NEW DEFINITION OF $\mathcal{N}_F^{\alpha}$-LAPLACE CONFORMABLE TRANSFORM AND THEIR APPLICATIONS

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Abstract

Using the new definition of the $\mathcal{N}_F^{\alpha}$-derivative function introduced by Juan E. Nápoles Valdés and al. (2020), we provide a new definition for the $\mathcal{N}_F^{\alpha}$-Laplace transform and $\mathcal{N}_F^{\alpha}$-Laplace conformable transform. Additionally, we establish several important results related to these new transforms. We also give a new definition of convolution related to this $\mathcal{N}_F^{\alpha}$-derivative and we show that it is commutative and associative.

1. Introduction

Differentiation and integration emerged in literature during Newton's and Leibniz's eras. Fourier, who introduced integral representation in 1822, is also known as the first to define arbitrary positive order. Later, many more mathematicians studied similar fields, among them were Kommum [9], Hadamard, Riesz [[10], [11]], and others.

We note that many applications of integral transform in sciences and technology and some of the current transforms being used are the Sumudu Conformable transforms, Laplace Conformable transform and Fourier Conformable transform ([21], [22], [5]).

By adding a fractional parameter α and a modulating function $F(t,\alpha)$, the conformable Laplace transform adds more flexibility and offers a more comprehensive framework for resolving issues that extend beyond traditional integer-order systems. This modification extends the traditional Laplace transform's application to fractional calculus, which is crucial for modeling complex networks, fluid dynamics and viscoelasticity while preserving many of its fundamental characteristics.

Several results from the conformable Laplace transform are examined in this article. The properties and theorems provided essential tools for handling fractional order differential equations, integrals, and convolutions within the framework of this unique transform.

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2020 Mathematics Subject Classification. 44A05; 44A15; 44A20; 45D05, 34K38.
Key words and phrases. Conformable derivative; Laplace transform.

By extending these classical results into the realm of conformable fractional calculus, this work provides a comprehensive overview of the capabilities of the conformable Laplace transform. The results presented not only generalize classical Laplace theory but also provide powerful tools for researchers and engineers working with systems characterized by fractional dynamics.

The paper is organized as follows: Definitions and basic notions, including an overview of the $\mathcal{N}_F^\alpha$-conformable derivative and its characteristics, are given in Section 2. The primary outcomes of the $\mathcal{N}_F^\alpha$-Laplace is covered in Section 3. We demonstrated a few results of the conformable $\mathcal{N}_F^\alpha$-Laplace in section 4. To demonstrate the dependability of the $\mathcal{N}_F^\alpha$-Laplace transform approach, we solve the $\mathcal{N}_F^\alpha$-harmonic oscillator in section 5 as an example:

$$\mathcal{N}_F^{(2\alpha)}w(t) + \gamma^2 w(t) = f(t) \quad (1.1)$$
$$w(0) = 1, \mathcal{N}_F^\alpha w(0) = 0$$

where $\mathcal{N}_F^{(\alpha)}$ is the $\mathcal{N}_F^\alpha$-derivative operator and γ is a constant, and the following first-order differential equation

$$y'(t) + a(2\alpha t + 1)y(t) = 2\alpha t^3 + t^2 + (2\alpha t + 1)\sin(t)$$

In the last section, we present the general conclusion.

2. BASIC NOTIONS

Definition 2.1. [6] The function f is $\mathcal{N}_F^\alpha$-derivative at t if the quotient

$$\frac{f\left(t + h\frac{1}{F(t,\alpha)}\right) - f(t)}{h}$$

has a limit when h tends to 0 . In this case, the limit is denoted

$$\mathcal{N}_F^\alpha f(t) := \lim_{h\to 0}\frac{f\left(t + h\frac{1}{F(t,\alpha)}\right) - f(t)}{h}$$

with $\alpha \in (0,1], F(t,\alpha) \neq 0$ and $t \in [0,+\infty[$.

1. Let $F(t,\alpha) = t^{1-\alpha}$. In this case, the $\mathcal{N}_F^\alpha$-derivative coincides with $\mathcal{T}^\alpha$.

$$\mathcal{T}^\alpha f(t) = \lim_{h\to 0}\frac{f\left(t + h\frac{1}{t^{\alpha-1}}\right) - f(t)}{h}$$

defined by Khalil et al. [13].

2. Let $F(t,\alpha) = \frac{1}{e^{t-\alpha}}$. In this case, the $\mathcal{N}_F^\alpha$-derivative coincides with N_1.

$$N_1 f(t) = \lim_{h\to 0}\frac{f(t + he^{t^{-\alpha}}) - f(t)}{h}$$

defined in [7].

3. Let $F(t,\alpha) = \ln(e + t^{-\alpha})$. In this case, the $\mathcal{N}_F^\alpha$-derivative coincides with ${}^{(\alpha)}\mathcal{D}_{\frac{1}{ln}}$.

$${}^{(\alpha)}\mathcal{D}_{\frac{1}{ln}} f(t) = \lim_{h\to 0}\frac{f\left(t + \frac{h}{\ln(e + t^{-\alpha})}\right) - f(t)}{h}$$

defined in [24].

Definition 2.2. Let $0 < \alpha \leq 1$ and $f: [0, +\infty[\to \mathbb{R}$.
(1) As we say, f is $\mathcal{N}_F^{\alpha}$-differentiable on $[0, +\infty[$ if f is $\mathcal{N}_F^{\alpha}$-differentiable at every point of $[0, +\infty[$.
(2) As we say, f is n times $\mathcal{N}_F^{\alpha}$-differentiable on $[0, +\infty[$ if f is continuous, $\forall j \in \{0, \dots n\} \mathcal{N}_F^{(j\alpha)} f(t) = \mathcal{N}_F^{\alpha}(\mathcal{N}_F^{\alpha} \dots (\mathcal{N}_F^{\alpha}(f)))(t)$, j times, exist for all $t \in]0, +\infty[$ and $\mathcal{N}_F^{(j\alpha)} f(0) = \lim_{t \to 0^+} \mathcal{N}_F^{(j\alpha)} f(t)$ exists.

Theorem 2.3. [6] Let α be in $(0,1]$ and $f, g: [0, +\infty) \to \mathbb{R}$ $\mathcal{N}_F^{\alpha}$-differentiable.

(1) $\mathcal{N}_F^{\alpha}(af + bg)(t) = a\mathcal{N}_F^{\alpha}(f)(t) + b\mathcal{N}_F^{\alpha}(g)(t), a, b \in \mathbb{R}$.

