}\right)^\sigma\right)^{-q_3}, \quad \sigma > 0. \tag{4.2}$$

Proof. The equation 2.8 can be written as:

$$E_{\sigma,\eta}^{q_3}(\omega t^\sigma) = \frac{1}{\Gamma(q_3)} \sum_{j=0}^{\infty} \frac{\Gamma(q_3 + j)}{\Gamma(\sigma j + \eta)} \frac{(\omega t^\sigma)^j}{j!}, \quad z, \sigma, \eta, q_3 \in \mathbb{C}, \Re(\sigma) > 0. \tag{4.3}$$

Taking $\mathbb{K}\mathbb{H}$ - transform of PFI operator using (2.5) and (4.3), we have:

$$\mathbb{K}\mathbb{H} [e_{\sigma,\eta,\omega}^{q_3}(t, \omega)] = \mathbb{K}\mathbb{H} [t^{\eta-1} E_{\sigma,\eta}^{q_3}(\omega t^\sigma)] = \mathbb{K}\mathbb{H} \left[\frac{1}{\Gamma(q_3)} \sum_{j=0}^{\infty} t^{j\sigma+\eta-1} \frac{\Gamma(q_3 + j)}{\Gamma(\sigma j + \eta)} \frac{\omega^j}{j!} \right], \tag{4.4}$$

$$= \frac{1}{\Gamma(q_3)} \sum_{j=0}^{\infty} \frac{\Gamma(q_3 + j)}{\Gamma(\sigma j + \eta)} \frac{\omega^j}{j!} \mathbb{K}\mathbb{H} [t^{j\sigma+\eta-1}], \tag{4.5}$$

by using the Property (4.1), we obtain:

$$\mathbb{K}\mathbb{H} [t^{\eta-1} E_{\sigma,\eta}^{q_3}(\omega t^\sigma)] = \frac{1}{\Gamma(q_3)} \sum_{j=0}^{\infty} \left(\frac{uv}{s}\right)^{j\sigma+\eta-1} \Gamma(j\sigma + \eta) \frac{\Gamma(q_3 + j)}{\Gamma(\sigma j + \eta)} \frac{\omega^j}{j!}, \tag{4.6}$$

$$= \left(\frac{uv}{s}\right)^{\eta-1} \sum_{j=0}^{\infty} \frac{\Gamma(q_3 + j)}{\Gamma(q_3)} \frac{\left(\omega \left(\frac{uv}{s}\right)^\sigma\right)^j}{j!}, \tag{4.7}$$

therefore:

$$\mathbb{K}\mathbb{H} [t^{\eta-1} E_{\sigma,\eta}^{q_3}(\omega t^\sigma)] = \left(\frac{uv}{s}\right)^{\eta-1} \left(1 - \omega \left(\frac{uv}{s}\right)^\sigma\right)^{-q_3}, \tag{4.8}$$

□

Remark 4.3. Lemma 4.2 generalizes some of the results that already exist, as shown in the following table:

Khalouta	$\mathcal{H}(s, u, v)$	$\mathbb{K}\mathbb{H} [t^{\eta-1} E_{\sigma,\eta}^{q_3}(\omega t^\sigma)] = \left(\frac{uv}{s}\right)^{\eta-1} \left(1 - \omega \left(\frac{uv}{s}\right)^\sigma\right)^{-\eta}$
Laplace	$\frac{1}{s} \mathcal{H}(s, 1, 1)$	$\mathcal{L} [t^{\eta-1} E_{\sigma,\eta}^{q_3}(\omega t^\sigma)] = \left(\frac{1}{s}\right)^\eta (1 - \omega s^{-\sigma})^{-\eta}$ [21, 13]
Sumudu	$\mathcal{H}(1, u, 1)$	$\mathcal{S} [t^{\eta-1} E_{\sigma,\eta}^{q_3}(\omega t^\sigma)] = u^{\eta-1} (1 - \omega u^\sigma)^{-\eta}$ [15, 10]
Elzaki	$u^2 \mathcal{H}(1, u, 1)$	$E [t^{\eta-1} E_{\sigma,\eta}^{q_3}(\omega t^\sigma)] = u^{\eta+1} (1 - \omega u^\sigma)^{-\eta}$ [11]
Shehu	$\frac{u}{s} \mathcal{H}(s, u, 1)$	$\mathcal{S}\mathcal{H} [t^{\eta-1} E_{\sigma,\eta}^{q_3}(\omega t^\sigma)] = \left(\frac{u}{s}\right)^\eta \left(1 - \omega \left(\frac{u}{s}\right)^\sigma\right)^{-\eta}$ [3]
Sawi	$\frac{1}{u} \mathcal{H}(1, u, 1)$	$\mathcal{S}\mathcal{A} [t^{\eta-1} E_{\sigma,\eta}^{q_3}(\omega t^\sigma)] = u^{\eta-2} (1 - \omega u^\sigma)^{-\eta}$ [19]

