Stochastic Analysis of COVID-19 Epidemic Model

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Received:12/01/2025 Revised:10/03/2025 Accepted:29/03/2025

Abstract

In our work, we focused on examining the epidemic transmission dynamics of the SEIAR stochastic model for COVID-19. We prove the existence and uniqueness of a global positive solution which allowed us to better understand the underlying mechanisms of disease spread and explore the concepts of disease extinction and persistence within the stochastic model. By identifying key parameters and variables, we were able to determine potential scenarios in which the disease could die out naturally over time, as well as situations where it could persist in a sustainable manner within the population studied. We also carried out numerical simulations which allowed us to validate our theoretical results and visualize in a concrete way how the parameters of the model affect the dynamics of the disease. By analyzing these simulations, we were able to identify patterns of spread and assess the conditions under which a disease might disappear or persist in the study population.

Key words and phrases: COVID-19, Stochastic Analysis, Extinction, Persistence, Numerical Simulation.

1 Introduction

Mathematical modeling of infectious diseases is an essential tool in understanding, predicting and controlling the spread of these diseases within a population [31, 24, 26, 7, 13].
In this field, environmental noise and variability significantly contribute to the spread of infectious diseases [25]. For example, diseases such as COVID-19, malaria and hepatitis B serve as notable instances of this phenomenon. So that these diseases occur and spread stochastically due to the unpredictability of interactions between individuals [3, 1].

Many researchers have studied the Covid-19 infection, some of which we highlight. In [23] the authors provided a stochastic and deterministic model to study infected cases of COVID-19 in the UAE. The authors in [5] studied COVID-19 infection using symptomatic and asymptomatic classes. A mathematical study on COVID-19 was also performed by [4] using fractional derivative modeling. In [22] the authors propose a new model where the total population $N(t)$ is divided into: susceptible class (S), exposed class (E), symptomatic infectious class (I), super-spreaders class (P), asymptomatic infectious class (A), hospitalized (H), recovery class (R), and mortality class (F). In [18] the researchers proposed the SAIQH model to study the spread of COVID-19, based on a model [17]. Researchers in [9] investigated the effectiveness of an epidemic modeling approach to the spread of COVID-19 using the SIR model with the aim of providing a theoretical framework for investigating its spread within the community. In [2] the authors constructed a SEIR model by considering vaccination and isolation factors as typical parameters for studying the spread of COVID-19.

There is also some scientific research that uses the concept of stochastic differential equations to better understand the infection of COVID-19. In [28], the authors formulated the stochastic SIR model, then paid full attention to conditions sufficient for extinction and persistence and examined the threshold of the proposed COVID-19 random model, when noise was small or large. In [11] authors developed a discrete-time stochastic epidemic model with binomial distributions to study the dynamics of the disease transmission. The impact of the epidemic on healthcare in India was also studied using the stochastic SEIR model by [8].

Our research focuses on studying the dynamics of COVID-19 infection by means of a stochastic mathematical model, some sufficient conditions on extinction and persistence have been obtained.

The paper is organized as follows, In the section 2, we formulated the COVID-19 model using stochastic differential equations based on the deterministic model of [5] then, we show the existence and uniqueness of a global positive solution of the stochastic model in section 3. In the section 4, we study the extinction of the disease, and the persistence in the section 5. In the section 6, we have carried out some numerical simulations to illustrate the obtained theoretical results and discuss these results. Finally, in section 7 we conclude with a conclusion about what we have done in this paper.

2 Model formulation

To investigate COVID-19 pandemic-related disease progression, authors in [5] present the following compartmental deterministic model for disease transmission

$$\begin{cases} \frac{dS}{dt} = \Lambda - \beta_I S(I + kA) - d_p S \\ \frac{dE}{dt} = \beta_I S(I + kA) - (1-\alpha) w_p E - \alpha w'_p E - d_p E \\ \frac{dI}{dt} = (1-\alpha) w_p E - (\lambda_p + d_p + \mu) I + \xi A \\ \frac{dA}{dt} = \alpha w'_p E - (\lambda'_p + d_p + \nu) A - \xi A \\ \frac{dR}{dt} = \lambda_p I + \lambda'_p A - d_p R \end{cases} \tag{1}$$

Where the total population $N(t)$ has been divided into five classes :

$S(t)$: represent the susceptible individuals, those who have not yet been in contact with the virus.

$E(t)$: represent individuals who have been exposed to the virus, but are currently in the incubation phase, not yet able to transmit the disease.

$I(t)$: The symptomatic infectious class, includes people with symptoms and able to transmit the disease.

$A(t)$: The asymptomatic infectious class, includes people who do not show symptoms and are able to transmit the disease.

$R(t)$: The eliminated class, includes individuals who have recovered from the disease.

The parameters used in model (1) are shown in the table 1 below

Parameter	description
Λ	The recruitment rate
β_I	The transmission rate
k	Infection rate adjustment factor for asymptomatic cases.
d_p	The natural death rate
w_p	The progression rate from the exposed class to the symptomatic class
w'_p	The progression rate from the exposed class to the asymptomatic class
α	The proportion of individuals who become asymptomatic
ξ	The rate at which asymptomatic individuals might develop symptoms
λ_p	The recovered rate of symptomatic individuals
λ'_p	The recovered rate of asymptomatic individuals
μ	The death rate related to the disease of symptomatic individuals
ν	The death rate related to the disease of asymptomatic individuals

Table 1: Parameter used in the model

The model (1) is studied in the invariant set Ω, where

$$\Omega = \left\{ (S, E, I, A, R) : S > 0, E \geq 0, I \geq 0, A \geq 0, R \geq 0\,; S + E + I + A + R \leq \frac{\Lambda}{d_p} \right\}$$

On the other hand, it is essential to note that any system is constantly influenced by external disturbances. These disturbances, or environmental noise, can bring an increased level of realism compared to deterministic systems, thus conferring a key element of veracity.
We introduce randomness into the model (1) by replacing the parameter βdt by $\beta dt + \sigma dB(t)$, where $B(t)$ is independent standard brownian motions and $\sigma^2 > 0$ represent the intensities of $B(t)$.
So, the stochastic model corresponding to the deterministic model (1) is given in the following form:

$$\begin{cases} dS = (\Lambda - \beta_I S(I + kA) - d_p S)dt - \sigma S(I + kA)dB(t) \\ dE = (\beta_I S(I + kA) - (1 - \alpha)w_p E - \alpha w_p' E - d_p E)dt + \sigma S(I + kA)dB(t) \\ dI = ((1 - \alpha)w_p E - (\lambda_p + d_p + \mu)I + \xi A)dt \\ dA = (\alpha w_p' E - (\lambda_p^{'} + d_p + \nu)A - \xi A)dt \\ dR = (\lambda_p I + \lambda_p^{'} A - d_p R)dt \end{cases} \tag{2}$$

Given the biological importance of the categories $(S(t), E(t), I(t), A(t), R(t))$, our goal is to examine the model in the positive quadrant.

$$\mathbb{R}_+^5 = \left\{ (S, E, I, A, R) \in \mathbb{R}^5 : S > 0, E \geq 0, I \geq 0, A \geq 0, R \geq 0 \right\}$$

Next, we introduce some fundamental concepts regarding stochastic differential equations [21].
In our work, let $(\Omega, \{F_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{F_t\}_{t\geq 0}$ satisfying the usual conditions (*i.e.* it is right continuous and F_0 contains all $P-$null sets). Denote

$$\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_i > 0 \text{ for all } 1 \leq i \leq n\}$$

In general, consider the d-dimensional stochastic differential equation

$$dx(t) = f(x(t), t) + g(x(t), t)dB(t), \quad \text{for } t \geq t_0 \tag{3}$$

With initial value $x(t_0) = x_0 \in \mathbb{R}^n$. Let $0 < s \leq \infty$ and $\mathbb{S}_s = \{x \in \mathbb{R}^n : |x| < s\}$ Define the differential operator L associated with equation (3) by