(2) $\mathcal{N}_F^{\alpha}(\lambda) = 0, \lambda \in \mathbb{R}$.

(3) $\mathcal{N}_F^{\alpha}(fg)(t) = \mathcal{N}_F^{\alpha}(f)(t)g(t) + f(t)\mathcal{N}_F^{\alpha}(g)(t)$.

(4) $\mathcal{N}_F^{\alpha}\left(\frac{f}{g}\right)(t) = \frac{g(t)\mathcal{N}_F^{\alpha}(f)(t) - f(t)\mathcal{N}_F^{\alpha}(g)(t)}{g^2(t)}$.

(5) Additionally, $\mathcal{N}_F^{\alpha}(f)(t) = \frac{f'(t)}{F(t,\alpha)}$ if f is differentiable , with $t > 0$.

Example 2.4:

1. Let f and $F(t, \alpha)$ be two real functions defined on $[0, +\infty[$ by $f(t) = e^t$ and $F(t, \alpha) = \frac{1}{ch(\alpha t)}$, where $\alpha \in (0,1]$. Then

$$\mathcal{N}_F^{\alpha} f(t) := \operatorname{ch}(\alpha t) e^t = \frac{e^{(1+\alpha)t} + e^{(1-\alpha)t}}{2}$$

for all $t > 0$. 2. Let f and $F(t, \alpha)$ be two real functions defined on $[0, +\infty[$ by $f(t) = \frac{\alpha}{2}t^4 + \frac{t^3}{3} + 2\alpha t^2 + 2t$ and $F(t, \alpha) = 2\alpha t + 1$, where $\alpha \in (0,1]$. Then

$$\mathcal{N}_F^{\alpha} f(t) := t^2 + 2$$

for all $t > 0$.
Definition 2.5. ($\mathcal{N}_F^{\alpha}$-integral)
Let f be a real function taking its values in a segment $[0, t]$ and $\alpha \in (0,1]$. Then the $\mathcal{N}_F^{\alpha}$-integral of f on $[0, t]$, defined and denoted

$$\mathcal{J}_F^{\alpha} f(t) = \int_0^t F(\nu, \alpha) f(\nu) d\nu, t \in [0, +\infty[$$

Example 2.6. Let $F(t, \alpha) = 2\alpha t + 1$. Then

$$\mathcal{J}_F^{\alpha}(e^t) = e^t - 1 + 2\alpha(te^t - e^t + 1)$$

Lemma 2.7. [6] Consider the continuous function $f: [0, +\infty) \to \mathbb{R}$ with $0 < \alpha \leq 1$. Then, for all $t \in (0, +\infty)$

$$\mathcal{N}_F^{\alpha}(\mathcal{J}_F^{\alpha}(f))(t) = f(t)$$

Example 2.8. Consider $F(t, \alpha) = 2\alpha t + 1$. By Example 2.6 and Theorem 2.3

$$\mathcal{N}_F^{\alpha}(\mathcal{J}_F^{\alpha}(e^t)) = \frac{[e^t - 1 + 2\alpha(te^t - e^t + 1)]'}{2\alpha t + 1} = e^t$$

Lemma 2.9. [6] Consider $\alpha \in (0,1]$ and the $\mathcal{N}_F^{\alpha}$-differentiable $f: [0, +\infty) \to \mathbb{R}$.

$$\mathcal{J}_F^{\alpha}(\mathcal{N}_F^{\alpha} f)(t) = f(t) - f(0), t > 0$$

Consider the following continuous function $F(t,\alpha)$ such that $F(t,\alpha) > 0$ for all $t > 0$ and $G_\alpha(t)$ its primitive function, verifies $G_\alpha(0) = 0$ and $\lim_{t\to+\infty} G_\alpha(t) = +\infty$, where $0 < \alpha \leq 1$. For example

$$F(t,\alpha) = \text{ch}(\alpha t) \text{ and } G_\alpha(t) = \frac{sh(\alpha t)}{\alpha}$$

We will demonstrate various findings and evidence of $\mathcal{N}_F^{\alpha}$-Laplace transform.

3. $\mathcal{N}_F^{\alpha}$-LAPLACE TRANSFORM

Definition 3.1. Let us consider $\alpha \in (0,1]$ and the function $f: [0,\infty) \to \mathbb{R}$. The definition of the $\mathcal{N}^{\alpha}_F$ Laplace transform of f is

$$\mathcal{L}_F^{\alpha}(f(t))(s) = \int_0^{\infty} e^{-sG\alpha(t)} f(t) F(t,\alpha) dt \qquad (3.1)$$

assuming the integral converges .

Example 3.2. Let $0 < \alpha \leq 1, F(t,\alpha) = t^{\alpha-1}, G_\alpha(t) = \frac{t^\alpha}{\alpha}$ and $f(t) = e^{-\frac{t^\alpha}{\alpha}}$. Then

$$\mathcal{L}_F^{\alpha}(f(t))(s) = \frac{1}{1+s}, s > -1$$

Example 3.3. Let $0 < \alpha \leq 1 F(t,\alpha) = 2\alpha t + 1, G_\alpha(t) = \alpha t^2 + t$ and $f(t) = t^2$. For $t \in [0, +\infty[$

$$\begin{aligned}
\mathcal{L}_F^{\alpha}(f(t))(s) &= \int_0^{\infty} e^{-sG\alpha(t)} f(t) F(t,\alpha) dt \\
&= \int_0^{\infty} e^{-s(\alpha t^2 + t)} f(t)(2\alpha t + 1) dt \\
&= \int_0^{\infty} e^{-su} f\left(\frac{1}{2\alpha}(\sqrt{4\alpha u + 1} - 1)\right) du \\
&= \int_0^{\infty} e^{-su} \left(\frac{1}{2\alpha^2} + \frac{1}{\alpha} u - \frac{1}{2\alpha^2}\sqrt{4\alpha u + 1}\right) du
\end{aligned}$$

on the other hand, for $s > 0$

$$\int_0^{\infty} e^{-su}\left(\frac{1}{2\alpha^2} + \frac{1}{\alpha} u\right) du = \frac{1}{2\alpha^2 s} + \frac{1}{\alpha s^2}$$

and using variable change $v = \sqrt{4\alpha u + 1}$ and integrating by parts, we find that