Table 2: Lemma 4.2, and special cases with some integral transformations

Lemma 4.4. The \mathbb{KH} - transform of PFD (2.10) is:

$$\mathbb{KH} \left\{ \mathbf{D}_{\sigma, \eta, \omega, 0^+}^{q_3} \mathscr{W}(t), s, u, v \right\} = \left[\frac{s}{uv} \right]^\eta \left(1 - \omega \left(\frac{s}{uv} \right)^\eta \right)^{q_3} \mathscr{K}(s, u, v) - \sum_{j=0}^{r-1} \left[\frac{s}{uv} \right]^{r-k} \left[\mathbf{D}_{\sigma, j-r+\eta, \omega, 0^+}^{q_3} \mathscr{W}(t) \right]_{t=0^+}. \quad (4.9)$$

Proof. Taking the \mathbb{KH} - transform on the both side of (2.10), we obtain:

$$\begin{aligned} \mathbb{KH} \left\{ \mathbf{D}_{\sigma, \eta, \omega, 0^+}^{q_3} \mathscr{W}(t), s, u, v \right\} &= \mathbb{KH} \left\{ \frac{d^r}{dt^r} \mathbf{E}_{\sigma, r-\eta, \omega, 0^+}^{-q_3} \mathscr{W}(t), s, u, v \right\}, \\ &= \mathbb{KH} \left\{ \frac{d^r}{dt^r} \left(\mathbf{e}_{\sigma, r-\eta, \omega, 0^+}^{-q_3} * \mathscr{W} \right) (t), s, u, v \right\}, \end{aligned}$$

and then using theorems (3.2) and (3.4), we get:

$$\begin{aligned} \mathbb{KH} \left\{ \mathbf{D}_{\sigma, \eta, \omega, 0^+}^{q_3} \mathscr{W}(t), s, u, v \right\} &= \left[\frac{s}{uv} \right]^r \mathbb{KH} \left\{ \left(\mathbf{e}_{\sigma, r-\eta, \omega, 0^+}^{-q_3} * \mathscr{W} \right) (t), s, u, v \right\} \\ &\quad - \sum_{j=0}^{r-1} \left[\frac{s}{uv} \right]^{r-j} \left[\left(\frac{d}{dt} \right)^{(j)} \mathbf{E}_{\sigma, r-\eta, \omega, 0^+}^{-q_3} \mathscr{W}(t) \right]_{t=0^+}, \\ &= \left(\frac{s}{uv} \right)^\eta \left(1 - \omega \left(\frac{uv}{s} \right)^\sigma \right)^{q_3} \mathscr{K}(s, u, v) \\ &\quad - \sum_{j=0}^{r-1} \left(\frac{s}{uv} \right)^{r-j} \left[\left(\frac{d}{dt} \right)^{(j)} \mathbf{E}_{\sigma, j-(j-r+\eta), \omega, 0^+}^{-q_3} \mathscr{W}(t) \right]_{t=0^+}, \\ &= \left(\frac{s}{uv} \right)^\eta \left(1 - \omega \left(\frac{uv}{s} \right)^\sigma \right)^{q_3} \mathscr{K}(s, u, v) \\ &\quad - \sum_{j=0}^{r-1} \left(\frac{s}{uv} \right)^{r-j} \left[\mathbf{D}_{\sigma, j-r+\eta, \omega, 0^+}^{q_3} \mathscr{W}(t) \right]_{t=0^+}. \end{aligned}$$

□

Remark 4.5. The generalization of Lemma 4.4 encompasses several of our previously established findings, thereby facilitating the derivation of additional results. For instance:

1. Taking $v = 1$ in Eq. (4.9), we acquire the Shehu transform [4].
2. Taking $s = v = 1$ for the Elzaki transform [33].
3. Taking $u = v = 1$ for the Laplace transform [13].