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{n} f_i(x,t)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n}[g^T(x,t)g(x,t)]_{i,j}\frac{\partial^2}{\partial x_i \partial x_j}$$

If L acts on a function $V \in C^{2,1}(\mathbb{S}_s \times \mathbb{R}_+; \mathbb{R}_+)$, then

$$LV(x,t) = V_t(x,t) + V_x(x,t)f(x,t) + \frac{1}{2}\text{trace}[g^T(x,t)V_{xx}(x,t)g(x,t)],$$

Where $V_t = \frac{\partial V}{\partial t}, V_x = (\frac{\partial V}{\partial x_1}, ..., \frac{\partial V}{\partial x_d}), V_{xx} = (\frac{\partial^2 V}{\partial x_i \partial x_j})_{d\times d}$. By Itô's formula $x(t) \in \mathbb{S}_s$, then

$$dV(x(t),t) = LV(x(t),t) + V_x(x(t),t)g(x(t),t)dB(t)$$

3 Existence and uniqueness of positive solution

In this section, we present here the results of the following theorem, which attest to the existence of a unique global positive solution for the system (2) using the Lyapunov analysis method which mentioned in [10].

Theorem 3.1. *there is a unique solution $(S(t), E(t), I(t), A(t), R(t))$ of the system (2) on $t \geq 0$ for any value $(S(0), E(0), I(0), A(0), R(0)) \in \mathbb{R}^5_+$, and the solution will remain in $\mathbb{R}^5_+$ with probability 1, namely, $(S(t), E(t), I(t), A(t), R(t)) \in \mathbb{R}^5_+$, for all $t \geq 0$ almost surely.*

Proof. Since the coefficients of the equations are locally Lipschitz continuous for any given initial value $(S(0), E(0), I(0), A(0), R(0)) \in \mathbb{R}^5_+$, there is a unique local solution $(S(t), E(t), I(t), A(t), R(t)) \in \mathbb{R}^5_+$ on $t \in [0, \tau_e)$, where τ_e is the explosion time [21].
To show this solution is global, we need to show that $\tau_e = \infty$ a.s. Let $k_0 \geq 0$ be sufficiently large so that $S(0), E(0), I(0), A(0)$ and $R(0)$ all lie within the interval $\left[\frac{1}{k_0}, k_0\right]$.
For each integer $k \geq k_0$, define the stopping time

$$\tau_k = inf\left\{t \in [0,\tau_e) : \min\{(S(t), E(t), I(t), A(t), R(t)\} \leq \frac{1}{k} \text{ or } \max\{(S(t), E(t), I(t), A(t), R(t)\} \geq k\right\}$$

We use the notation $inf\emptyset = \infty$ thought in this study, where $\emptyset$ is an empty set. τ_k increases as k approaches to ∞. the setting of $\tau_\infty = lim_{k\to+\infty}\tau_k$ along with the use of $\tau_\infty \leq \tau_e$ a.s proves that τ_∞ is $= \infty$ a.s, and the solution of the system (2) lie in $\mathbb{R}^5_+$ a.s, $\forall t \geq 0$.
Also for the completion of conclusion, we will prove $\tau_e = \infty$ a.s. If this is not true, then

there exist a pair of constants $T > 0$ and $\epsilon \in (0,1)$ such that

$$P\left\{\tau_\infty \leq T\right\} > \epsilon$$

Hence there is an integer $k_1 \geq k_0$ such that

$$P\left\{\tau_k \leq T\right\} \geq \epsilon \quad \textit{for all} \quad k \geq k_1 \tag{3.1}$$

For $t \leq \tau_k$, we can see, for each k,

$$dN(t) = [\Lambda - d_p N - \mu I - \nu A]dt \leq [\Lambda - d_p N]dt$$

And so

$$N(t) \leq \begin{cases} \frac{\Lambda}{d_p} & \text{if} \quad N(0) \leq \frac{\Lambda}{d_p} \\ N(0) & \text{if} \quad N(0) > \frac{\Lambda}{d_p} \end{cases}$$

Define a C^2 function $V : \mathbb{R}_+^5 \to \mathbb{R}_+$, by

$$V(S,E,I,A,R) = (S-1-\ln S)+(E-1-\ln E)+(I-1-\ln I)+(A-1-\ln A)+(R-1-\ln R)$$

Since the function $u - 1 - \ln u \geq 0$ for all $u > 0$, so the non-negativity of the function V is assured. Applying the Itô formula, we obtain

$$dV(S,E,I,A,R) = LVdt + \sigma(I+kA)\left(1 - \frac{S}{E}\right)dB(t)$$

Where $LV : \mathbb{R}_+^5 \to \mathbb{R}_+$ is defined by

$$\begin{aligned} LV = \Lambda - d_p S - \frac{\Lambda}{S} + \beta_I(I+kA) + 4d_p + \frac{\sigma^2}{2}(I+kA)^2 - d_p E - \frac{\beta_I S(I+kA)}{E} \\ + (1-\alpha)w_p + \alpha w_p' + \frac{\sigma^2}{2E^2}S^2(I+kA)^2 - (d_p+\nu)A - \frac{E}{A}\alpha w_p' + (\lambda_p^{'} + \nu) + \xi \\ - (d_p+\mu)I - \frac{E}{I}(1-\alpha)w_p + (\lambda_p + \mu) - \frac{A}{I}\xi - d_p R - \frac{I}{R}\lambda_p - \frac{I}{R}\lambda_p^{'} \end{aligned}$$

From the idea in [6], we pose : $f(x) = -\beta_I x + \frac{\sigma^2}{2}x^2$ $\left(\text{where}, x = \frac{S(I+kA)}{E}\right)$is decreasing on $\left[0, \frac{\beta_I}{\sigma^2}\right]$ and increasing on $\left[\frac{\beta_I}{\sigma^2}, \frac{\Lambda}{d_p}\right]$, satisfied

$$f(0) = 0, \quad f\left(\frac{\beta_I}{\sigma^2}\right) = -\frac{\beta_I^2}{2\sigma^2} < 0, \quad f\left(\frac{\Lambda}{d_p}\right) = -\beta_I\frac{\Lambda}{d_p} + \frac{\sigma^2}{2}\frac{\Lambda^2}{{d^2}_p}$$

We see that, $f(x) \leq \max\left\{0, \left(-\beta_I\frac{\Lambda}{d_p} + \frac{\sigma^2}{2}\frac{\Lambda^2}{{d^2}_p}\right)\right\}$ and since $N(t) \leq \frac{\Lambda}{d_p}$ and every solution of system (2) is bounded by $\frac{\Lambda}{d_p}$, we find that

$$\begin{aligned} LV &\leq \Lambda + (1-\alpha)w_p + \alpha w_p' + \lambda_p^{'} + \nu + \xi + 4d_p + \beta_I(1+k)\frac{\Lambda}{d_p} + (1+k)^2\frac{\sigma^2\Lambda^2}{2d_p^2} \\ &+ \max\left\{0, \left(-\beta_I\frac{\Lambda}{d_p} + \frac{\sigma^2}{2}\frac{\Lambda^2}{{d^2}_p}\right)\right\} \\ &= K \end{aligned}$$

On the other hand, we have $\left(1 - \frac{S}{E}\right) \leq (S+E+1)$

Therefore, if $t_1 \leq T$,

$$\int_0^{\tau_k \wedge t_1} dV(S(t), E(t), I(t), A(t), R(t)) \leq \int_0^{\tau_k \wedge t_1} K dt + \int_0^{\tau_k \wedge t_1} \sigma(I + kA)\left(S + E + 1\right) dB(t).$$