$$\int_0^{\infty} e^{-su}\sqrt{4\alpha u+1}du = \frac{e^{\frac{s}{4\alpha}}}{2\alpha}\int_1^{\infty} e^{-\frac{s}{4\alpha}t^2}t^2dt$$
$$= \frac{e^{\frac{s}{4\alpha}}}{2\alpha}[\frac{2\alpha}{s}e^{\frac{-s}{4\alpha}} + \frac{2\alpha}{s}\int_1^{+\infty} e^{\frac{-s}{4\alpha}t^2}dt]$$
$$= \frac{1}{s} + \frac{e^{\frac{s}{4\alpha}}}{s}[\sqrt{\frac{\alpha\pi}{s}} - \sqrt{\frac{\alpha\pi}{s}}\operatorname{erf}(\sqrt{\frac{s}{4\alpha}})]$$

then

$$\mathcal{L}_F^{\alpha}(t^2)(s) = \frac{1}{\alpha s^2} - \frac{1}{2\alpha^2}(\frac{e^{\frac{s}{4\alpha}}}{s}[\sqrt{\frac{\alpha\pi}{s}} - \sqrt{\frac{\alpha\pi}{s}}\operatorname{erf}(\sqrt{\frac{s}{4\alpha}})])$$

where erf is the error function.

Theorem 3.4. Let $\mu, k \in \mathbb{R}$ and $0 < \alpha \leq 1$. Then

(1)

$$\mathcal{L}_F^{\alpha}\{k\}(s) = \frac{k}{s}, s > 0$$

(2)

$$\mathcal{L}_F^{\alpha}\{e^{\mu G_\alpha(t)}\}(s) = \frac{1}{s-\mu}, s > \mu$$

(3)

$$\mathcal{L}_F^{\alpha}\{\sin(\mu G_\alpha(t))\}(s) = \frac{\mu}{s^2+\mu^2}, s > 0$$

(4)

$$\mathcal{L}_F^{\alpha}\{\cos(\mu G_\alpha(t))\}(s) = \frac{s}{s^2+\mu^2}, s > 0$$

(5)

$$\mathcal{L}_F^{\alpha}\{\sinh(\mu G_\alpha(t))\}(s) = \frac{\mu}{s^2-\mu^2}, s > |\mu|$$

(6)

$$\mathcal{L}_F^{\alpha}\{\cosh(\mu G_\alpha(t))\}(s) = \frac{s}{s^2-\mu^2}, s > |\mu|$$

Proof. In all properties, we use the variable change $u = G_\alpha(t)$.

1. For $s > 0$, we have

$$\mathcal{L}_F^{\alpha}\{k\}(s) = \int_0^{+\infty} e^{-sG_\alpha(t)}kF(t,\alpha)dt = k\int_0^{+\infty} e^{-su}du = k\,[-\frac{1}{s}e^{-su}]_0^{+\infty} = \frac{k}{s}$$

2. For $s > \mu$, we have

$$\mathcal{L}_F^\alpha\{e^{\mu G_\alpha(t)}\}(s) = \int_0^{+\infty} e^{-(s-\mu)u}du = [-\frac{1}{s-\mu}e^{-(s-\mu)u}]_0^{+\infty} = \frac{1}{s-\mu}$$

3. Let $s > 0$.

$$\mathcal{L}_F^\alpha\{\sin(\mu G_\alpha(t))\}(s) = \int_0^{+\infty} e^{-su}\sin(\mu u)du = \frac{\mu}{s^2+\mu^2}$$

4. Let $s > 0$.

$$\mathcal{L}_F^\alpha\{\cos(\mu G_\alpha(t))\}(s) = \int_0^{+\infty} e^{-su}\cos(\mu u)du = \frac{s}{s^2+\mu^2}$$

5. For $s > |\mu|$, we have

$$\begin{aligned}\mathcal{L}_F^\alpha\{\sinh(\mu G_\alpha(t))\}(s) &= \int_0^{+\infty} e^{-sG_\alpha(t)}\sinh(\mu G_\alpha(t))F(t,\alpha)dt \\ &= \int_0^{+\infty} e^{-su}\frac{e^{\mu u}-e^{-\mu u}}{2}du \\ &= \frac{1}{2}(\int_0^{+\infty} e^{-(s-\mu)u}du - \int_0^{+\infty} e^{-(s+\mu)u}du) \\ &= \frac{1}{2}(\frac{1}{s-\mu} - \frac{1}{s+\mu}) = \frac{\mu}{s^2-\mu^2}\end{aligned}$$

6. For $s > |\mu|$, we have

$$\begin{aligned}\mathcal{L}_F^\alpha\{\cosh(\mu G_\alpha(t))\}(s) &= \int_0^{+\infty} e^{-sG_\alpha(t)}\cosh(\mu G_\alpha(t))F(t,\alpha)dt \\ &= \int_0^{+\infty} e^{-su}\frac{e^{\mu u}+e^{-\mu u}}{2}du \\ &= \frac{1}{2}(\int_0^{+\infty} e^{-(s-\mu)u}du + \int_0^{+\infty} e^{-(s+\mu)u}du) \\ &= \frac{1}{2}(\frac{1}{s-\mu} + \frac{1}{s+\mu}) = \frac{s}{s^2-\mu^2}\end{aligned}$$

Theorem 3.5. Let us consider $\alpha \in (0,1]$ and the function $f: [0,\infty) \to \mathbb{R}$. Then

$$\mathcal{L}_F^\alpha(f(t))(s) = \mathcal{L}(f(G_\alpha^{-1}(u))(s), s > 0$$

Proof. Applying Definition 3.1 and letting $G_\alpha(t) = u$

$$\begin{aligned}\mathcal{L}_F^\alpha(f(t))(s) &= \int_0^{\infty} e^{-sG_\alpha(t)}f(t)F(t,\alpha)dt \\ &= \int_0^{\infty} e^{-su}f(G_\alpha^{-1}(u))du = \mathcal{L}(f(G_\alpha^{-1}(u))(s)\end{aligned}$$

Example 3.6. Let $0 < \alpha \leq 1, F(t,\alpha) = \frac{t^{\alpha-1}}{\mathrm{ch}\,(\frac{t^\alpha}{\alpha})}, G_\alpha(t) = \mathrm{sh}\,(\frac{t^\alpha}{\alpha})$. Then

$$\mathcal{L}_F^\alpha(f(t))(s) = \mathcal{L}\,(f\,(\mathrm{argsh}\,((\alpha u)^{\frac{1}{\alpha}}))\,(s)$$

Theorem 3.7. Asume that $f:[0,\infty)\to\mathbb{R}$ is a continuously $\mathcal{N}_F^\alpha$-differentiable function and $0<\alpha\leq 1$. Then

$$\mathcal{L}_F^\alpha(\mathcal{N}_F^\alpha f(t)))(s) = s\mathcal{L}_F^\alpha(f(t))(s) - f(0), s > 0$$