Lemma 4.6. The \mathbb{KH} - transform of RFPD (2.12) is given by:

$$\mathbb{KH} \left\{ {}^C \mathbf{D}_{\sigma, \eta, \omega, 0^+}^{q_3} \mathscr{W}(t), s, u, v \right\} = \left(\frac{s}{uv} \right)^\eta \left(1 - \omega \left(\frac{uv}{s} \right)^\eta \right)^{q_3} \mathscr{K}(s, u, v) - \sum_{j=0}^{r-1} \left[\frac{s}{uv} \right]^{\eta-k} \left(1 - \omega \left(\frac{uv}{s} \right)^\eta \right)^{q_3} \mathscr{W}^{(j)}(0). \quad (4.10)$$

Proof. In the same way as used in the previous proof, we obtain:

$$\begin{aligned} \mathbb{KH} \left\{ {}^C D_{\sigma, \eta, \omega, 0^+}^{q_3} \mathscr{W}(t), s, u, v \right\} &= \mathcal{K} \mathcal{H} \left[\mathbf{E}_{\sigma, r-\eta, \omega, 0^+}^{-q_3} \frac{d^r}{dx^r} \mathscr{W}(t) \right] (s, u, v) \\ &= \mathbb{KH} \left[\left(\mathbf{E}_{\sigma, r-\eta, \omega, 0^+}^{-q_3} * \frac{d^r}{dx^r} \mathscr{W} \right) (t) \right] (s, u, v), \\ &= \left(\frac{s}{uv} \right)^\eta \left(1 - \omega \left(\frac{uv}{s} \right)^\eta \right)^{q_3} \mathcal{K}(s, u, v) \\ &\quad - \sum_{j=0}^{r-1} \left[\frac{s}{uv} \right]^{\eta-j} \left(1 - \omega \left(\frac{uv}{s} \right)^\eta \right)^{q_3} \mathscr{W}^{(k)}(0), \end{aligned}$$

This is the desired result (4.10). □

Lemma 4.7. *The \mathbb{KH} - transform of HPFD (2.13) is expressed as:*

$$\begin{aligned} \mathbb{KH} \left(D_{\sigma, \omega, 0^+}^{q_3, \eta, q_4} \mathscr{W}(t) \right) (s, u, v) &= \left(\frac{uv}{s} \right)^{-\eta} \left(1 - \omega \left(\frac{uv}{s} \right)^\sigma \right)^{q_3} \mathbb{KH}[\mathscr{W}(t)](s, u, v) \\ &\quad - \left(\frac{s}{uv} \right)^{q_4(\eta-1)+1} \left(1 - \omega \left(\frac{uv}{s} \right)^\sigma \right)^{q_3 q_4} \left[\left(\mathbf{E}_{\sigma, (1-q_4)(1-\eta), \omega, 0^+}^{-q_3(1-q_4)} \mathscr{W} \right) (t) \right]_{t=0^+}. \end{aligned}$$

Proof. Taking \mathbb{KH} - transform on both side of HPFD (2.13), then using (2.5), (2.9), we get:

$$\begin{aligned} \mathbb{KH} \left(D_{\sigma, \omega, 0^+}^{q_3, \eta, q_4} \mathscr{W}(t) \right) (s, u, v) &= \mathbb{KH} \left[\left(\mathbf{E}_{\sigma, q_4(1-\eta), \omega, 0^+}^{-q_3 q_4} \frac{d}{dt} \left(\mathbf{E}_{\sigma, (1-q_4)(1-\eta), \omega, 0^+}^{-q_3(1-q_4)} \mathscr{W} \right) \right) (t) \right] (s, u, v), \\ &= \mathbb{KH} \left[\left(\mathbf{e}_{\sigma, q_4(1-\eta), \omega, 0^+}^{-q_3 q_4} * \frac{d}{dt} \left(\mathbf{E}_{\sigma, (1-q_4)(1-\eta), \omega, 0^+}^{-q_3(1-q_4)} \mathscr{W} \right) (t) \right) \right] (s, u, v), \end{aligned}$$

then, using (3.4) and (4.2), we have:

