GLOBAL SOLUTION AND ASYMPTOTIC BEHAVIOR OF REACTION DIFFUSION SYSTEM WITH NONLINEARITIES OF SIGN NON CONSTANT

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Abstract

The purpose of this current manuscript is to establish global existence and asymptotic behavior in time of solutions for the strongly coupled reaction-diffusion system with diagonal matrix of diffusion coefficients and verifie law of balance and nonlinearities non constant sign. Our techniques of proof are based on Lyapunov functional methods and some G^p estimates .

Keywords: Global Existence, Reaction Diffusion Systems, Lyapunov Functional.
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1 Introduction

In this paper, we study the following semilinear parabolic system :

$$\begin{cases} \frac{6m}{6t} - d_1 \mathrm{A} r = Z(t,x,r,z) & \text{in } \mathbb{R}^+ \times \Omega \\ \frac{6z}{6t} - d_2 \mathrm{A} z = \mu(t,x,r,z) & \text{in } \mathbb{R}^+ \times \Omega \end{cases} \tag{1.1}$$

Where Ω is a regular and bounded domain of R^n, $(n \geq 1)$, $r = r(t,x)$

$z = z(t,x)$, $x \in \Omega, t > 0$ are real valued functions, A denotes the Laplacian operator, and the constants of diffusion d_1, d_2 are assumed to be nonnegative.

System (1.1) is supplemented to the following boundary conditions

$$\frac{6r}{6\eta} = \frac{6z}{6\eta} = 0 \qquad \text{in } \mathbb{R}^+ \times 6\Omega \tag{1.2}$$

and the initial data

$$r(.,0) = r_0(.),\ z(.,0) = z_0(.) \qquad \text{in } \Omega \tag{1.3}$$

which are assumed to be nonnegative.

Concerning the functions Z and μ, we assume the following hypothesis:

(H1) $Z(t,x,r,s)$ and $\mu(t,x,r,s)$ are continuously differentiable on $(\mathbb{R}^+ \times \Omega \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, such that

$$Z(t,x,0,s) \geq 0, \mu(t,x,r,0) \geq 0, 6r, s \geq 0 \tag{1.4}$$

and

$$Z(t,x,r,s) + \mu(t,x,r,s) = 0, 6r, s \geq 0\ , t > 0, x \in \Omega$$

(H2) Assume further that

For all $r, s \geq 0$ and a real $m \geq 1$ such that:

$$\sup(|Z(t,x,r,s)|, |\mu(t,x,r,s)|) \leq K(r+s+1)^m, 6r, s \geq 0 \tag{1.5}$$

As well as, we suppose that, one of the following conditions is provided:

(C1) there exists an integer $6p \geq 2, k(p) > 0$ and positive numbers $({}_i(p))_{0 \leq i \leq p}$ such that

$$c^{i+1}\ {}_i(p)Z(t,x,r,s) + c^i\ {}_{i-1}(p)\mu(t,x,r,s) \leq k(p)(r+s+1) \tag{1.6}$$

with

$$c^{i+1}_i(p) \leq \frac{4(d_1 d_2)}{(d_1+d_2)^2} c^{i+2}_{i+1}(p) c^i_{i-1}(p) \tag{1.7}$$

(C2) there exist $k(1) > 0$ and positive number $({}_i(1))$ such that

$$\begin{gathered} c^2\ {}_1(1)Z(t,x,r,s) + c\ {}_0(1)\mu(t,x,r,s) \leq k(1)(r+s+1) \\ c\ ^0(1), c^2\ ^1(1) > 0 \end{gathered} \tag{1.8}$$

The above system (1.1)–(1.3) arises in physics, chemistry and various biological processes including population dynamics. (See [6], [22] and references therein).

Condition (1.2) means that there is no species of immoderation .

The main question we want to address is the existence of global and asymptotic behavior solutions for system (1.1)–(1.3).