Proof. Applying Definition 3.1 and Theorem 2.3

$$\mathcal{L}_F^\alpha(\mathcal{N}_F^\alpha f(t))(s) = \int_0^\infty e^{-sG\alpha(t)} f'(t)dt$$

Through part-by-part integration, we have:

$$\int_0^\infty e^{-sG\alpha(t)} f'(t)dt = -f(0) + s\int_0^\infty e^{-sG\alpha(t)} f(t)F(t,\alpha)dt$$

then

$$\mathcal{L}_F^\alpha(\mathcal{N}_F^\alpha f(t))(s) = s\mathcal{L}_F^\alpha(f(t))(s) - f(0)$$

Example 3.8. Let $F(t,\alpha)=\alpha, G_\alpha(t)=\alpha t$ and $f(t)=t^2$.
We have, for all $s>0$ and $t>0$

$$\mathcal{L}_F^\alpha(t^2)(s) = \frac{2}{\alpha^2 s^3}$$

and

$$\mathcal{L}_F^\alpha(\mathcal{N}_F^\alpha(t^2))(s) = \mathcal{L}_F^\alpha\left(\frac{2}{\alpha}t\right)(s) = \frac{2}{\alpha^2 s^2}$$

then

$$\mathcal{L}_F^\alpha(\mathcal{N}_F^\alpha f(t))(s) = s\mathcal{L}_F^\alpha(f(t))(s) - f(0)$$

Theorem 3.9. Let f be defined in $[0,+\infty[]$ as a n times continuously $\mathcal{N}_F^\alpha$ differentiable function. We have

$$\mathcal{L}_F^\alpha(\mathcal{N}_F^{(n\alpha)}(f(t)))(s) = s^n\mathcal{L}_F^\alpha(f(t))(s) - \sum_{k=0}^{n-1} s^k\mathcal{N}_F^{((n-1-k)\alpha)}f(0)$$

Proof. For $n=1$, see Theorem 3.7.
Suppose that

$$\mathcal{L}_F^\alpha(\mathcal{N}_F^{(n\alpha)}(f(t))(s)) = s^n\mathcal{L}_F^\alpha(f(t))(s) - \sum_{k=0}^{n-1} s^k\mathcal{N}_F^{((n-1-k)\alpha)}f(0)$$

and prove that

$$\mathcal{L}_F^\alpha(\mathcal{N}_F^{(n\alpha)}(f(t))(s)) = s^n\mathcal{L}_F^\alpha(f(t))(s) - \sum_{k=0}^{n-1} s^k\mathcal{N}_F^{((n-1-k)\alpha)}f(0)$$

By Theorem 3.7, we have

$$\mathcal{L}_F^{\alpha}(\mathcal{N}_F^{((n+1)\alpha)}(f(t))(s) = \mathcal{L}_F^{\alpha}(\mathcal{N}_F^{(\alpha)}[\mathcal{N}_F^{(n\alpha)}f(t))](s)$$
$$= s\mathcal{L}_F^{\alpha}(\mathcal{N}_F^{(n\alpha)}(f(t))(s) - \mathcal{N}_F^{(n\alpha)}(f(0)$$
$$= s\ [s^n\mathcal{L}_F^{\alpha}(f(t))(s) - \sum_{k=0}^{n-1} s^k\mathcal{N}_F^{((n-1-k)\alpha)}f(0)] - \mathcal{N}_F^{(n\alpha)}f(0)$$
$$= s^{n+1}\mathcal{L}_F^{\alpha}(f(t))(s) - \sum_{k=0}^{n-1} s^{k+1}\mathcal{N}_F^{((n-1-k)\alpha)}f(0)] - \mathcal{N}_F^{(n\alpha)}f(0)$$
$$= s^{n+1}\mathcal{L}_F^{\alpha}(f(t))(s) - \sum_{k=0}^{n} s^k\mathcal{N}_F^{((n-k)\alpha)}f(0)$$

Theorem 3.10. Assume that $0 < \alpha \leq 1$ and that $f: [0, \infty) \to \mathbb{R}$ is a specified continuous function. Then

$$\mathcal{L}_F^{\alpha}\left(\overset{\alpha}{\mathcal{I}_F} f(t)\right)(s) = \frac{1}{s}\overset{\alpha}{\mathcal{L}_F}(f(t))(s), s > 0.$$

Proof. Lemma 2.7 gives us

$$\mathcal{L}_F^{\alpha}(\mathcal{N}_F^{\alpha}\mathcal{I}_F^{\alpha}f(t))(s) = \mathcal{L}_F^{\alpha}(f(t))(s)$$

and according to theorem 3.7

$$\mathcal{L}_F^{\alpha}(\mathcal{N}_F^{\alpha}\mathcal{I}_F^{\alpha}f(t))(s) = s\mathcal{L}_F^{\alpha}(\mathcal{I}_F^{\alpha}(f)(t))(s) - \mathcal{I}_F^{\alpha}(f)(0) = s\mathcal{L}_F^{\alpha}(\mathcal{I}_F^{\alpha}(f)(t))(s)$$

So

$$\mathcal{L}_F^{\alpha}\left(\overset{\alpha}{\mathcal{I}_F} f(t)\right)(s) = \frac{1}{s}\overset{\alpha}{\mathcal{L}_F}(f(t))(s), \forall s > 0$$

Example 3.11. Let $G_\alpha(t), F(t,\alpha)$ and $f(t)$ defined in Example 3.8.
We have

$$\mathcal{L}_F^{\alpha}(t^2)(s) = \frac{2}{\alpha^2 s^3}$$

and

$$\overset{\alpha}{\mathcal{L}_F}(\overset{\alpha}{\mathcal{I}_F}(t^2))(s) = \overset{\alpha}{\mathcal{L}_F}\left(\frac{\alpha}{3}t^3\right)(s) = \frac{2}{\alpha^2 s^4}$$

then

$$\mathcal{L}_F^{\alpha}(\mathcal{I}_F^{\alpha}t^2)(s) = \frac{1}{s}\mathcal{L}_F^{\alpha}(t^2)(s).$$