This implies that,

$$\begin{aligned} \mathbb{E}\left[V\left(S\left(\tau_k \wedge t_1\right), E\left(\tau_k \wedge t_1\right), I\left(\tau_k \wedge t_1\right), A\left(\tau_k \wedge t_1\right), R\left(\tau_k \wedge t_1\right)\right)\right] &\leq V\left(S_0, E_0, I_0, A_0, R_0\right) \\ &+ \mathbb{E}\int_0^{\tau_k \wedge t_1} K dr \\ &+ \mathbb{E}\int_0^{\tau_k \wedge t_1} \sigma(I + kA)\left(S + E + 1\right) dB(r) \end{aligned} \tag{3.2}$$

Set $\Omega_k = \{\tau_k \leq T\}$ for $k \geq k_1$ and by (3.1), $P\left(\Omega_k\right) \geq \epsilon$. Note that, for every $\omega \in \Omega_k$, there is at least one of $S\left(\tau_k, \omega\right), E\left(\tau_k, \omega\right), I\left(\tau_k, \omega\right), A\left(\tau_k, \omega\right)$ and $R\left(\tau_k, \omega\right)$ equals either k or $1/k$, and, hence, $V\left(S\left(\tau_k, \omega\right), E\left(\tau_k, \omega\right), I\left(\tau_k, \omega\right), A\left(\tau_k, \omega\right), R\left(\tau_k, \omega\right)\right)$ is no less than

$$k - 1 - \log k \quad \text{or} \quad \frac{1}{k} - 1 - \log\frac{1}{k} = \frac{1}{k} - 1 + \log k$$

Consequently,

$$V\left(S\left(\tau_k, \omega\right), E\left(\tau_k, \omega\right), I\left(\tau_k, \omega\right), A\left(\tau_k, \omega\right), R\left(\tau_k, \omega\right)\right) \geq (k - 1 - \log k) \wedge \left(\frac{1}{k} - 1 + \log k\right)$$

It then follows from (3.1) and (3.2) that

$$\begin{aligned} V\left(S_0, E_0, I_0, A_0, R_0\right) + KT &\geq \mathbb{E}\left[1_{\Omega_k(\omega)} V\left(S\left(\tau_k, \omega\right), E\left(\tau_k, \omega\right), I\left(\tau_k, \omega\right), A\left(\tau_k, \omega\right), R\left(\tau_k, \omega\right)\right)\right] \\ &\geq \epsilon\left[(k - 1 - \log k) \wedge \left(\frac{1}{k} - 1 + \log k\right)\right] \end{aligned}$$

where $1_{\Omega_k(\omega)}$ is the indicator function of Ω_k. Letting $k \to \infty$ leads to the contradiction $\infty > V\left(S_0, E_0, I_0, A_0, R_0\right) + KT = \infty$. So we must, therefore, have $\tau_\infty = \infty$ a.s.

□

Remark 1. *From the theorem 3.1, for any initial value* $(S(0), E(0), I(0), A(0), R(0)) \in \mathbb{R}^5_+$, *there is a unique global solution* $(S(t), E(t), I(t), A(t), R(t)) \in \mathbb{R}^5_+$, *almost surely of the system (2). Hence*

$$d(S+E+I+A+R) \leq (\Lambda - d_p(S+E+I+A+R))$$

And

$$S(t)+E(t)+I(t)+A(t)+R(t) \leq \frac{\Lambda}{d_p} + e^{-d_p t}\left(S(0)+E(0)+I(0)+A(0)+R(0) - \frac{\Lambda}{d_p}\right)$$

If $S(0)+E(0)+I(0)+A(0)+R(0) \leq \frac{\Lambda}{d_p}$, *then* $S(t)+E(t)+I(t)+A(t)+R(t) \leq \frac{\Lambda}{d_p}$ *a.s, so the region*

$$\Omega^* = \left\{(S,E,I,A,R) : S > 0, E \geq 0, I \geq 0, A \geq 0, R \geq 0\,; S+E+I+A+R \leq \frac{\Lambda}{d_p} a.s\right\}$$

Is a positively invariant set of the system (2) on Ω^*,
In our paper, we assume that $(S(0), E(0), I(0), A(0), R(0)) \in \Omega^*$

4 Extinction

In a population system when we study disease extinction, we look at how the spread of disease can decrease over time until it is no longer present in the population.
For this purpose, we show the following definition

Definition 1. *[16, 30] the population* $x(t)$ *is said to be exponentially extinct if*

$$\limsup_{t\to\infty} \frac{\log x(t)}{t} < 0, a.s$$

So, we investigate the conditions for the extinction of the disease. For convenience we introduce the notation:

$$\langle x(t) \rangle = \frac{1}{t}\int_0^t x(r)dr$$

This notation represents the average population $x(t)$ over time t given its evolution over the period $[0,t]$. This can help us understand how the population mean changes over time in the population system we are studying.

Lemma 1. *(Strong Law of Large Numbers)[21] Let* $M = \{M_t\}_{t\geq 0}$ *be a real-value continuous local martingale vanishing at* $t = 0$*. then*

$$\lim_{t\to+\infty} \langle M, M\rangle_t = \infty \quad a.s, \quad \Longrightarrow \quad \lim_{t\to+\infty} \frac{M_t}{\langle M, M\rangle_t} = 0 \quad a.s$$

and also

$$\limsup_{t\to\infty} \frac{\langle M, M\rangle_t}{t} < \infty \quad a.s, \quad \Longrightarrow \quad \lim_{t\to+\infty} \frac{M_t}{t} = 0 \quad a.s$$

Theorem 4.1. *Let us assume that* $(S(t), E(t), I(t), A(t), R(t))$ *the solution of the system (2) with initial conditions* $(S(0), E(0), I(0), A(0), R(0)) \in \Omega^*$*, we have*

$$\lim_{t\to+\infty} \frac{S(t) + E(t) + I(t) + A(t) + R(t)}{t} = 0, \quad a.s$$

We give the following results about the extinction of the disease

Theorem 4.2. *Let* $(S(t), E(t), I(t), A(t), R(t))$ *be the solution of system (2) with initial value* $(S(0), E(0), I(0), A(0), R(0)) \in \Omega^*$*. If* $\sigma^2 > \frac{\beta_I^2}{2d_p}$*, then*

$$\limsup_{t\to\infty} \frac{\log(E(t) + I(t) + A(t) + R(t))}{t} < 0, \quad a.s$$

Namely, $E(t), I(t), A(t)$ *and* $R(t)$ *tend to zero exponentially a.s. That is to say, the exposed, symptomatic, asymptomatic and recovered individuals go to the extinction almost surely. Which means*

$$\lim_{t\to+\infty} E(t) = 0, \quad \lim_{t\to+\infty} I(t) = 0, \quad \lim_{t\to+\infty} A(t) = 0, \quad \lim_{t\to+\infty} R(t) = 0 \quad a.s$$

In addition

$$\lim_{t\to+\infty} \langle S(t)\rangle = \frac{\Lambda}{d_p}, \quad a.s$$

Proof. Define the C^2 function $V : \mathbb{R}_+^4 \to \mathbb{R}$ such that $V = \ln \Phi$ where $\Phi(t) = E(t) + I(t) +$

$A(t)+R(t)$, By Itô's formula applied to the second equation of the system (2), we get

$$\begin{aligned}
dV &= \frac{d\Phi}{\Phi} - \frac{1}{2\Phi^2}(d\Phi)^2 \\
&= \frac{1}{\Phi}\left(\beta_I S(I+kA) - d_pE - (\mu+d_p)I - (\nu+d_p)A - d_pR - \frac{\sigma^2 S^2(I+kA)^2}{2(E+I+A+R)^2}\right)dt \\
&+ \sigma\frac{S(I+kA)}{\phi}dB(t) \\
&= \left(\beta_I \frac{S(I+kA)}{E+I+A+R} - d_p\frac{E+I+A+R}{E+I+A+R} - \mu\frac{I}{E+I+A+R} - \nu\frac{A}{E+I+A+R}\right)dt \\
&- \frac{\sigma^2 S^2(I+kA)^2}{2(E+I+A+R)^2}dt + \sigma\frac{S(I+kA)}{E+I+A+R}dB(t)
\end{aligned}$$