We are interested with global existence in time of positive solutions to problem (1.1)-(1.3). When the reaction Z is constant sign, then global existence is immediate and many results have been obtained. Note that, Alikakos [1], he showed the existence of global bounded solutions if $Z(r,z) = -\mu(r,z) = -rz^{\sigma}$ for $1 < \sigma < \frac{n+2}{n}$. The extension of this result for $\sigma > 1$ is obtained by Masuda [16]. Then Haraux and Youkana [4], generalized the result of Masuda via the functional of Lyapunov by putting $Z(r,z) = -\mu(r,z) = -r\mathrm{W}(z)$ where W is a nonlinear function satisfying the condition:

$$\lim_{G_2 \to +\infty} \frac{[\log(1+\mathrm{W}(\ _2))]}{_2} = 0.$$

Barabanova , generalize the result of Haraux and Youkana concerning the global existence of nonnegative solutions of a reaction-diffusion equation with exponential nonlinearity.

There is also another very powerful method that relies on compact semigroups, which is the method we will use in this work. For a better understanding, we send the reader to the works of Moumeni and Salah Derradji [17]. Later on, a more general model was studied by Haraux and Kirane [3] . They took different diffusion coefficients for the two equations and general nonlinear terms. They proved the existence of global bounded solutions and investigated their asymptotic behavior (see [7], [8], [11], [12] and [21]). It is clear that here, the strict control of mass is satisfied

$$Z(t,x,r,s) + \mu(t,x,r,s) = 0, 6r, s \geq 0\ , t > 0, x \in \Omega. \tag{1.9}$$

which gives, with homogenous boundary conditions (1.2) , the total mass of the solution is invariant.

In diagonal case, when the reaction is constant sign, the existence in time of a global solution is trivial: It is the case for example when the reaction Z positive, then by the maximum principle we get the uniform boundeness of the component r and using the the strict balance law (1.9), we deduce the boundeness z and the global existence in time follows automatically. Also when the coefficients of diffusion

$d_1 = d_2$ global existence is deduced by summing the two equations, using the positivity of the solutions and applying the maximum principle. In addition, of $d_1 = d_2$ the reaction is not constant sign and the maximum principle isn't applicable, Kouachi and many authers show this situation (see [14], [10], [23]) for which only two main properties hold: Quasi positivity and balance law but with two difficulties: They change sign and with unlimited polynomialgrowth. We over come the first difficulty by fixing the reaction sign after some time and the second one by using a judious polynomial Lyapunov functional . The reactions terms chang sign signifies that none of the equations is nice in the sense that neither r and z is a priori boundede or at least boundede in some G^p space for p large to apply the well known regularizing effect and deduce the global existence of strong solutions in time for problem (1.1)-(1.3). We should remark that the global existence of solutions with quadratic nonlinearities was proved by [23] in the whole space as well as for boundede domains and without using the control mass condition.

The remainder of the current manuscript is organized as follows. In section 2, we introduce some various preliminaries and give a local existence result. In section 3, the global existence of strong solution. Finally, asymptotic behavior of solution are studied in section 4 .

2 Local existence

Throughout this work, we denote by $\|.\|_p, p \in [1;+\infty)$ the norm in G^p and $\|.\|_\infty$ the norm in $C(\Omega)$ or G^∞, respectively, defined by $\|r\|_p = \left(\int_\Omega |r|^p dx\right)^{\frac{1}{p}}$ and $\|r\|_\infty = ess\sup_{x\in\Omega} |r(x)|$.