Theorem 3.12. Consider two differentiable functions, $G_1^{\alpha}(t)$ and $G_2^{\alpha}(t)$, such that $G_1'^{\alpha}(t) = F_1(t,\alpha)$ and $G_2'^{\alpha}(t) = F_2(t,\alpha)$. Then

$$\overset{\alpha}{\mathcal{L}}_{F_1+F_2}(f(t))(s) = \overset{\alpha}{\mathcal{L}}_{F_1}(e^{-sG_2^{\alpha}(t)} f(t))(s) + \overset{\alpha}{\mathcal{L}}_{F_2}(e^{-sG_1^{\alpha}(t)} f(t))(s)$$

where $\alpha \in (0,1]$.
Proof. Definition 3.1 states that we have

$$\mathcal{L}^{\alpha}_{F_1+F_2}(f(t))(s)=\int_0^{\infty} e^{-s(G_1^{\alpha}(t)+G_2^{\alpha}(t))} f(t)(F_1(t,\alpha)+F_2(t,\alpha))dt$$
$$=\int_0^{\infty} e^{-sG_1^{\alpha}(t)}\,(e^{-sG_2^{\alpha}(t)} f(t))F_1(t,\alpha)dt+\int_0^{\infty} e^{-sG_2^{\alpha}(t)}\,(e^{-sG_1^{\alpha}(t)} f(t))F_2(t,\alpha)dt$$
$$=\mathcal{L}^{\alpha}_{F_1}(e^{-sG_2^{\alpha}(t)} f(t))(s)+\mathcal{L}^{\alpha}_{F_2}(e^{-sG_1^{\alpha}(t)} f(t))(s)$$

Example 3.13.

Let $0<\alpha\leq 1, f(t)=t^2$,

$$G_1^{\alpha}(t)=\begin{cases}\alpha-\alpha t, 0\leq t<1\\ 0, t\geq 1\end{cases}, F_1(t,\alpha)=\begin{cases}-\alpha, 0\leq t<1\\ 0, t\geq 1\end{cases} \tag{3.2}$$

and

$$G_2^{\alpha}(t)=\begin{cases}0, 0\leq t<1\\ \alpha t+\alpha, t\geq 1\end{cases}, F_2(t,\alpha)=\begin{cases}0, 0\leq t\leq 1\\ \alpha, t>1\end{cases}$$

functions, we have

$$\mathcal{L}^{\alpha}_{F_1+F_2}(f(t))(s)=\int_0^{+\infty} e^{-\alpha s|s-1|}t^2F(t,\alpha)dt$$
$$=-\alpha e^{-\alpha s}\int_0^1 e^{(\alpha s)t}t^2dt+\alpha e^{\alpha s}\int_1^{+\infty} e^{-(\alpha s)t}t^2dt$$
$$=\mathcal{L}^{\alpha}_{F_1}(e^{-sG_2^{\alpha}(t)}(t^2)(s)+\mathcal{L}^{\alpha}_{F_2}(e^{-sG_1^{\alpha}(t)}(t^2)(s)$$
$$=\frac{4}{\alpha s^2}+\frac{2}{\alpha^2 s^3}e^{-\alpha s}$$

We have for $t\in[0,+\infty[, G_1^{\alpha}(t)+G_2^{\alpha}(t)=\alpha|t-1|, F_1(t,\alpha)+F_2(t,\alpha)=\alpha\text{sign}(t-1)$

Theorem 3.14. Let $n\geq 2$ and $i\in\{1,2,\dots,n\}$. Asume that $G_i^{\alpha}(t)$ derivables

$$\mathcal{L}^{\alpha}_{(\sum_{i=1}^n F_i)}(f(t))(s)=\sum_{i=1}^{n}\mathcal{L}^{\alpha}_{F_i}(e^{-s[\sum_{j=1,j\neq i}^n G_j^{\alpha}(t)]}f(t))\,(s)$$

where $\alpha\in(0,1]$ and $G_i'^{\alpha}(t)=F_i(t,\alpha)$.

$$F(t,\alpha)=F_1(t,\alpha)+F_2(t,\alpha)=\begin{cases}-\alpha, 0\le t<1\\ \alpha, t>1.\\ 0, t=1,\end{cases}$$

$$\mathcal{L}^{\alpha}_{F_1}\,(e^{-sG^{\alpha}_2(t)}\,(t^2)(s)=-\alpha\int_0^1 e^{-s(\alpha-\alpha t)}t^2dt$$

$$=-\alpha e^{-\alpha s}\int_0^1 e^{(\alpha s)t}t^2dt$$

$$=-\frac{1}{s}+\frac{2}{\alpha s^2}-\frac{2}{\alpha^2 s^3}+\frac{2}{\alpha^2 s^3}e^{-\alpha s}.$$

$$\mathcal{L}^{\alpha}_{F_2}\,(e^{-sG^{\alpha}_1(t)}\,(t^2)(s)=\alpha\int_1^{+\infty} e^{-s(\alpha t-\alpha)}t^2dt$$

$$=\alpha e^{\alpha s}\int_1^{+\infty} e^{-(\alpha s)t}t^2dt$$

$$=\frac{1}{s}+\frac{2}{\alpha s^2}+\frac{2}{\alpha^2 s^3}.$$

Proof. demonstration by recurrence.
For $n=2$, see Theorem 3.12.
Suppose that

$$\mathcal{L}^{\alpha}_{(\sum_{i=1}^{n}F_i)}(f(t))(s)=\sum_{i=1}^{n}\mathcal{L}^{\alpha}_{F_i}(e^{-s[\sum_{j=1,j\neq i}^{n}G^{\alpha}_j(t)]}f(t))\,(s)$$

and prove that

$$\mathcal{L}^{\alpha}_{(\sum_{i=1}^{n+1}F_i)}(f(t))(s)=\sum_{i=1}^{n+1}\mathcal{L}^{\alpha}_{F_i}(e^{-s[\sum_{j=1,j\neq i}^{n+1}G^{\alpha}_j(t)]}f(t))\,(s)$$