$$\begin{aligned} \mathbb{KH} \left(D_{\sigma, \omega, 0^+}^{q_3, \eta, q_4} \mathscr{W}(t) \right) (s, u, v) &= \frac{uv}{s} \mathbb{KH} \left[\mathbf{e}_{\sigma, q_4(1-\eta), \omega, 0^+}^{-q_3 q_4} \right] (s, u, v) \mathbb{KH} \left[\frac{d}{dt} \left(\mathbf{E}_{\sigma, (1-q_4)(1-\eta), \omega, 0^+}^{-q_3(1-q_4)} \mathscr{W} \right) (t) \right] (s, u, v), \\ &= \left(\frac{uv}{s} \right)^{q_4(1-\eta)} \left[1 - \omega \left(\frac{uv}{s} \right)^\sigma \right]^{q_3 q_4} \\ &\quad \left\{ \left(\frac{uv}{s} \right)^{(1-q_4)(1-\eta)-1} \left(1 - \omega \left(\frac{uv}{s} \right)^\sigma \right)^{q_3(1-q_4)} \mathbb{KH}[\mathscr{W}(t)](s, u, v) - \frac{s}{uv} \left[\left(\mathbf{E}_{\sigma, (1-q_4)(1-\eta), \omega, 0^+}^{-q_3(1-q_4)} \mathscr{W} \right) (t) \right]_{t=0^+} \right\} \\ &= \left(\frac{uv}{s} \right)^{-\eta} \left(1 - \omega \left(\frac{uv}{s} \right)^\sigma \right)^{q_3} \mathbb{KH}[\mathscr{W}(t)](s, u, v) \\ &\quad - \left(\frac{s}{uv} \right)^{q_4(\eta-1)+1} \left(1 - \omega \left(\frac{uv}{s} \right)^\sigma \right)^{q_3 q_4} \left[\left(\mathbf{E}_{\sigma, (1-q_4)(1-\eta), \omega, 0^+}^{-q_3(1-q_4)} \mathscr{W} \right) (t) \right]_{t=0^+}. \end{aligned}$$

□

Lemma 4.8. *The \mathbb{KH} - transform of the RHPFD operator (2.14) is given by:*

$$\begin{aligned} \mathbb{KH} \left({}^C D_{\sigma, \omega, 0^+}^{q_3, \eta, q_4} \mathscr{W}(t) \right) (s, u, v) &= \left(\frac{s}{uv} \right)^\eta \left(1 - \omega \left(\frac{uv}{s} \right)^\sigma \right)^{q_3} \mathbb{KH}[\mathscr{W}(t)](s, u, v) \\ &\quad - \left(\frac{uv}{s} \right)^{-\eta} \left(1 - \omega \left(\frac{uv}{s} \right)^\sigma \right)^{q_3} \mathscr{W}(0^+). \end{aligned}$$

Proof. By similar calculations to those in Lemma 4.7, we obtain:

$$\begin{aligned} \mathbb{KH} \left({}^C \mathbf{D}_{\sigma, \omega, 0^+}^{q_3, \eta, q_4} \mathscr{W}(t) \right) (s, u, v) &= \mathbb{KH} \left[\left(\mathbf{E}_{\sigma, 1-\eta, \omega, 0^+}^{-q_3} \frac{d}{dt} \mathscr{W} \right) (t) \right] (s, u, v), \\ &= \mathbb{KH} \left[\left(\mathbf{e}_{\sigma, 1-\eta, \omega, 0^+}^{-q_3} * \frac{d}{dt} \mathscr{W} \right) (t) \right] (s, u, v), \\ &= \frac{uv}{s} \left(\frac{uv}{s} \right)^{-\eta} \left(1 - \omega \left(\frac{uv}{s} \right)^\sigma \right)^{q_3} \left[\frac{s}{uv} \mathcal{H} \mathcal{H} [\mathscr{W}] (s, u, v) - \frac{s}{uv} \mathscr{W} (0^+) \right], \\ &= \left(\frac{s}{uv} \right)^\eta \left(1 - \omega \left(\frac{uv}{s} \right)^\sigma \right)^{q_3} \mathbb{KH} [\mathscr{W}(t)] (s, u, v) \\ &\quad - \left(\frac{uv}{s} \right)^{-\eta} \left(1 - \omega \left(\frac{uv}{s} \right)^\sigma \right)^{q_3} \mathscr{W} (0^+). \end{aligned}$$

□

5 Application of FEL equation

In this section, we consider the following FEL equation, which includes the HPFD operator. [14, 13].

Theorem 5.1. Consider the FEL equation:

$$\mathbf{D}_{q_1, \omega, 0^+}^{q_3, q_2, q_4} \mathcal{R}(\mathcal{T}) = \mathcal{F}(\mathcal{T}) + c_1 \mathbf{E}_{q_1, q_2, \omega, 0^+}^\delta \mathcal{R}(\mathcal{T}), \quad (5.1)$$

$$\mathbf{E}_{q_1, (1-q_4)(1-q_2), \omega, 0^+}^{-q_3(1-q_4)} \mathcal{F} \Big|_{\mathcal{T}=0^+} = c_2, \quad (5.2)$$