The study of local existence and uniqueness of solutions $(r;z)$ of (1.1)-(1.3) follows from the basic existence theory for parabolic semi linear equations (detect, e.g., [2], [10]). Consequently, for any initial data in G^∞ there exists a $T_{\max} \in (0;+\infty]$ such that (1.1)-(1.3) has a unique classical solution on $(0, T_{\max}[\times \Omega$. Arthromere,

if $T_{\max} < \infty$ then $\lim_{t\to T_{\max}} \{\|r(t,.)\|_\infty, \|z(t,.)\|_\infty\} = +\infty$

For the record, if there exists a positive constant C such that

$\|r(t,.)\|_\infty + \|z(t,.)\|_\infty \le C \; \forall t \in [0, T_{\max})$ therefore $T_{\max} = +\infty$

Remark2.1

Underneath condition (H1), it follows from the invariant region mode that system (1.1)–(1.3) processes positivity. Particularly, if the initial data r_0 and z_0 in (1.3) are nonnegative, thereupon the functions r and z of the corresponding solution of (1.1)–(1.3) are as well as nonnegative on $]0, T_{\max}[\times\Omega$. See [10].

3 Statement of the main results

3.1 Existence of global solutions

In this section, we state and prove our global existence result of system (1.1)–(1.3). Our main theorem reads as follows.

Theorem3.1

Let $p > \frac{mn}{2}$. We assume that the condition (1.5) holds and one of the conditions (1.6) or (1.8) are satisfied.

At that point, the solution $(r(t,.), z(t,.))$ of (1.1)–(1.3) with initial positive condition in $G^\infty(\Omega)$ exists globally in time.

We sign that to prove Theorem 3.1 it is sufficient to derive a uniform estimate of $\sup(\|Z(t,x,r,z)\|_q, \|\mu(t,x,r,z)\|_q)$ for some $q > n/2$. (See [10] for more details).

The following lemma is the key element in the proof of the Theorem 3.1.

Lemma3.1

Let $(r(t,.), z(t,.))$ be the solution of (1.1)–(1.3). If one of the conditions (1.6) or (1.8) has been satisfied , there would exist an integer $p \ge 1$

and a continous function $K_p : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\sup(\|r(t,.)\|_p, \|z(t,.)\|_p) \le K_p(t), t \le T_{\max}$.

Proof

Let us consider the function H_p defined by

$$H_p(t) = \int_\Omega \sum_{i=1}^{p} (c^{i+1} C_p^i \theta_i(p) r^i z^{p-i}) dx = \int_\Omega \sum_{i=1}^{p} (\vartheta_i(p) r^i z^{p-i}) dx \tag{3.1}$$

where

$$\vartheta_i(p) = c^{i+1} C_p^i \theta_i(p) \tag{3.2}$$

Differentiating G with respect to t takings

$$
\begin{aligned}
H_p'(t) &= \int_\Omega \left[\sum_{i=1}^{p} (i\sigma_i(p) r^{i-1} z^{p-i}) r_t + \sum_{i=0}^{p-1} ((p-i)\sigma_i(p) r^i z^{p-i-1}) z_t \right] dx \\
&= \int_\Omega \sum_{i=1}^{p} (i\sigma_i(p) r^{i-1} z^{p-i})(d_1 \Delta r + Z(t,x,r,z))dx + \\
&\quad \int_\Omega \sum_{i=0}^{p-1} ((p-i)\sigma_i(p) r^i z^{p-i-1})(d_2 \Delta z + \mu(t,x,r,z))dx
\end{aligned}
$$

A simple data processing leads

$$
\begin{aligned}
H_p'(t) &= \int_\Omega \sum_{i=1} (i\sigma_i(p) r^{i-1} z^{p-i})(d_1 \Delta r + Z(t,x,r,z))dx + \\
&\quad \int_\Omega \sum_{i=1}^{p} ((p-i+1)\sigma_{i-1}(p) r^{i-1} z^{p-i})(d_2 \Delta z + \mu(t,x,r,z))dx
\end{aligned}
$$