We know that

$$\frac{\beta_I S(I+kA)}{E+I+A+R} - \frac{\sigma^2 S^2(I+kA)^2}{2(E+I+A+R)^2} = -\frac{\sigma^2}{2}\left(\frac{S(I+kA)}{E+I+A+R} - \frac{\beta_I}{\sigma^2}\right)^2 + \frac{\beta_I^2}{2\sigma^2} \leq \frac{\beta_I^2}{2\sigma^2}$$

And

$$\frac{I+kA}{E+I+A+R} \leq 1+k$$

Then we get

$$dV \leq \left(\frac{\beta_I^2}{2\sigma^2} - d_p\right)dt + \sigma(1+k)SdB(t) \tag{4}$$

We integrate the previous equation in the interval $[0,t]$, and divide by t, we get

$$\frac{\log\Phi(t)}{t} \leq \frac{\beta_I^2}{2\sigma^2} - d_p + \frac{\log\Phi(0)}{t} + \frac{\sigma(1+k)}{t}M(t)$$

Where, $M(t) = \int_0^t S(r)dB(r)$, moreover

$$\limsup_{t\to\infty} \frac{\langle M, M\rangle_t}{t} \leq \frac{\Lambda^2}{d_p^2} < \infty$$

By the lemma 1 and the result in theorem 4.1, we obtain

$$\lim_{t\to+\infty} \frac{M_t}{t} = 0 \quad \text{and} \quad \lim_{t\to+\infty} \frac{\log\Phi(0)}{t} = 0 \quad a.s$$

If $\sigma^2 > \frac{\beta_I^2}{2d_p}$ is satisfied, we get

$$\limsup_{t\to\infty} \frac{\log\Phi(t)}{t} \leq \frac{\beta_I^2}{2\sigma^2} - d_p < 0 \quad a.s \tag{5}$$

Remark 2. *[14] From the theorem (4.2), we get that there is an constant $\gamma > 0$ such that*

$$\limsup_{t\to\infty} \frac{\log \Phi(t)}{t} \leq -\gamma$$

that is to say, for a arbitrary small constant $0 < \epsilon < min(\frac{1}{2}, \frac{\epsilon}{2})$, there exists a positive constant $T_1 = T_1(\omega)$ and a set Ω_ϵ such that $P(\Omega_\epsilon) \geq 1 - \epsilon$ and $\log \Phi(t) \leq -\frac{\gamma}{2}t$ for $t \geq T_1$, $\omega \in \Omega_\epsilon$, and so $\Phi(t) \leq e^{-\frac{\gamma}{2}t}$. Therefore

$$\limsup_{t\to\infty} \Phi(t) \leq 0 \quad a.s$$

Which together with the positive of the solution implies

$$\lim_{t\to+\infty} \Phi(t) = \lim_{t\to+\infty} (E(t) + I(t) + A(t) + R(t)) = 0 \quad a.s$$

We conclude that,

$$\lim_{t\to+\infty} E(t) = 0 \quad a.s \tag{6}$$

$$\lim_{t\to+\infty} I(t) = 0 \quad a.s \tag{7}$$

$$\lim_{t\to+\infty} A(t) = 0 \quad a.s \tag{8}$$

$$\lim_{t\to+\infty} R(t) = 0 \quad a.s \tag{9}$$

On the other side, we integrate the system (2) then dividing by t, we obtain

$$\frac{S(t)-S(0)}{t} = \Lambda - \beta_I \langle S(t)(I(t)+kA(t))\rangle - d_p\langle S(t)\rangle - \frac{\sigma}{t}\int_0^t S(t)(I(t)+kA(t))dB(r) \tag{10}$$

$$\begin{aligned} \frac{E(t)-E(0)}{t} &= \beta_I \langle S(t)(I(t)+kA(t))\rangle - ((1-\alpha)w_p + \alpha w_p' + d_p)\langle E(t)\rangle \\ &\quad + \frac{\sigma}{t}\int_0^t S(t)(I(t)+kA(t))dB(r) \end{aligned} \tag{11}$$

$$\frac{I(t)-I(0)}{t} = (1-\alpha)w_p\langle E(t)\rangle - (\lambda_p + d_p + \mu)\langle I(t)\rangle + \xi\langle A(t)\rangle \tag{12}$$

$$\frac{A(t)-A(0)}{t} = \alpha w_p'\langle E(t)\rangle - (\lambda_p' + d_p + \nu)\langle A(t)\rangle - \xi\langle A(t)\rangle \tag{13}$$

$$\frac{R(t)-R(0)}{t} = \lambda_p\langle I(t)\rangle + \lambda_p'\langle A(t)\rangle - d_p\langle R(t)\rangle \tag{14}$$

We collect the first two equations (10-11), we get

$$\varphi(t) = \Lambda - d_p\langle S(t)\rangle - ((1-\alpha)w_p + \alpha w_p' + d_p)\langle E(t)\rangle$$

Where, $\varphi(t) = \frac{S(t)-S(0)}{t} + \frac{E(t)-E(0)}{t}$, then we find

$$\langle S(t)\rangle = \frac{1}{d_p}\Big(\Lambda - ((1-\alpha)w_p + \alpha w_p^{'} + d_p)\langle E(t)\rangle - \varphi(t)\Big) \tag{15}$$

According to the theorem 4.1 and the result (6), we conclude that,

$$\lim_{t\to+\infty}\langle S(t)\rangle = \frac{\Lambda}{d_p} \quad a.s$$

Remark 3. *This result indicates that the time mean of the susceptible population $S(t)$ converges to $\frac{\Lambda}{d_p}$ in the long run. This implies that the susceptible population stabilizes at a constant value equal to $\frac{\Lambda}{d_p}$ after the disease has died out.*

□

5 Persistence

Several concepts of stochastic persistence are present in [30, 19], but in this context we adopt the concept of stochastic strongly persistence in mean. Before establishing the main conclusion of this section, we introduce the definition of stochastic strongly persistence in mean.

Definition 2. *the population $x(t)$ is said to be strongly persistence in the mean if* $\liminf_{t\to\infty}\langle x(t)\rangle > 0, a.s$

The previous definition is commonly used in [7, 10, 29, 15, 27, 20]. So, our system (2) is said to be strongly persistence in the mean, if

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t I(r)dr > 0 \quad \text{and} \quad \liminf_{t\to\infty}\frac{1}{t}\int_0^t A(r)dr > 0 \quad a.s$$

Remark 4. *This expression means that in the considered model, the disease persists on average if the lower limit of the average number of symptomatic and asymptomatic individuals over a long period of time is greater than zero with a probability 1. In other words, this means that the disease persists and spreads in the population, because there are always enough infected and asymptomatic people to maintain the transmission of the disease.*

Lemma 2. *[14] Let $x \in C[\Omega\times[0,\infty),\mathbb{R}]$. If there exist positive constant λ,μ such that*

$$\log x(t) \geq \lambda t - \mu\int_0^t x(s)ds + F(t), \quad a.s.$$

For all $t \geq 0$*, where* $F \in C[\Omega \times [0,\infty), \mathbb{R}]$ *and* $\lim_{t\to\infty} \frac{F(t)}{t} = 0$*, then*

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t x(s)ds \geq \frac{\lambda}{\mu}, \quad a.s.$$

Assume that $\frac{\Lambda}{d_p} > \frac{(1-\alpha)w_p+\alpha w_p^{'}+d_p}{\beta_I}$ and $\sigma^2 > \frac{2\beta_I d_p}{\Lambda}$, define the stochastic basic reproduction number R_0^s such as

$$R_0^s = \frac{\beta_I \frac{\Lambda}{d_p}}{\frac{\sigma^2\Lambda^2(k^2+2k+2)}{2d_p^2} + (1-\alpha)w_p + \alpha w_p^{'} + d_p}$$