where $\mathcal{R} \in L_1[0, \infty)$; $q_2 \in (0, 1)$, $q_4 \in [0, 1]$; $\omega, c_1 \in \mathbb{C}$; $\mathcal{T}, q_1 > 0, c_2, q_3, \delta \geq 0$. Then, the solution of this Cauchy problem is given as:

$$\mathcal{R}(\mathcal{T}) = \sum_{j=0}^{\infty} \mathbf{E}_{q_1, q_2(2j+1), \omega, 0^+}^{q_3+j(\delta+q_3)} \mathcal{R}(\mathcal{T}) + c_2 \sum_{j=0}^{\infty} c_1^j t^{q_4(1-q_2)+q_2(2j+1)-1} \mathbf{E}_{q_1, q_4(1-q_2)+q_2(2j+1)}^{q_3-q_3q_4+j(\delta+q_3)} (\omega t^{q_1}). \quad (5.3)$$

Proof. By using \mathbb{KH} - transform on both side of the Equation (5.1), we have:

$$\mathbb{KH} \left(\mathbf{D}_{q_1, \omega, 0^+}^{q_3, q_2, q_4} \mathcal{R}(\mathcal{T}) \right) (s, u, v) = \mathcal{H}_{\mathcal{F}}(s, u, v) + c_1 \mathbb{KH} \left(\mathbf{E}_{q_1, q_2, \omega, 0^+}^\delta \mathcal{R}(\mathcal{T}) \right) (s, u, v),$$

where $\mathcal{H}_{\mathcal{R}}(s, u, v)$ and $\mathcal{H}_{\mathcal{F}}(s, u, v)$ are the $\mathcal{H} \mathcal{H}$ - transform of $\mathcal{R}(\mathcal{T})$ and $\mathcal{F}(\mathcal{T})$, respectively. Then, using (2.9), (3.6), (2.5) and (5.2), we get:

$$\mathbb{KH} \left(\mathbf{D}_{q_1, \omega, 0^+}^{q_3, q_2, q_4} \mathcal{R}(\mathcal{T}) \right) (s, u, v) = \mathcal{H}_{\mathcal{F}}(s, u, v) + c_1 \left(\frac{uv}{s} \right)^{q_2} \left(1 - \omega \left(\frac{uv}{s} \right)^{q_1} \right)^{-\delta} \mathcal{H}_{\mathcal{R}}(s, u, v).$$

Using the results of lemma 4.7, we get the following:

$$\begin{aligned} \left(\frac{uv}{s} \right)^{-q_2} \left[1 - \omega \left(\frac{uv}{s} \right)^{q_1} \right]^{q_3} \mathcal{H}_{\mathcal{R}}(s, u, v) - c_2 \left(\frac{uv}{s} \right)^{q_4(1-q_2)} \left[1 - \omega \left(\frac{uv}{s} \right)^{q_1} \right]^{q_3q_4} \\ = \mathcal{H}_{\mathcal{F}}(s, u, v) + c_1 \left(\frac{uv}{s} \right)^{q_2} \left(1 - \omega \left(\frac{uv}{s} \right)^{q_1} \right)^{-\delta} \mathcal{H}_{\mathcal{R}}(s, u, v), \end{aligned}$$

so that:

$$\mathcal{H}_{\mathcal{R}}(s, u, v) = \frac{1}{\left[1 - \frac{c_1 \left(\frac{uv}{s}\right)^{q_2} \left(1 - \omega\left(\frac{uv}{s}\right)^{q_1}\right)^{-\delta}}{\left(\frac{uv}{s}\right)^{-q_2} \left[1 - \omega\left(\frac{uv}{s}\right)^{q_1}\right]^{q_3}} \right]} \frac{\mathcal{H}_{\mathcal{F}}(s, u, v) + c_2 \left(\frac{uv}{s}\right)^{q_4(1-q_2)} \left[1 - \omega\left(\frac{uv}{s}\right)^{q_1}\right]^{q_3 q_4}}{\left(\frac{uv}{s}\right)^{-q_2} \left[1 - \omega\left(\frac{uv}{s}\right)^{q_1}\right]^{q_3}},$$

if $\left| \frac{c_1 \left(\frac{uv}{s}\right)^{q_2} \left(1 - \omega\left(\frac{uv}{s}\right)^{q_1}\right)^{-\delta}}{\left(\frac{uv}{s}\right)^{-q_2} \left[1 - \omega\left(\frac{uv}{s}\right)^{q_1}\right]^{q_3}} \right| \leq 1$, the next step is valid, i.e.