We have, by Theorem 3.12 and recurrence hypotheses

$$\mathcal{L}^{\alpha}_{(\sum_{i=1}^{n+1}F_i)}(f(t))(s)=\mathcal{L}^{\alpha}_{(F_{n+1}+\sum_{i=1}^{n}F_i)}(f(t))(s)$$

$$=\mathcal{L}^{\alpha}_{F_{n+1}}(e^{-s[\sum_{j=1}^{n}G^{\alpha}_j(t)]}f(t))\,(s)+\mathcal{L}^{\alpha}_{(\sum_{i=1}^{n}F_i)}(e^{-sG^{\alpha}_{n+1}(t)}f(t))(s)$$

$$=\mathcal{L}^{\alpha}_{F_{n+1}}\,(e^{-s[\sum_{j=1,j\neq n+1}^{n+1}G^{\alpha}_j(t)]}f(t))\,(s)+\sum_{i=1}^{n}\mathcal{L}^{\alpha}_{F_i}\,(e^{-s[\sum_{j=1,j\neq i}^{n}G^{\alpha}_j(t)]}\,(e^{-sG^{\alpha}_{n+1}(t)}f(t)))\,(s)$$

$$=\mathcal{L}^{\alpha}_{F_{n+1}}\,(e^{-s[\sum_{j=1,j\neq n+1}^{n+1}G^{\alpha}_j(t)]}f(t))\,(s)+\sum_{i=1}^{n}\mathcal{L}^{\alpha}_{F_i}\,(e^{-s[\sum_{j=1,j\neq i}^{n+1}G^{\alpha}_j(t)]}f(t))\,(s)$$

$$=\sum_{i=1}^{n+1}\mathcal{L}^{\alpha}_{F_i}(e^{-s[\sum_{j=1,j\neq i}^{n+1}G^{\alpha}_j(t)]}f(t))\,(s)$$

Next, we give a new definition of convolution and we show that it is commutative and associative.
Definition 3.15. The $\mathcal{N}^{\alpha}_F$-convolution product of two functions f and g, is another function, which is generally noted $f *^{\alpha}_F g$ and which is defined by:

$$(f *^{\alpha}_F g)(t)=\int_0^t f(s)g[G^{-1}_{\alpha}(G_{\alpha}(t)-G_{\alpha}(s))]F(s,\alpha)ds, t\ge 0$$

Proposition 3.16. The $\mathcal{N}_F^{\alpha}$-convolution product is commutative:

$$(f *_F^\alpha g) = (g *_F^\alpha f)$$

Proof. This is easily seen by operating the following change of variable: $= G_\alpha^{-1}(G_\alpha(t) - G_\alpha(s))$,

$$(f *_F^\alpha g)(t) = \int_0^t f(s) g[G_\alpha^{-1}(G_\alpha(t) - G_\alpha(s))] F(s,\alpha) ds$$

$$= \int_0^t f[G_\alpha^{-1}(G_\alpha(t) - G_\alpha(u))] g(u) F(u,\alpha) du$$

$$= (g *_F^\alpha f)(t)$$

Example 3.17. For $F(t,\alpha) = \alpha$ and $G_\alpha(t) = \alpha t + 1$.
and
where $\alpha \in (0,1]$ and $f(t) = t^2$ and $g(t) = t$.
Thus
Note that condition $G_\alpha(0) = 0$ is not verified, so it is necessary.
Example 3.18. For $F(t,\alpha) = \alpha$, $G_\alpha(t) = \alpha t$, we have $G_\alpha(0) = 0$,
and

$$(f *_F^\alpha g)(t) \neq (g *_F^\alpha f)(t).$$

$$(g *_F^\alpha f)(t) = \frac{\alpha}{12} t^4$$

where $\alpha \in (0,1]$ and $f(t) = t^2$ and $g(t) = t$.
Thus

$$(f *_F^\alpha g)(t) = \frac{\alpha}{12} t^4 - \frac{1}{3} t^3$$

$$(g *_F^\alpha f)(t) = \frac{\alpha}{12} t^4 + \frac{1}{2\alpha} t^2 - \frac{1}{3} t^3$$

$$(f *_F^\alpha g)(t) = \frac{\alpha}{12} t^4$$

$$(f *_F^\alpha g)(t) = (g *_F^\alpha f)(t).$$

Proposition 3.19. The $\mathcal{N}_F^{\alpha}$-convolution product is associative:

$$(f *_F^\alpha g) *_F^\alpha h = f *_F^\alpha (g *_F^\alpha h).$$

Proof. The $\mathcal{N}_F^{\alpha}$-convolution product is associative when considering integrable functions

$$[(f *_F^\alpha g) *_F^\alpha h](t) = \int_0^t (f *_F^\alpha g)(s) h[G_\alpha^{-1}(G_\alpha(t) - G_\alpha(s))] F(s,\alpha) ds$$

$$= \int_0^t \left[\int_0^s f(u) g[G_\alpha^{-1}(G_\alpha(s) - G_\alpha(u))] F(u,\alpha) du\right] h[G_\alpha^{-1}(G_\alpha(t) - G_\alpha(s))] F(s,\alpha) ds$$

$$= \int_0^t \int_0^s f(u) g[G_\alpha^{-1}(G_\alpha(s) - G_\alpha(u))] h[G_\alpha^{-1}(G_\alpha(t) - G_\alpha(s)) F(u,\alpha) F(s,\alpha) du ds$$

$$= \int_0^t f(u) \left[\int_u^t g[G_\alpha^{-1}(G_\alpha(s) - G_\alpha(u))] h[G_\alpha^{-1}(G_\alpha(t) - G_\alpha(s)) F(s,\alpha) ds\right] F(u,\alpha) du$$

by making a change of variable $v = G_\alpha^{-1}(G_\alpha(s) - G_\alpha(u))$, we obtain

$$\begin{aligned}
&(f *_F^\alpha g) *_F^\alpha h(t) \\
&= \int_0^t f(u)\, [\int_0^{G_\alpha^{-1}(G_\alpha(t)-G_\alpha(u))} g(v)h[G_\alpha^{-1}(G_\alpha(t) - G_\alpha(v) - G_\alpha(u))F(v,\alpha)dv]F(u,\alpha)du \\
&= \int_0^t f(u)(g *_F^\alpha h)(G_\alpha^{-1}(G_\alpha(t) - G_\alpha(u)))F(u,\alpha)du \\
&= [f *_F^\alpha (g *_F^\alpha h)](t)
\end{aligned}$$

Thus

$$(f *_F^\alpha g) *_F^\alpha h = f *_F^\alpha (g *_F^\alpha h).$$

Theorem 3.20. The $\mathcal{N}^\alpha$-Laplace transform changes the $\mathcal{N}^\alpha$-convolution product into a product:

$$\mathcal{L}_F^\alpha((f *_F^\alpha g)(t))(s) = \mathcal{L}_F^\alpha(f(t))(s) \cdot \mathcal{L}_F^\alpha(g(t))(s).$$