Theorem 5.1. *Let* $(S(t), E(t), I(t), A(t), R(t))$ *be the solution of system (2) with initial value* $(S(0), E(0), I(0), A(0), R(0)) \in \Omega^*$*. If* $R_0^s > 1$*, Then the system (2) has the following property:*

$$\liminf_{t\to\infty}\langle E(t)\rangle > 0 \quad a.s$$

We also conclude

$$\liminf_{t\to\infty}\langle A(t)\rangle > 0 \quad , \quad \liminf_{t\to\infty}\langle I(t)\rangle > 0 \quad a.s$$

$$\liminf_{t\to\infty}\langle R(t)\rangle > 0 \quad , \quad \liminf_{t\to\infty}\langle S(t)\rangle > 0 \quad a.s$$

Proof. Define a C^2 function $U : \mathbb{R}_+^4 \to \mathbb{R}$ as follows

$$U(S,E,I,A) = -\ln S - \ln E + \frac{\beta}{\lambda_p + d_p + \mu} I + \frac{\beta}{\lambda_p^{'} + d_p + \nu} A$$

Applying the Itô formula to the function $U(t)$ we obtain

$$\begin{aligned}
dU =& -\frac{dS}{S} + \frac{(dS)^2}{2S^2} - \frac{dE}{E} + \frac{(dE)^2}{2E^2} + \frac{\beta_I}{\lambda_p + d_p + \mu} dI + \frac{\beta_I}{\lambda_p^{'} + d_p + \nu} dA \\
=& \left(-\frac{\Lambda}{S} + d_p + \frac{\sigma^2 (I+kA)^2}{2} - \frac{\beta_I S(I+kA)}{E} + \frac{\sigma^2 S^2 (I+kA)^2}{2E^2} + (1-\alpha)w_p + \alpha w_p^{'} + d_p \right) dt \\
&+ \left(\frac{\beta_I (1-\alpha) w_p}{\lambda_p + d_p + \mu} + \frac{\beta_I \alpha w_p^{'}}{\lambda_p^{'} + d_p + \nu} \left(\frac{\xi}{\lambda_p + d_p + \mu} + k \right) \right) E dt \\
&+ \sigma (I+kA) \left(1 - \frac{S}{E} \right) dB(t)
\end{aligned}$$

Motivated by the idea in [6], we use the fact that the function $f(x) = -\beta_I x + \frac{\sigma^2}{2} x^2$ where $x = \frac{S(I+kA)}{E}$ is decreasing on $\left[0, \frac{\beta_I}{2\sigma^2}\right]$ and increasing on $\left[\frac{\beta_I}{2\sigma^2}, \frac{\Lambda}{d_p}\right]$, such as $f(x) \leq f(\frac{\Lambda}{d_p}) =$

$-\beta_I \frac{\Lambda}{d_p} + \frac{\sigma^2}{2}\frac{\Lambda^2}{d_p^2}$ Then,

$$\begin{aligned}
dU &\leq \left(\frac{\sigma^2(I+kA)^2}{2} - \beta_I \frac{\Lambda}{d_p} + \frac{\sigma^2}{2}\frac{\Lambda^2}{d_p^2} + (1-\alpha)w_p + \alpha w_p^{'} + d_p\right) dt \\
&+ \left(\frac{\beta_I(1-\alpha)w_p}{\lambda_p + d_p + \mu} + \frac{\beta_I \alpha w_p^{'}}{\lambda_p^{'} + d_p + \nu}\left(\frac{\xi}{\lambda_p + d_p + \mu} + k\right)\right) Edt \\
&+ \sigma(I+kA)\,(S+E+1)\,dB(t) \\
&\leq \left(\frac{\sigma^2\Lambda^2(k^2+2k+2)}{2d_p^2} - \beta_I \frac{\Lambda}{d_p} + (1-\alpha)w_p + \alpha w_p^{'} + d_p\right) dt \\
&+ \left(\frac{\beta_I(1-\alpha)w_p}{\lambda_p + d_p + \mu} + \frac{\beta_I \alpha w_p^{'}}{\lambda_p^{'} + d_p + \nu}\left(\frac{\xi}{\lambda_p + d_p + \mu} + k\right)\right) Edt \\
&+ \sigma(I+kA)\,(S+E+1)\,dB(t) \\
&= -\left(\frac{\sigma^2\Lambda^2(k^2+2k+2)}{2d_p^2} + (1-\alpha)w_p + \alpha w_p^{'} + d_p\right)(R_0^s - 1)dt \\
&+ \left(\frac{\beta_I(1-\alpha)w_p}{\lambda_p + d_p + \mu} + \frac{\beta_I \alpha w_p^{'}}{\lambda_p^{'} + d_p + \nu}\left(\frac{\xi}{\lambda_p + d_p + \mu} + k\right)\right) Edt \\
&+ \sigma(I+kA)\,(S+E+1)\,dB(t)
\end{aligned}$$

We integrate the previous inequality and divide by t we get

$$\begin{aligned}
\frac{\ln E(t)}{t} \geq &\left(\frac{\sigma^2\Lambda^2(k^2+2k+2)}{2d_p^2} + (1-\alpha)w_p + \alpha w_p^{'} + d_p\right)(R_0^s - 1) \\
&- \left(\frac{\beta_I(1-\alpha)w_p}{\lambda_p + d_p + \mu} + \frac{\beta_I \alpha w_p^{'}}{\lambda_p^{'} + d_p + \nu}\left(\frac{\xi}{\lambda_p + d_p + \mu} + k\right)\right)\langle E\rangle + F(t) - \sigma\frac{M(t)}{t}
\end{aligned}$$

Where $F(t) = \left(\frac{\ln E(0)}{t} - \frac{\ln S(t) - \ln S(0)}{t} + \frac{\beta}{\lambda_p + d_p + \mu}\frac{I(t) - I(0)}{t} + \frac{\beta}{\lambda_p^{'} + d_p + \nu}\frac{A(t) - A(0)}{t}\right)$ and $M(t) = \int_0^t (I+kA)\,(S+E+1)\,dB(r)$

Moreover

$$\limsup_{t\to\infty} \frac{\langle M, M\rangle_t}{t} \leq \left(\frac{\Lambda(\Lambda + d_p)(1+k)}{d_p^2}\right)^2 < \infty$$

By the lemma 1 and the result in theorem 4.1, we obtain

$$\lim_{t\to+\infty} \frac{M_t}{t} = 0 \quad \text{and} \quad \lim_{t\to+\infty} F(t) = 0 \quad a.s$$

Its follows from the Lemma (2) that

$$\liminf_{t\to\infty}\langle E(t)\rangle \geq \frac{\left(\frac{\sigma^2\Lambda^2(k^2+2k+2)}{2d_p^2} + (1-\alpha)w_p + \alpha w_p^{'} + d_p\right)(R_0^s - 1)}{\frac{\beta_I(1-\alpha)w_p}{\lambda_p + d_p + \mu} + \frac{\beta_I \alpha w_p^{'}}{\lambda_p^{'} + d_p + \nu}\left(\frac{\xi}{\lambda_p + d_p + \mu} + k\right)} > 0 \quad a.s. \tag{16}$$

Since the condition in the theorem (5.1) holds.