Against the above equality, as a deduction

$$
\begin{aligned}
H_p'(t) &= \int_\Omega \sum_{i=1}^{p} d_1 i\sigma_i(p) r^{i-1} z^{p-i} \Delta r dx + \int_\Omega \sum_{i=1}^{p} d_2 (p-i+1)\sigma_{i-1}(p) r^{i-1} z^{p-i} \Delta z dx \\
&\quad + \int_\Omega \sum_{i=1}^{p} i\sigma_i(p) r^{i-1} z^{p-i} Z(t,x,r,z)dx + \int_\Omega \sum_{i=1}^{p} (p-i+1)\sigma_{i-1}(p) r^{i-1} z^{p-i} \mu(t,x,r,z)dx
\end{aligned}
\tag{3.3}
$$

At this point, we distinguish two cases:
Case 1: When $p = 1$, we get from (3.3)

$$
\begin{aligned}
H_1'(t) &= \int_\Omega \sum_{i=1}^{p} d_1 \sigma_1(1) \Delta r dx + \int_\Omega \sum_{i=1}^{p} d_2 \sigma_0(1) \Delta z dx \\
&\quad + \int_\Omega \sum_{i=1}^{p} \sigma_1(1) Z(t,x,r,z)dx + \int_\Omega \sum_{i=1}^{p} \sigma_0(1) \mu(t,x,r,z)dx
\end{aligned}
\tag{3.4}
$$

By a simple use of Green's formula we attain:

$$
\begin{aligned}
H_1'(t) &= \int_\Omega \sum_{i=1}^{p} \sigma_1(1) Z(t,x,r,z)dx + \int_\Omega \sum_{i=1}^{p} \sigma_0(1) \mu(t,x,r,z)dx \\
&= \int_\Omega \sum_{i=1}^{p} c^2{}_1(1) Z(t,x,r,z)dx + \int_\Omega \sum_{i=1}^{p} c{}_0(1) \mu(t,x,r,z)dx
\end{aligned}
$$

Using condition (1.8) we deduce,

$$
\begin{aligned}
H_1'(t) &\leq k(1) \int_\Omega (r + z + 1)dx \\
&= k(1) \int_\Omega (r+z)dx + k(1) mes(\Omega)
\end{aligned}
$$

Hence, the functional H_1 satisfies

$$
H_1'(t) \leq k_1(1) H_1 + k_2(1), \forall t < T_{\max}
$$

with

$$k_1(1) = \frac{k(1)}{\min(\sigma_1(1), \sigma_0(1))}, k_2(1) = k(1)mes(\Omega)$$

Which gives us, by a simple integration of the above inequality

$$H_1(t) \leq [H_1(0) + \frac{k_2(1)}{k_1(1)}]\exp(k_1(1) \leq t) - \frac{k_2(1)}{k_1(1)}]$$

It's not hard to note that from (3.1) we gain

$$\begin{aligned} H_1(t) &\geq \min(\sigma_1(1), \sigma_0(1)) \int_\Omega (r+z)dx \\ &\geq \min(\sigma_1(1), \sigma_0(1)) \sup(|r(t,.)|_1, |z(t,.)|_1) \end{aligned}$$

Consequently, we obtain

$$\sup(|r(t,.)|_1, |z(t,.)|_1) \leq k_1(t), \forall t < T_{\max} \tag{3.5}$$

where

$$k_1(t) = \frac{1}{\min(\sigma_1(1), \sigma_0(1))} \times [H_1(0) + \frac{k_2(1)}{k_1(1)}]\exp(k_1(1) \leq t) - \frac{k_2(1)}{k_1(1)}]$$

Case 2: When $p \geq 2$,we earn

$$\begin{aligned} \Gamma &= \int_\Omega \sum_{i=1}^{p} d_1 i\sigma_i(p) r^{i-1} z^{p-i} \Delta r dx + \int_\Omega \sum_{i=1}^{p} d_2(p-i+1)\sigma_{i-1}(p) r^{i-1} z^{p-i} \Delta z dx \\ &= \int_\Omega \sum_{i=1}^{p} \Delta\{d_1 i\sigma_i(p) r\} r^{i-1} z^{p-i} dx + \int_\Omega \sum_{i=1}^{p} \Delta\{d_2(p-i+1)\sigma_{i-1}(p) z\} r^{i-1} z^{p-i} dx \end{aligned}$$