Proof. We have

$$\begin{aligned}
\mathcal{L}_F^\alpha((f *_F^\alpha g)(t))(s) &= \int_0^{+\infty} e^{-sG\alpha(t)}(f *_F^\alpha g)(t)F(t,\alpha)dt \\
&= \int_0^{+\infty} e^{-sG_\alpha(t)}\, [\int_0^t f(u)g[G_\alpha^{-1}(G_\alpha(t) - G_\alpha(u))]F(u,\alpha)du]\, F(t,\alpha)dt \\
&= \int_0^{+\infty} f(u)F(u,\alpha)\, [\int_u^{+\infty} e^{-sG\alpha(t)}g[G_\alpha^{-1}(G_\alpha(t) - G_\alpha(u))]F(t,\alpha)dt)]\, du
\end{aligned}$$

By variable making change $v = G_\alpha^{-1}(G_\alpha(t) - G_\alpha(s))$,

$$\begin{aligned}
\mathcal{L}_F^\alpha((f *_F^\alpha g)(t))(s) &= \int_0^{+\infty} f(u)F(u,\alpha)\, [\int_0^{+\infty} e^{-s(G\alpha(v)+G_\alpha(u))}g(v)F(v,\alpha)dv]\, du \\
&= \int_0^{+\infty} f(u)e^{-sG\alpha(u)}F(u,\alpha)du \cdot \int_0^{+\infty} f(v)e^{-sG\alpha(v)}F(v,\alpha)dv \\
&= \mathcal{L}_F^\alpha(f(t))(s) \cdot \mathcal{L}_F^\alpha(g(t))(s)
\end{aligned}$$

Example 3.21. Consider the functions defined in Example 3.18.

We have $(g *_F^\alpha f)(t) = (f *_F^\alpha g)(t) = \frac{\alpha}{12}t^4$ and

$$\mathcal{L}_F^\alpha((f *_F^\alpha g)(t))(s) = \mathcal{L}_F^\alpha\, (\frac{\alpha}{12}t^4)\, (s) = \frac{2}{\alpha^3 s^5}$$

However

$$\mathcal{L}_F^\alpha(f(t))(s) = \mathcal{L}_F^\alpha(t^2)(s) = \frac{2}{\alpha^2 s^3}$$

and

$$\mathcal{L}_F^{\alpha}(g(t))(s) = \mathcal{L}_F^{\alpha}(t)(s) = \frac{1}{\alpha s^2}$$

And thus

$$\mathcal{L}_F^{\alpha}((f *_F^{\alpha} g)(t))(s) = \mathcal{L}_F^{\alpha}(f(t))(s) \cdot \mathcal{L}_F^{\alpha}(g(t))(s)$$

4. $\mathcal{N}_F^{\alpha}$-Laplace Conformable transform

Definition 4.1. Consider the function f defined on $[0, +\infty)$. The definition of the $\mathcal{N}_F^{\alpha}$-Laplace Conformable transform of f is

$$_c\mathcal{L}_F^{\alpha}\{f(t)\}(s) = \int_0^{\infty} e^{-sG_{\alpha}(\frac{t^{\alpha}}{\alpha})} f(t)F\left(\frac{t^{\alpha}}{\alpha}, \alpha\right) t^{\alpha-1}dt \quad (4.1)$$

assuming the integral converges .

Theorem 4.2. Consider the function f defined on $[0, +\infty)$. Then

$$_c\mathcal{L}_F^{\alpha}\{f(t)\}(s) = \mathcal{L}_F^{\alpha}\{f((\alpha t)^{\frac{1}{\alpha}})\}(s)$$

Proof. by applications of definitions 4.1 and letting $v = \frac{t^{\alpha}}{\alpha}$ we have:

$$_c\mathcal{L}_F^{\alpha}\{f(t)\}(s) = \int_0^{\infty} e^{-sG_{\alpha}(v)} f((\alpha v)^{\frac{1}{\alpha}}) F(v, \alpha)dv$$

by Definition of $\mathcal{N}_F^{\alpha}$-Laplace transform of f

$$_c\mathcal{L}_F^{\alpha}\{f(t)\}(s) = \mathcal{L}_F^{\alpha}\{f((\alpha t)^{\frac{1}{\alpha}})\}(s)$$

Theorem 4.3. Consider the $\mathcal{N}_F^{\alpha}$-differentiable function f and $\alpha \in (0,1]$. Then

$$_c\mathcal{L}_F^{\alpha}[\mathcal{N}_F^{\alpha}f(t)](s) = s\ _c\mathcal{L}_F^{\alpha}(f(t))(s) - f(0), s > 0$$

Proof. Let us apply the previous theorem, then

$$_c\mathcal{L}_F^{\alpha}[\mathcal{N}_F^{\alpha}f(t)](s) = \mathcal{L}_F^{\alpha}(\mathcal{N}_F^{\alpha}f((\alpha t)^{\frac{1}{\alpha}}))(s)$$

by Theorem 3.7

$$\mathcal{L}_F^\alpha\left(\mathcal{N}_F^\alpha f\left((\alpha t)^{\frac{1}{\alpha}}\right)\right)(s) = s\mathcal{L}_F^\alpha\left(f\left((\alpha t)^{\frac{1}{\alpha}}\right)\right)(s) - f(0)$$

Thus, by Theorem 4.2

$$_c\mathcal{L}_F^\alpha(\mathcal{N}_F^\alpha f(t))(s) = s\,_c\mathcal{L}_F^\alpha(f(t))(s) - f(0)$$

Theorem 4.4. Consider the function f defined on $[0,+\infty)$ and $\alpha \in (0,1]$. Then

$$_c\mathcal{L}_F^\alpha[\mathcal{J}_F^\alpha f(t)](s) = \frac{_c\mathcal{L}_F^\alpha(f(t))(s)}{s}, s > 0$$

Proof. Let's apply Theorems 4.2 and 3.10

$$_c\mathcal{L}_F^\alpha(\mathcal{J}_F^\alpha f(t))(s) = \mathcal{L}_F^\alpha\left(\mathcal{J}_F^\alpha f\left((\alpha t)^{\frac{1}{\alpha}}\right)\right)(s)$$

$$= \frac{\mathcal{L}_F^\alpha\left(f\left((\alpha t)^{\frac{1}{\alpha}}\right)\right)(s)}{s}$$

$$= \frac{\mathcal{C}_F^\alpha(f(t))(s)}{s}$$

Theorem 4.5. Consider two differentiable functions, $G_1^\alpha(t)$ and $G_2^\alpha(t)$, such that $G_1'^\alpha(t) = F_1(\iota,\alpha)$ and $G_2'^\alpha(t) = F_2(\iota,\alpha)$. Then

$$_c\mathcal{L}_{F_1+F_2}^\alpha(f(t))(s) = {_c\mathcal{L}_{F_1}^\alpha}\left(e^{-sG_2^\alpha(t)} f(t)\right)(s) + {_c\mathcal{L}_{F_2}^\alpha}\left(e^{-sG_1^\alpha(t)} f(t)\right)(s)$$

where $\alpha \in (0,1]$.