From the equation (13), we have

$$\langle A(t)\rangle = \frac{\alpha w_p^{'}}{\lambda_p^{'}+d_p+\nu+\xi}\langle E(t)\rangle - \frac{1}{\lambda_p^{'}+d_p+\nu+\xi}\frac{A(t)-A(0)}{t}$$

Applying the limit and according to the theorem 4.1 and the result in equation (16), we find that

$$\begin{aligned}\liminf_{t\to\infty}\langle A(t)\rangle &= \frac{\alpha w_p^{'}}{\lambda_p^{'}+d_p+\nu+\xi}\liminf_{t\to\infty}\langle E(t)\rangle - \frac{1}{\lambda_p^{'}+d_p+\nu+\xi}\liminf_{t\to\infty}\frac{A(t)-A(0)}{t}\\ &\geq \frac{\alpha w_p^{'}}{\lambda_p^{'}+d_p+\nu+\xi}\left(\frac{\left(\frac{\sigma^2\Lambda^2(k^2+2k+2)}{2d_p^2}+(1-\alpha)w_p+\alpha w_p^{'}+d_p\right)(R_0^s-1)}{\frac{\beta_I(1-\alpha)w_p}{\lambda_p+d_p+\mu}+\frac{\beta_I\alpha w_p^{'}}{\lambda_p^{'}+d_p+\nu}\left(\frac{\xi}{\lambda_p+d_p+\mu}+k\right)}\right) > 0 \quad a.s\end{aligned} \tag{17}$$

In the same way, according to the equation (12) we have

$$\langle I(t)\rangle = \frac{(1-\alpha)w_p}{\lambda_p+d_p+\mu}\langle E(t)\rangle + \frac{\xi}{\lambda_p+d_p+\mu}\langle A(t)\rangle - \frac{1}{\lambda_p+d_p+\mu}\frac{I(t)-I(0)}{t}$$

Applying the limit and according to the theorem 4.1 and the previous results

$$\liminf_{t\to\infty}\langle E(t)\rangle > 0 \quad \text{and} \quad \liminf_{t\to\infty}\langle A(t)\rangle > 0 \quad a.s$$

We find that

$$\begin{aligned}\liminf_{t\to\infty}\langle I(t)\rangle &= \frac{(1-\alpha)w_p}{\lambda_p+d_p+\mu}\liminf_{t\to\infty}\langle E(t)\rangle + \frac{\xi}{\lambda_p+d_p+\mu}\liminf_{t\to\infty}\langle A(t)\rangle\\ &- \liminf_{t\to\infty}\frac{1}{\lambda_p+d_p+\mu}\frac{I(t)-I(0)}{t}\\ &\geq \left(\frac{(1-\alpha)w_p}{\lambda_p+d_p+\mu}+\frac{\xi}{(\lambda_p+d_p+\mu)}\frac{\alpha w_p^{'}}{(\lambda_p^{'}+d_p+\nu+\xi)}\right)\\ &\times\left(\frac{\left(\frac{\sigma^2\Lambda^2(k^2+2k+2)}{2d_p^2}+(1-\alpha)w_p+\alpha w_p^{'}+d_p\right)(R_0^s-1)}{\frac{\beta_I(1-\alpha)w_p}{\lambda_p+d_p+\mu}+\frac{\beta_I\alpha w_p^{'}}{\lambda_p^{'}+d_p+\nu}\left(\frac{\xi}{\lambda_p+d_p+\mu}+k\right)}\right) > 0 \quad a.s\end{aligned} \tag{18}$$

In addition, we have

$$\langle R(t)\rangle = \frac{\lambda_p}{d_p}\langle I(t)\rangle + \frac{\lambda_p^{'}}{d_p}\langle A(t)\rangle + \frac{1}{d_p}\frac{R(t)-R(0)}{t}$$

Applying the limit and according to the theorem 4.1 and the results

$$\liminf_{t\to\infty}\langle I(t)\rangle > 0 \quad \text{and} \quad \liminf_{t\to\infty}\langle A(t)\rangle > 0 \quad a.s$$

We find that

$$\begin{aligned}
\liminf_{t\to\infty}\langle R(t)\rangle =& \frac{\lambda_p}{d_p}\liminf_{t\to\infty}\langle I(t)\rangle + \frac{\lambda_p^{'}}{d_p}\liminf_{t\to\infty}\langle A(t)\rangle + \frac{1}{d_p}\liminf_{t\to\infty}\frac{R(t)-R(0)}{t}\\
\geq& \left(\frac{\lambda_p}{d_p}\frac{(1-\alpha)w_p}{\lambda_p+d_p+\mu} + \frac{\lambda_p\xi}{d_p(\lambda_p+d_p+\mu)}\frac{\alpha w_p^{'}}{(\lambda_p^{'}+d_p+\nu+\xi)} + \frac{\lambda_p^{'}\alpha w_p^{'}}{d_p(\lambda_p^{'}+d_p+\nu+\xi)}\right)\\
&\times\left(\frac{\left(\frac{\sigma^2\Lambda^2(k^2+2k+2)}{2d_p^2}+(1-\alpha)w_p+\alpha w_p^{'}+d_p\right)(R_0^s-1)}{\frac{\beta_I(1-\alpha)w_p}{\lambda_p+d_p+\mu}+\frac{\beta_I\alpha w_p^{'}}{\lambda_p^{'}+d_p+\nu}\left(\frac{\xi}{\lambda_p+d_p+\mu}+k\right)}\right) > 0 \quad a.s
\end{aligned} \tag{19}$$

Moreover, from the equation (15)and according to the theorem 4.1 and the result

$$\liminf_{t\to\infty}\langle E(t)\rangle > 0 \quad a.s$$

We find that

$$\begin{aligned}
\liminf_{t\to\infty}\langle S(t)\rangle =& \frac{\Lambda}{d_p} - \frac{(1-\alpha)w_p+\alpha w_p^{'}+d_p}{d_p}\liminf_{t\to\infty}\langle E(t)\rangle - \frac{1}{d_p}\liminf_{t\to\infty}\varphi(t)\\
\geq& \frac{\Lambda}{d_p} - \left(\frac{(1-\alpha)w_p+\alpha w_p^{'}+d_p}{d_p}\right)\\
&\times\left(\frac{\left(\frac{\sigma^2\Lambda^2(k^2+2k+2)}{2d_p^2}+(1-\alpha)w_p+\alpha w_p^{'}+d_p\right)(R_0^s-1)}{\frac{\beta_I(1-\alpha)w_p}{\lambda_p+d_p+\mu}+\frac{\beta_I\alpha w_p^{'}}{\lambda_p^{'}+d_p+\nu}\left(\frac{\xi}{\lambda_p+d_p+\mu}+k\right)}\right) > 0 \quad a.s
\end{aligned} \tag{20}$$

According to the results (16, 17, 18, 19, 20) we conclude the proof of theorem 5.1.

Remark 5. *These results show that the variables related to the different categories of exposed, infected symptomatic, infected asymptomatic, recovered and susceptible individuals remain positive as time progresses towards infinity. This suggests long-term persistence of the disease.*

□

6 Numerical simulation and discussion

This section presents several numerical simulations aimed at clarifying and validating the theoretical results obtained using the parameter values of our model shown in Table 2. The main objective is to analyze the dynamics of the proposed model in different scenarios, using the Milstein higher-order method. This numerical approach is recognized for its efficiency in taking into account stochastic aspects [12]. The discretization of the system (2) is expressed in the following form:

$$\begin{aligned}
S_{i+1} &= S_i + [\Lambda - \beta_I S_i(I_i + kA_i) - d_p S_i]\,\Delta t - \sigma S_i(I_i + kA_i)\sqrt{\Delta t}\,\epsilon_i \\
&\quad + \frac{\sigma^2}{2} S_i(I_i + kA_i)^2(\epsilon_i^2 - 1)\Delta t \\
E_{i+1} &= E_i + \left[\beta_I S_i(I_i + kA_i) - (1-\alpha)w_p E_i - \alpha w_p' E_i - d_p E_i\right]\Delta t + \sigma S_i(I_i + kA_i)\sqrt{\Delta t}\,\epsilon_i \\
&\quad + \frac{\sigma^2}{2} S_i(I_i + kA_i)^2(\epsilon_i^2 - 1)\Delta t \\
I_{i+1} &= I_i + [(1-\alpha)w_p E_i - (\lambda_p + d_p + \mu)I_i + \xi A_i]\,\Delta t \\
A_{i+1} &= A_i + \left[\alpha w_p' E_i - (\lambda_p' + d_p + \nu)A_i - \xi A_i\right]\Delta t \\
R_{i+1} &= R_i + \left[\lambda_p I_i + \lambda_p' A_i - d_p R_i\right]\Delta t
\end{aligned}$$