Which impllies

$$\Gamma = \sum_{i=1}^{p} \int_\Omega \Delta\{d_1 i\sigma_i(p) r\} r^{i-1} z^{p-i} dx + \sum_{i=1}^{p} \int_\Omega \Delta\{d_2(p-i+1)\sigma_{i-1}(p) z\} r^{i-1} z^{p-i} dx$$

Therefore, Green's formula gives

$$\Gamma = -\sum_{i=1}^{p} \int_\Omega \partial\{d_1 i\sigma_i(p) r\} \partial\{r^{i-1} z^{p-i}\} dx + \sum_{i=1}^{p} \int_\Omega \partial\{d_2(p-i+1)\sigma_{i-1}(p) z\} \partial\{r^{i-1} z^{p-i}\} dx$$

Then,

$$\begin{aligned} \Gamma = &- \int_\Omega \sum_{i=1}^{p} d_1(i-1) i\sigma_i(p) r^{i-2} z^{p-i} \partial^2 r dx + \int_\Omega \sum_{i=1}^{p-1} d_1 i(p-i) r^{i-1} z^{p-i-1} \partial r \partial z dx \\ &+ \int_\Omega \sum_{i=1}^{p} d_2(i-1)(p-i+1)\sigma_{i-1}(p) r^{i-2} z^{p-i} \partial r \partial z dx \\ &+ \int_\Omega \sum_{i=1}^{p-1} d_2(p-i+1)(p-i)\sigma_{i-1}(p) r^{i-1} z^{p-i-1} \partial^2 z dx \end{aligned}$$

Away,

$$\Gamma = -\int_\Omega \sum_{i=1}^{p-1} [d_1 i(i+1)6_{i+1}(p)\partial^2 r + (d_1+d_2)i(p-i)6_i(p)\partial r\partial z + d_2(p-i+1)(p-i)6_{i-1}(p)\partial^2 z] r^{i-1}z^{p-i-1}dx$$

Thus, the relation (3.3) comes

$$H_p'(t) = \int_\Omega \sum_{i=1}^{p} i6_i(p)r^{i-1}z^{p-i}Z(t,x,r,z)dx + \int_\Omega \sum_{i=1}^{p}(p-i+1)6_{i-1}(p)r^{i-1}z^{p-i}\mu(t,x,r,z)dx$$
$$-\int_\Omega \{\sum_{i=1}^{p-1}[d_1 i(i+1)6_{i+1}(p)\partial^2 r + (d_1+d_2)i(p-i)6_i(p)\partial r\partial z + d_2(p-i+1)(p-i)6_{i-1}(p)\partial^2 z] r^{i-1}z^{p-i-1}\}dx$$

Also,

$$6_i(p) = c^{i+1}C_p^i \,{}_i(p),\ i = 0,\dots,p$$

Accordingly,

$$H_p'(t) = \int_\Omega \sum_{i=1}^{p} ic^{i+1}C_p^i \,{}_i(p)r^{i-1}z^{p-i}Z(t,x,r,z)dx + \int_\Omega \sum_{i=1}^{p}(p-i+1)c^iC_p^{i-1}\,{}_{i-1}(p)r^{i-1}z^{p-i}\mu(t,x,r,z)dx$$
$$-\int_\Omega \{\sum_{i=1}^{p-1}[d_1 i(i+1)c^{i+2}C_p^{i+1}\,{}_{i+1}(p)\partial^2 r + (d_1+d_2)i(p-i)c^{i+1}C_p^i\,{}_i(p)\partial r\partial z + d_2(p-i+1)(p-i)c^iC_p^{i-1}\,{}_{i-1}(p)\partial^2 z] r^{i-1}z^{p-i-1}\}dx$$