$$\begin{aligned} \mathcal{H}_{\mathcal{R}}(s, u, v) &= \sum_{j=0}^{\infty} c_1^j \left(\frac{uv}{s}\right)^{(2q_2)j} \left(1 - \omega\left(\frac{uv}{s}\right)^{q_1}\right)^{-(\delta+q_3)n} \left[\frac{\mathcal{H}_{\mathcal{F}}(s, u, v) + c_2 \left(\frac{uv}{s}\right)^{q_4(1-q_2)} \left[1 - \omega\left(\frac{uv}{s}\right)^{q_1}\right]^{q_3 q_4}}{\left(\frac{uv}{s}\right)^{-q_2} \left[1 - \omega\left(\frac{uv}{s}\right)^{q_1}\right]^{q_3}} \right], \\ &= c_2 \sum_{n=0}^{\infty} c_1^j \left(\frac{uv}{s}\right)^{q_2(2j+1)+q_4(1-q_2)} \left(1 - \omega\left(\frac{uv}{s}\right)^{q_1}\right)^{-(\delta+q_3)j+q_3 q_4 - q_3} \\ &\quad + \mathcal{H}_{\mathcal{F}}(s, u, v) \sum_{j=0}^{\infty} c_1^j \left(\frac{uv}{s}\right)^{q_2(2j+1)} \left(1 - \omega\left(\frac{uv}{s}\right)^{q_1}\right)^{-(\delta+q_3)j - q_3}. \end{aligned}$$

Finally, we obtain the required solution (5.3), by using the inverse of $\mathbb{K}\mathbb{H}$ - transform. □

Theorem 5.2. Consider the following Cauchy problem:

$${}^C \mathbf{D}_{q_1, \omega, 0^+}^{q_3, q_2, q_4} \mathcal{W}(x, t) = c_2 \frac{\partial^2}{\partial t^2} Y(x, t), \quad t > 0, x \in \mathbb{R}, \tag{5.4}$$

$$\mathcal{W}(\mathcal{X}, t)|_{t=0^+} = G(\mathcal{X}), \tag{5.5}$$

$$\lim_{\mathcal{X} \rightarrow \pm\infty} \mathcal{W}(\mathcal{X}, t) = 0, \tag{5.6}$$

where $G \in L_1[0, \infty)$; $q_2 \in (0, 1)$, $q_4 \in [0, 1]$; $\omega \in \mathbb{R}$, $c_2, q_1 > 0$, $q_3 \geq 0$, the solution of this problem, is given by:

$$\mathcal{W}(\mathcal{X}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} E_{q_1, q_2, n+1}^{q_3 n} (\omega t^{q_1}) (-c_2 t^{q_3} \eta^2)^n e^{-i\eta \mathcal{X}} G(\eta) d\eta. \tag{5.7}$$

Proof. Let $\mathcal{Y}(\eta, t)$ and $\overline{\mathcal{K}}(\eta, s, u, v)$ are the Fourier transform of $\mathcal{W}(\mathcal{X}, t)$ with respect to the time variable \mathcal{X} and the $\mathbb{K}\mathbb{H}$ - transform of $\mathcal{Y}(\eta, t)$ with respect to the time variable t respectively. So, Taking the Fourier transform on both said of Equation (5.4), we get:

$${}^C \mathbf{D}_{q_1, \omega, 0^+}^{q_3, q_2, q_4} (\mathcal{Y}(\mathcal{X}, t)) = -c_2 \eta^2 \mathcal{Y}(\mathcal{X}, t), \quad t > 0, \mathcal{X} \in \mathbb{R}. \tag{5.8}$$

Then,

$$\begin{aligned} \left(\frac{uv}{s}\right)^{-q_2} \left(1 - \omega\left(\frac{uv}{s}\right)^{q_1}\right)^{q_3} \overline{\mathcal{K}}(\eta, s, u, v) \\ - \left(\frac{uv}{s}\right)^{1-q_2} \left(1 - \omega\left(\frac{uv}{s}\right)^{q_1}\right)^{q_3} G(\eta) = -c_2 \eta^2 \mathcal{Y}(\mathcal{X}, t), \end{aligned} \tag{5.9}$$

so that:

$$\overline{\mathcal{K}}(\eta, s, u, v) = \left[\frac{\left(\frac{s}{uv}\right)}{G(\eta)} + \frac{c_2 \eta^2}{\left(\frac{uv}{s}\right)^{1-q_2} \left(1 - \omega \left(\frac{uv}{s}\right)^{q_1}\right)^{q_3} G(\eta)} \right]^{-1} = \left(\frac{uv}{s}\right) G(\eta) \frac{1}{\left(1 + \frac{c_2 \eta^2}{\left(\frac{uv}{s}\right)^{-q_2} \left(1 - \omega \left(\frac{uv}{s}\right)^{q_1}\right)^{q_3}}\right)},$$

if $\frac{c_2 \eta^2}{\left(\frac{uv}{s}\right)^{-q_2} \left(1 - \omega \left(\frac{uv}{s}\right)^{q_1}\right)^{q_3}} < 1$, the next step is valid, i.e.