Where ϵ_i is mutually independent $N(0,1)$ random variable, where the step size is shown by $\Delta t = 0.01$.

parameter	Extinction	Persistence
Λ	0.1	0.1
β_I	0.03	0.2
ω_p	0.14	1.5
$\omega_p^{'}$	0.05	1.5
α	0.2	0.2
γ_p	0.005	0.02
$\gamma_p^{'}$	0.004	0.02
ξ	0.02	0.01
k	0.4	0.4
d_p	0.05	0.05
ν	0.03	0.03
μ	0.02	0.02
σ	0.5	0.14

Table 2: Values of the parameters used in the simulation

6.1 Results

We solve the stochastic model (2.2) using different parameter values to simulate several scenarios and obtain the required graphical results. In figures 1–2, the subgraphs represent the population of susceptible, exposed, infected symptomatic, infected asymptomatic, and the recovery individuals. Also represents the simulation of the stochastic model versus the deterministic model.

The figure 1, in order to demonstrate the extinction property, as stated in Theorem 4.2, it is essential to carefully choose specific parameter values, thus ensuring that $\sigma^2 > \frac{\beta_I^2}{2d_p}$.
Figure 2 shows us the long-term persistence of the disease within the population, as explained in Theorem 5.1, so that values of the parameters are chosen that confirm that $R_0^s > 1$.
The figure 3 represents the impact of transmission rate β_I on the dynamics of the stochastic model under different value of the parameter β_I. These simulation results provide an in-depth understanding of epidemic dynamics and valuable tools to guide public health decisions.

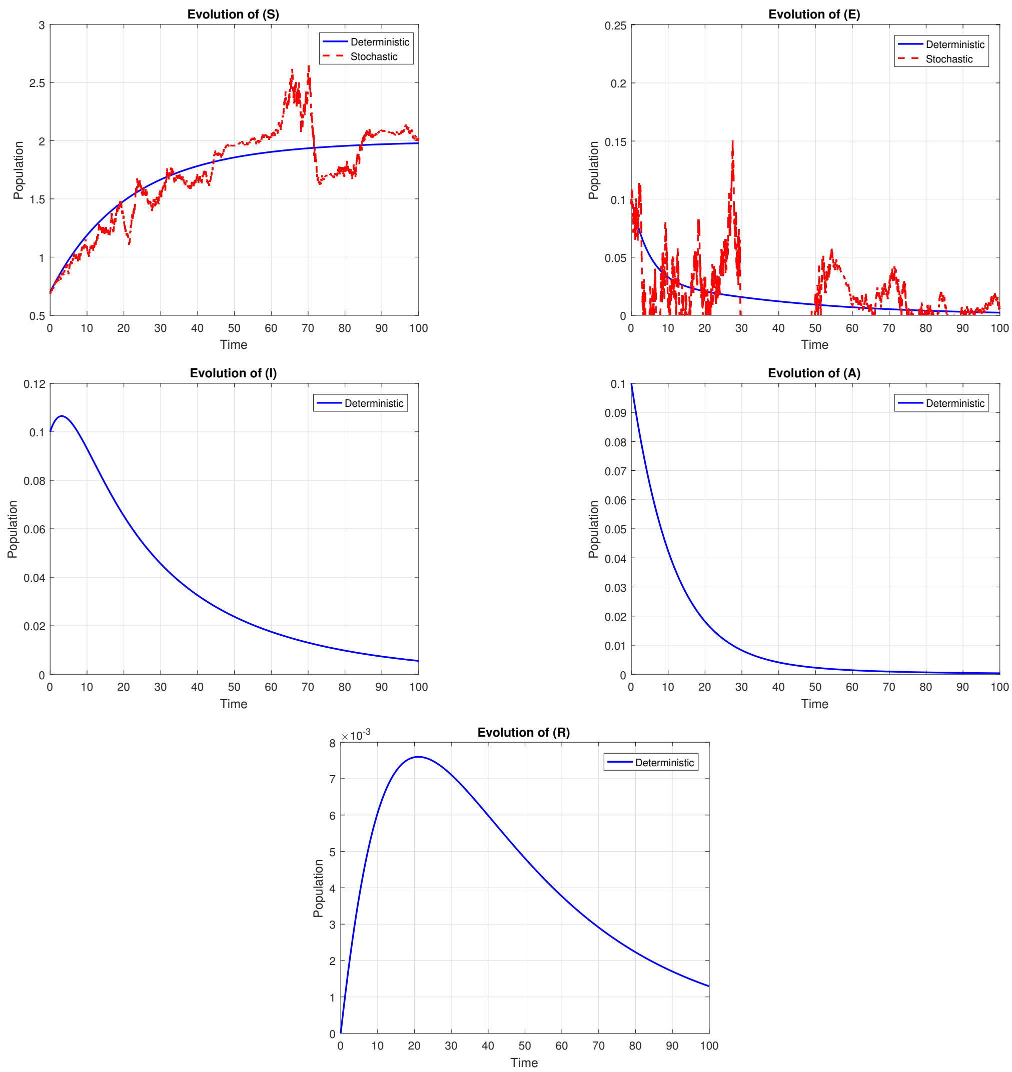

Figure 1: Dynamics of the solution $(S(t), E(t), I(t), A(t), R(t))$ with initial condition (0.7,0.1,0.1,0.1,0). This numerical simulation show that the disease dies out when $\sigma^2 > \frac{\beta_I^2}{2d_p}$.