Practicing the gate that :

$$iC_p^i = (p-i+1)C_p^{i-1} = pC_{p-1}^{i-1} \quad 6i = 1,\dots,p$$

and in the same manner

$$i(i-1)C_p^{i+1} = i(p-i)C_p^i = (p-i)(p-i+1)C_p^{i-1} = p(p-1)C_{p-2}^{i-2}$$

As a deduction,

$$H_p'(t) = \int_\Omega \{\sum_{i=1}^{p} pC_{p-1}^{i-1}[c^{i+1}\,{}_i(p)Z(t,x,r,z)dx + c^i\,{}_{i-1}(p)\mu(t,x,r,z)](r^{i-1}z^{p-i})dx$$
$$-p(p-1)\int_\Omega \{\sum_{i=1}^{p-1} C_{p-2}^{i-1}[d_1c^{i+2}\,{}_{i+1}(p)\partial^2 r + (d_1+d_2)c^{i+1}\,{}_i(p)\partial r\partial z + d_2c^i\,{}_{i-1}(p)\partial^2 z] r^{i-1}z^{p-i-1}\}dx$$

The quadratic forms

$$d_1c^{i+2}\,{}_{i+1}(p)\partial^2 r + (d_1+d_2)c^{i+1}\,{}_i(p)\partial r\partial z + d_2c^i\,{}_{i-1}(p)\partial^2 z$$

are positive since from (1.8) we gain

$$c^{i+1}\,{}_i(p) \le \frac{4(d_1d_2)}{(d_1+d_2)^2}c^{i+2}\,{}_{i+1}(p)c^i\,{}_{i-1}(p)$$

Hence,

$$H_P'(t) \le \int_\Omega \left\{ \sum_{i=1} pC_{p-1}^{i-1} [c^{i+1}{}_i(p)Z(t,x,r,z)dx + c^i{}_{i-1}(p)\mu(t,x,r,z)](r^{i-1}z^{p-i})dx \right.$$

In view of assumptions (1.6), we deduce :

$$H_P'(t) \le k'(p) \int_\Omega \left[\sum_{i=1}^{p} (r+z+1) C_{p-1}^{i-1} r^{i-1} z^{p-i} \right] dx$$

To prove that the functional G is uniformly bounded on the interval $[0, T^*]$
First, we write:

$$H_P'(t) \le k'(p) \int_\Omega \left[\sum_{i=1}^{p} C_{p-1}^{i-1} r^i z^{p-i} + \sum_{i=1} C_{p-1}^{i-1} r^{i-1} z^{p-i+1} + \sum_{i=1} C_{p-1}^{i-1} r^{i-1} z^{p-i} \right] dx$$

$$H_P'(t) \le k'(p) \int_\Omega \left[\sum_{i=1} C_{p-1}^{i-1} r^i z^{p-i} + \sum_{i=0}^{p-1} C_{p-1}^{i} r^i z^{p-i} + \sum_{i=0}^{p-1} C_{p-1}^{i} r^i z^{p-i-1} \right] dx$$

$$H_P'(t) \le k'(p) \int_\Omega \left[\sum_{i=0}^{p} C_p^i r^i z^{p-i} + \sum_{i=0}^{p-1} C_{p-1}^{i} r^i z^{p-i-1} \right] dx$$

In conjunction with

$$\sum_{i=0}^{p-1} C_{p-1}^{i} r^i z^{p-i-1} = (r+z)^{p-1}$$

Rothermere, the last inequality can be written as

$$H_p'(t) \le k_1(p) H_P(t) + k'(p) \int_\Omega (r+z)^{p-1}$$

Exploiting Hôlder's inequality to the second term in the right hand side of
the above inequality, we obtain

$$H_P'(t) \le k_1(p) H_P(t) + k'(p)(mes\Omega)^{\frac{1}{p}} \left(\int_\Omega (r+z)^p \, dx \right)^{\frac{(p-1)}{p}}$$

Then the following inequality holds,

$$(r+z)^p = \sum_{i=0}^{p} C_p^i r^i z^{p-i} \le \frac{\sup_{0\le i\le p} C_p^i}{\min_{0\le i\le p} C_p^i \, i^2} \sum_{i=0}^{p} 6(p) r^i z^{p-i}$$