$$\begin{aligned} \overline{\mathcal{K}}(\eta, s, u, v) &= \sum_{j=0}^{\infty} \left(\frac{uv}{s}\right) G(\eta) \left(\frac{-c_2 \eta^2}{\left(\frac{uv}{s}\right)^{-q_2} \left(1 - \omega \left(\frac{uv}{s}\right)^{q_1}\right)^{q_3}}\right)^j, \\ &= \sum_{j=0}^{\infty} (-c_2)^j \eta^{2j} \left(\frac{uv}{s}\right)^{1+q_2j} \left(1 - \omega \left(\frac{uv}{s}\right)^{q_1}\right)^{-q_3j} G(\eta). \end{aligned}$$

Using the \mathbb{KH} - transform, it produces:

$$\mathcal{Y}(k, t) = \sum_{j=0}^{\infty} (-c_2)^j \eta^{2j} G(\eta) E_{q_1, q_2, j+1}^{q_3, j}(\omega t^{q_1}). \quad (5.10)$$

Finally, by using the \mathbb{KH} - transform inverse formula for (5.10) we obtain the desired solution (5.7). \square

6 Conclusion

This study introduces some new results for the Khalouta integral transform for both the FHPD and its regularized form. We investigate its application in solving the FEL equation, demonstrating its effectiveness through a range of scenarios. Our study highlights the practicality of the Khalouta transform in addressing fractional differential equations, thereby enhancing its significance in mathematical analysis.

References

- [1] Om.P. Agrawal, *Fractional optimal control of a distributed system using eigenfunctions*, ASME J Comput Nonlinear Dyn **3(2)** (2008), 021204 (6 pages).
- [2] D. Baleanu, K. Diethelm, E. Scalas, J.J. Trujillo, *Fractional Calculus: Models and Numerical Methods*, Series on Complexity, Nonlinearity and Chaos, World Scientific Publishing, Boston, Mass, USA, 2012.
- [3] R. Belgacem, D. Baleanu, A. Bokhari, *Shehu Transform and Applications to Caputo-Fractional Differential Equations*, Int J Anal Appl, **6**, (2019), 917-927.
- [4] R. Belgacem, A. Bokhari, B. Sadaoui, *Shehu Transform of Hilfer-Prabhakar Fractional Derivatives and Applications on some Cauchy Type Problems*, Adv. Theory Nonlinear Anal. Appl. **5** (2021), 203–214.