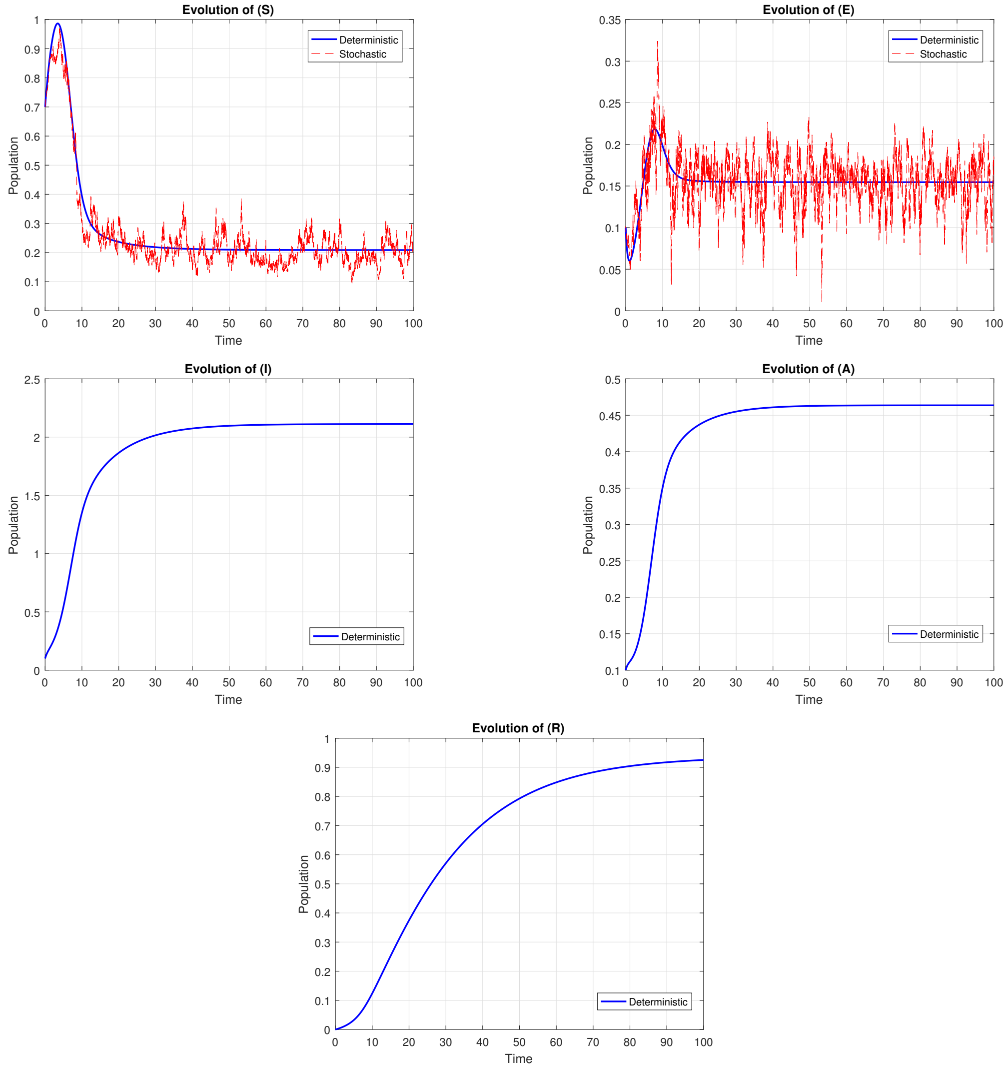

Figure 2: Dynamics of the solution $(S(t), E(t), I(t), A(t), R(t))$ with initial condition (0.7,0.1,0.1,0.1,0). This numerical simulation show that the disease persist when $R_0^s > 1$.

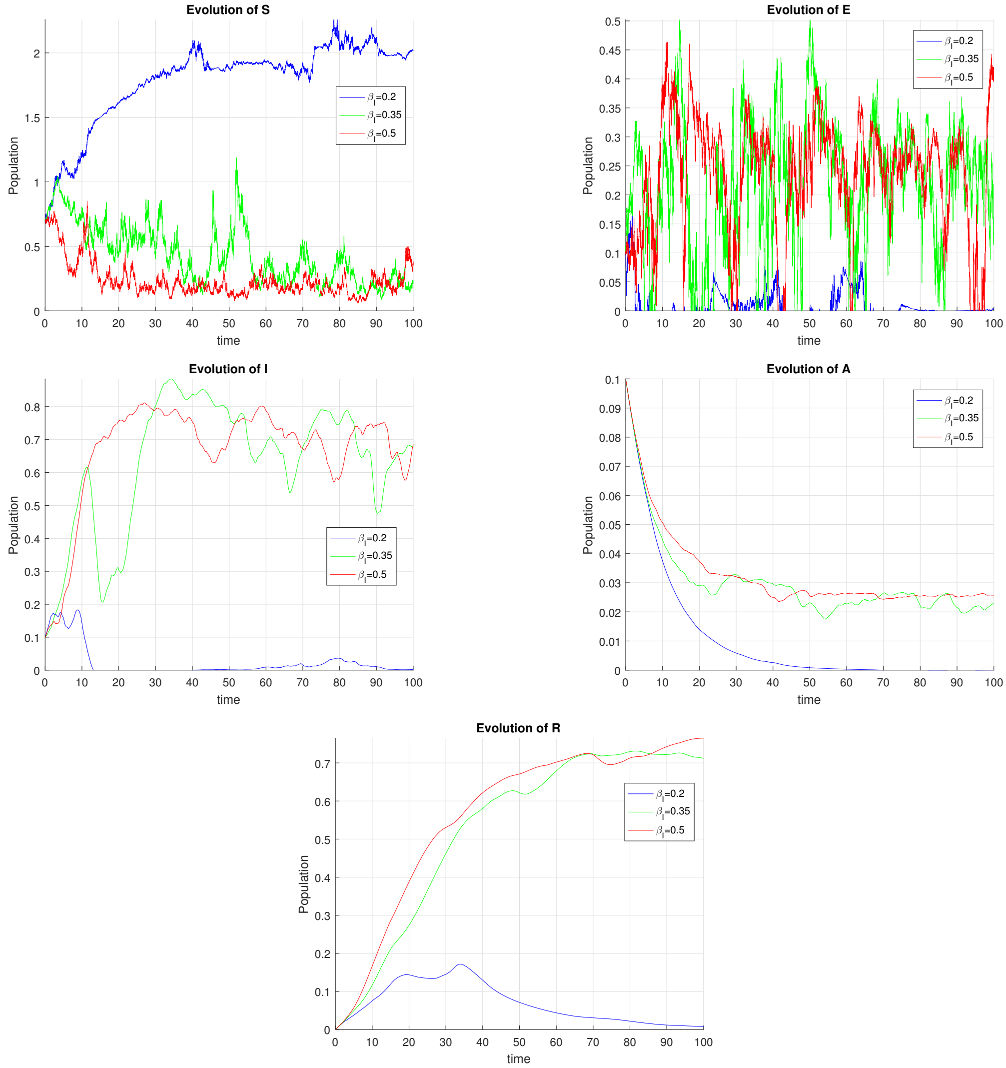

Figure 3: Dynamics of the solution $(S(t), E(t), I(t), A(t), R(t))$ with initial condition (0.7,0.1,0.1,0.1,0) and different values of the transmission rate $\beta_I = [0.2, 0.3, 0.5]$

6.2 Discussion

Using the values for the extinction scenario shown in Table 2, the condition $\sigma^2 > \frac{\beta_I^2}{2d_p}$ gives $0.25 > 0.009$. Thus, the model (illustrated in Figure 1) reveals a fascinating property of extinction, both in deterministic and stochastic scenarios. The disease free equilibrium is globally asymptotically stable, implying that the stochastic solution fluctuates around the deterministic steady-state value. Specifically, some compartments, such as exposed, symptomatic, asymptomatic, and recovered individuals, initially experience an increase before gradually decreasing to zero. In contrast, the compartment of sensitive individuals shows an upward trend before stabilizing around a specific value, given by $\frac{\Lambda}{d_p} = 2$.

The simulation in Figure 2 highlights the persistence property, demonstrating of the infection to maintain itself and avoid extinction over the long term. The introduction of stochasticity generates fluctuations around the deterministic solution, revealing the impact of random events on disease dynamics. For that, when taking the values of the parameters of disease persistence we found that $R_0^s = 1.5479 > 1$. This means that the infection has the ability to persist within the population over an extended period of time, as demonstrated by Theorem 5.1.

Figure 3 shows that the higher the transmission rate, the faster the disease spreads. The number of susceptible people (S) decreases rapidly, the number of infected (I) and asymptomatic (A) peaks earlier and higher, and more people become infected, become infected, and recover, increasing the number of susceptible (E) and recovered (R).

7 Conclusion

In conclusion, Our study provides a comprehensive analysis of a stochastic SEIAR model for COVID-19 where stochasticity is related to transmission rate, shedding light on the complex dynamics of epidemic transmission under the influence of random perturbations. The theoretical proof of the existence and uniqueness of a global positive solution has reinforced the reliability of the model in capturing the essential features of the disease spread. Through the identification of critical parameters, we have outlined the conditions under which the disease may either become extinct if ($\sigma^2 > \frac{\beta_I^2}{2d_p}$) or persist if ($R_0^s > 1$) within the population. Then we studied the effect of the transmission rate β_I on the dynamics of the disease, so that the higher its value, the higher the number of infections, and vice versa.

The numerical simulations carried out have further validated our theoretical findings, offering insightful visualizations of the stochastic dynamics and highlighting the influence of key model parameters on the epidemic trajectory. These results emphasize the importance of early interventions and control measures in mitigating the spread of the disease.

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