So , we have

$$H_P'(t) \le k_1(p) H_P(t) + k'(p)(mes\Omega)^{\frac{1}{p}} \left(\frac{\sup_{0\le i\le p} C_p^i}{\min_{0\le i\le p} C_p^i \, i^2} \right)^{\frac{(p-1)}{p}} (H_p(t))^{\frac{(p-1)}{p}} \qquad 6t < T_{\max}$$

Thence, $H_P(t)$ the functional satisfies the following differential inequality:

$$H_P'(t) \le k_1(p) H_P(t) + k_2(p)(H_p(t))^{\frac{(p-1)}{p}} \qquad 6t < T_{\max} \tag{3.6}$$

where

$$k_2(p) = k'(p)(mes\Omega)^{\frac{1}{p}}\left(\frac{\sup_{0\le i\le p} C_p^i}{\min_{0\le i\le p} C_p^i K^{i^2}}\right)^{\frac{(p-1)}{p}}$$

which gives us, by a simple integration

$$(H_p(t))^{\frac{1}{p}} \le \left((H_p(t)(0))^{\frac{1}{p}} + \frac{k_2'(p)}{k_1'(p)} \exp(k_1'(p)t) - \frac{k_2'(p)}{k_1'(p)}\right)$$

with

$$k_1'(p) = \frac{k_1(p)}{p} \qquad k_2'(p) = \frac{k_2(p)}{p}$$

By manipulating the inequality

$$H_p(t) = \int_\Omega \left(\sum_{i=0}^{p} c^{i+1} C_p^i\, {}_i(p) r^i z^{p-i}\right)dx \ge \int_\Omega (6_p(p)r^p + 6_0(p)z^p)dx \tag{3.7}$$

accordingly

$$H_p(t) \ge \min(6_0(p), 6_p(p)) \sup\left(\int_\Omega r^p dx, \int_\Omega z^p dx\right)$$

Consequently,

$$(H_p(t))^{\frac{1}{p}} \ge [\min(6_0(p), 6_p(p))]^{\frac{1}{p}} \sup\left(\left(\int_\Omega r^p dx\right)^{\frac{1}{p}}, \left(\int_\Omega z^p dx\right)^{\frac{1}{p}}\right)$$

And forevermore,

$$\sup(|r(t,.)|_p, |z(t,.)|_p) \le \frac{(H_p(t))^{\frac{1}{p}}}{[\min(6_0(p), 6_p(p))]^{\frac{1}{p}}} \quad 6t < T_{\max} \tag{3.8}$$

With (3.7) and (3.8) we obtain :

$$\sup(|r(t,.)|_p, |z(t,.)|_p) \le k(t) \qquad 6t < T_{\max} \tag{3.9}$$

where

$$k(t) = \frac{1}{[\min(6_0(p), 6_p(p))]^{\frac{1}{p}}}\left((H_p(t)(0))^{\frac{1}{p}} + \frac{k_2'(p)}{k_1'(p)} \exp(k_1'(p)t) - \frac{k_2'(p)}{k_1'(p)}\right)$$

The proof of Lemma 3.1 is complete.
Proof of theorem3.1
From (1.6)we have

$$\sup(|Z(t,x,r,z)|, |\mu(t,x,r,z)|) \le K(r+z+1)^m$$

Then, it follows that

$$\sup\left(\int_\Omega |Z(t,x,r,z)|^{\frac{p}{m}} dx, \int_\Omega |\mu(t,x,r,z)|^{\frac{p}{m}} dx\right) \le K^{\frac{p}{m}} \int_\Omega (r+z+1)^p dx$$

which implies :

$$\sup(|Z(t,x,r,z)|^{\frac{p}{m}}_{\frac{p}{m}}, |\mu(t,x,r,z)|^{\frac{p}{m}}_{\frac{p}{m}}) \le K^{\frac{p}{m}} \int_\Omega (r+z+1)^p dx \tag{3.10}$$