- [5] R. Belgacem, A. Bokhari, D. Baleanu, S. Djilali. New generalized integral transform via Dzherbashian-Nersesian fractional operator. *An International Journal of Optimization and Control: Theories Applications (IJOCTA)*, 14(2) (2024)., 90–98. <https://doi.org/10.11121/ijocta.1449>
- [6] A. Bokhari, D. Baleanu, R. Belgacem. Regularized Prabhakar derivative for partial differential equations. *Computational Methods for Differential Equations*, 10(3) (2022)., 726-737. doi: 10.22034/cmde.2021.39677.1736
- [7] A. Bokhari, D. Baleanu, R. Belgacem, *Application of Shehu transform to Atangana-Baleanu derivatives*, *J MATH COMPUT SCI-JM*, **20**, (2019), 101–107.
- [8] A. Bokhari, R. Belgacem, S. Kumar, D. Baleanu, S. Djilali, *Projectile motion using three parameter Mittag-Leffler function calculus*, *Mathematics and Computers in Simulation* 195 (2022) 22–30.
- [9] M. Caputo, *Linear model of dissipation whose Q is almost frequency independent-II*, *Geophysical Journal of the Royal Astronomical Society*, **13**, (1967), 529–539.
- [10] V. B. L. Chaurasia and Ravi Shanker Dubey and Fethi Bin Muhammad Belgacem. Fractional radial diffusion equation analytical solution via Hankel and Sumudu transforms. *MESA*, 3 (2012): 179-188.
- [11] Dubey, V.P., Singh, J., Dubey, S., Kumar, D. (2023). Some Integral Transform Results for Hilfer-Prabhakar Fractional Derivative and Analysis of Free-Electron Laser Equation. *Iranian Journal of Science*, 47, 1333–1342.
- [12] A. Erdelyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, *Higher Transcendental Functions*, McGraw-Hill, New York. 1955.
- [13] R. Garra, R. Goreno, F. Polito, Z. Tomovski, *Hilfer-Prabhakar Derivative and Some Applications*, *Appl Math Comput.* 1, (2014);242,576-589.
- [14] V. Gill, J. Singh, Y. Singh, *Analytical solution of generalized space-time fractional advection-dispersion equation via coupling of Sumudu and Fourier transforms*, *Front Phys.* 2019.
- [15] V. G. Gupta, B. Sharma, *Application of Sumudu Transform in Reaction-Diffusion Systems and Nonlinear Waves*, *Applied Mathematical Sciences*, Vol. 4, 2010, no. 9, 435 - 446.
- [16] R. Hilfer, *Threefold introduction to fractional derivatives*, *Anomalous transport, foundations and application* Wiley-VCH, Weinheim, Germany, (2008), 17–73.
- [17] Hilfer, R., *Applications of Fractional Calculus in Physics*, World Scientific Publishing Company, Singapore-New jersey-Hong Kong (2000), 87-130.
- [18] Khalid, M., Alha, S. New generalized integral transform on Hilfer-Prabhakar fractional derivatives and its applications. *Int. J. Dynam. Control* 12, 24-31 (2024). <https://doi.org/10.1007/s40435-023-01235-0>
- [19] Khalid, M. and Alha, S., *On Hilfer-Prabhakar fractional derivatives Sawi transform and its applications to fractional differential equations*. arXiv preprint arXiv:2301.06797, (2023)
- [20] Khalouta A. A New Exponential Type Kernel Integral Transform: Khalouta Transform and Its Application. Vol. LVII (2023).

- [21] A.A. Kilbas, M. Saigo, R.K. Saxena, Generalized Mittag-Leffler function and generalized fractional calculus operators, *Integr. Transf. Spec. Funct.* 15 (1) (2004) 31-49.
- [22] A.A. Kilbas AA, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Math Stud, 2006.
- [23] Kumawat N, Shukla A, Mishra MN, Sharma R and Dubey RS (2024) Khalouta transform and applications to Caputo-fractional differential equations. *Front. Appl. Math. Stat.* 10:1351526. doi: 10.3389/fams.2024.1351526
- [24] F. Liu, P. Zhuang, K. Burrage, *Numerical methods and analysis for a class of fractional advection-dispersion models*, *Computers and Mathematics with Applications*, **64** (2012), 2990-3007.
- [25] F. Mainardi, *Fractional Calculus and Waves in Linear Visco-elasticity: An Introduction to Mathematical Models*, Imperial College Press, London, 2010.
- [26] K.S. Miller, B. Ross, *An Introduction to the Fractional Integrals and Derivatives*, Theory and Applications. New York: Willey, 1993.
- [27] K.B. Oldham, J. Spanier, *The Fractional Calculus*, Academic, New York, 1974.
- [28] S. Panchal, A. Khandagale, P. Dole, *k-Hilfer-Prabhakar Fractional Derivatives and Applications*, *Indian journal of mathematics*, **59**, (2017), 367–383.
- [29] s. Panchal, A. Khandagale, P. Dole, *Sumudu Transform of Hilfer-Prabhakar Fractional Derivatives and with Applications*, *Proceeding of National Conference on Recent Trends in Mathematics*, **1**, (2017), 60–66.
- [30] i. Podlubny, *Fractional Differential Equations*. Academic Press, San Diego. 1999.
- [31] T.R. Prabhakar, *A singular integral equation with a generalized Mittag-Leffler function in the kernel*, *Yokohama Math J*, **19** (1971), 7–15.
- [32] S.G. Samko, A.A. Kilbas, O. I. Marichev, *Fractional Integrals and Derivatives-Theory and Applications*, Amsterdam: Gordon and Breach Science Publishers, 1993.
- [33] Y. Singh, V. Gill, S. Kundu, D. Kumar, *On the Elzaki transform and its applications in fractional free electron laser equation*, *Acta Univ Sapientiae Mathematica*, **11**, (2019), 419–429.
- [34] A. Wiman A, *Über den fundamental satz in der theorie der funktionen $E_\alpha(x)$* , *Acta Math*, **29**, (1905), 191–201.