Proof. Let's apply Theorems 4.2 and 3.12

$$_c\mathcal{L}_{F_1+F_2}^\alpha(f(t))(s) = \mathcal{L}_{F_1+F_2}^\alpha\left(f\left((\alpha t)^{\frac{1}{\alpha}}\right)\right)(s)$$

$$= \mathcal{L}_{F_1}^\alpha\left(e^{-sG_2^\alpha(t)} f\left((\alpha t)^{\frac{1}{\alpha}}\right)\right)(s) + \frac{1}{s}\mathcal{L}_{F_2}^\alpha\left(e^{-sG_1^\alpha(t)} f\left((\alpha t)^{\frac{1}{\alpha}}\right)\right)(s)$$

$$= {_c\mathcal{L}_{F_1}^\alpha}\left(e^{-sG_2^\alpha(t)}f(t)\right)(s) + {_c\mathcal{L}_{F_2}^\alpha}\left(e^{-sG_1^\alpha(t)}f(t)\right)(s).$$

Theorem 4.6. Let $n \geq 2$ and $i \in \{1,2,\dots,n\}$. Asume that $G_i^\alpha(t)$ derivables functions, we have

$$_c\mathcal{L}_{(\sum_{i=1}^\alpha F_i)}^\alpha(f(t))(s) = \sum_{i=1}^n {_c\mathcal{L}_{F_i}^\alpha}\left(e^{-s[\sum_{j=1,j\neq i}^n G_j^\alpha(t)]} f(t)\right)(s)$$

where $\alpha \in (0,1]$ and $G_i'^\alpha(t) = F_i(t,\alpha)$.

Proof. Let's apply Theorems 4.2 and 3.14

$$ {}_{c\mathcal{L}}\mathcal{L}^{\alpha}_{(\sum_{i=1}^{\alpha} F_i)}(f(t))(s) = \mathcal{L}^{\alpha}_{(\sum_{i=1}^{n} F_i)}\left(f\left((\alpha t)^{\frac{1}{\alpha}}\right)\right)(s) $$
$$ = \sum_{i=1}^{n} \mathcal{L}^{\alpha}_{F_i}\left(e^{-s[\sum_{j=1,j\neq i}^{n} G_j^{\alpha}(t)]} f\left((\alpha t)^{\frac{1}{\alpha}}\right)\right)(s) $$
$$ = \sum_{i=1}^{n} {}_c\mathcal{L}^{\alpha}_{F_i}\left(e^{-s[\sum_{j=1,j\neq i}^{n} G_j^{\alpha}(t)]} f(t)\right)(s). $$

Theorem 4.7. The ${}_c\mathcal{N}_F^{\alpha}$-Laplace transform changes the $\mathcal{N}_F^{\alpha}$-convolution product into a product:

$$ {}_c\mathcal{L}_F^{\alpha}((f \ast_F^{\alpha} g)(t))(u) = {}_c\mathcal{L}_F^{\alpha}(f(t))(u).\ {}_c\mathcal{L}_F^{\alpha}(g(t))(u). $$

Proof. By direct application of Theorems 4.2 and Theorem 3.20.

5. APPLICATION

1. Let's study the following differential equation:

$$ y'(t) + a(2\alpha t + 1)y(t) = 2\alpha t^3 + t^2 + (2\alpha t + 1)\sin(t) \qquad (5.1) $$

where $y(0) = b, t \geq 0$.

Put $F(t,\alpha) = 2\alpha t + 1$ and $G_\alpha(t) = \alpha t^2 + t$. So according to theorem 2.3, We replace the equation (5.1) with

$$ \mathcal{N}_F^{(\alpha)} y(t) + a y(t) = (t^2 + \sin(t)) $$

By utilizing the $\mathcal{N}_F^{\alpha}$-Laplace transform and Theorem 3.7, we have for $s > -a$

$$ \mathcal{L}_F^{\alpha}(y(t)) = \frac{b}{s+a} + \frac{1}{s+a}\mathcal{L}_F^{\alpha}(t^2 + \sin(t))(s) $$

By Theorem 3.4 and Theorem 3.20, we find

$$ y(t) = y_0\, e^{-a(\alpha t^2+t)} + \int_0^t e^{-a(\alpha s^2+s)}((t-s)^2 + \sin(t-s))ds. $$

2. We want to study the following $\mathcal{N}_F^{\alpha}$-harmonic oscillator equation:

$$ \mathcal{N}_F^{(2\alpha)} x(t) + \gamma^2 x(t) = f(t), x(0) = 1, \mathcal{N}_F^{\alpha} x(0) = 0 $$

where $F(t,\alpha) = 2\alpha t + 1$, $0 < \alpha \leq 1$ and $\gamma \neq 0$.
By applying the $\mathcal{N}_F^{\alpha}$-Laplace transform to this equation and using the fact that it is a linear operator, we have:

$$ \mathcal{L}_F^{\alpha}(\mathcal{N}_F^{(2\alpha)} x(t))(s) + \gamma^2 \mathcal{L}_F^{\alpha}(x(t))(s) = \mathcal{L}_F^{\alpha}(f(t))(s). $$

By Theorem 3.7, this equation transforms into

$$s^2\underset{F}{\mathcal{L}^\alpha}(x(t))(s) - \mathcal{N}_F x(0) - sx(0) + \gamma^2\underset{F}{\mathcal{L}^\alpha}(x(t))(s) = \underset{F}{\mathcal{L}^\alpha}(f(t))(s)$$
$$(s^2 + \gamma^2)\underset{F}{\mathcal{L}^\alpha}(x(t))(s) = s + \underset{F}{\mathcal{L}^\alpha}(f(t))(s)$$
$$\underset{F}{\mathcal{L}^\alpha}(x(t))(s) = \frac{s}{s^2+\gamma^2} + \frac{1}{s^2+\gamma^2}\underset{F}{\mathcal{L}^\alpha}(f(t))(s)$$

By Theorem 3.4 and Theorem 3.20, we find

$$x(t) = \cos(\gamma(\alpha t^2 + t)) + \frac{1}{\gamma}\int_0^t \sin(\gamma(\alpha t^2 + t))f(t-s)ds.$$

Acknowledgement:

The author T.Abdeljawad would like to thank Prince Sultan University for paying the APC and the support through the TAS research lab.

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