On the other hand, we get

$$\int_\Omega (r+z+1)^p\,dx = \int_\Omega \sum_{h=0}^{p} C_p^h (r+z)^h\,dx$$

$$\int_\Omega (r+z+1)^p\,dx = \int_\Omega [1+(r+z)^p]dx + \sum_{h=1}^{p-1} C_p^h \int_\Omega (r+z)^h$$

An application of Hôlder's inequality leads

$$\sum_{h=1}^{p-1} C_p^h \int_\Omega (r+z)^h \le \sum_{h=1}^{p-1} C_p^h \left[\int_\Omega 1^{\frac{p}{(p-k)}} dx\right]^{\frac{(p-h)}{p}} \left[\int_\Omega (r+z)^p dx\right]^{\frac{h}{p}}$$

Thence

$$\int_\Omega (r+z+1)^p\,dx \le mes(\Omega) + \int_\Omega (r+z)^p\,dx + \sum_{h=1}^{p-1} C_p^h \,(mes(\Omega))^{\frac{(p-h)}{p}} \left[\int_\Omega (r+z)^p\,dx\right]^{\frac{h}{p}} \tag{3.11}$$

Expoloiting (3.10) we acheive:

$$\left(\int_\Omega (r+z)^p\,dx\right)^{\frac{1}{p}} = |r(t,.)+z(t,.)|_p \le |r(t,.)|_p + |z(t,.)|_p \le 2k_p(t)$$

and the inequality (3.11) can be written as follows

$$\int_\Omega (r+z+1)^p\,dx \le mes(\Omega) + 2^p(k_p(t))^p + \sum_{h=1}^{p-1} C_p^h[(mes(\Omega))^{\frac{(p-k)}{p}}(2k_p(t))^h$$

$$\le \sum_{h=0}^{p} C_p^h[(mes(\Omega))^{\frac{(p-h)}{p}}(2k_p(t))^h$$

Therefore

$$\sup((|Z(t,x,r,z)|_{\frac{p}{m}}^{\frac{p}{m}}, |g(t,x,u,v)|_{\frac{p}{m}}^{\frac{p}{m}}) \le K^{\frac{p}{m}} \sum_{h=0}^{p} C_p^h[(mes(\Omega))^{\frac{(p-h)}{p}}(2k_p(t))^h \tag{3.12}$$

which gives that

$$\sup(|Z(t,x,r,z)|_{\frac{p}{m}}, |\mu(t,x,r,z)|_{\frac{p}{m}}) \le k_{p,m}(t) \quad \forall t < T_{\max} \tag{3.13}$$

where

$$k_{p,m}(t) = k\left[\sum_{h=0}^{p} 2^h C_p^h[(mes(\Omega))^{\frac{(p-h)}{p}}(k_p(t))^h\right]^{\frac{p}{m}}$$

Remark3.1

From both Lemma 3.1 and Theorem 3.1, we have obtained an uniform estimate of $\sup(|Z(t,x,u,v)|_q, |\mu(t,x,u,v)|_q)$ with $q = p/m > n/2$. By the preliminary remarks, we conclude that the solution of the given problem exists globally in time.

3.2 Asymptotic behavior of solution

In this section, we show our asymptotic behavior result of system (1.1)–(1.3). Thanks too Kirane [9]

Theorem4.1

the solution $v = (r, z)$ of the system (1)-(3) converges a constant vector of the form $\eta = (\eta_1, \eta_2)$ as $t \to \infty$, uniformly in $\overline{\Omega}$.

$$r \underset{t\to\infty}{\to} \eta_1$$

$$z \underset{t\to\infty}{\to} \eta_2$$

Furthermore, we have :$\eta_1 \geq 0, \eta_2 \geq 0, Z(t, x, \eta_1, \eta_2) = 0$
and $\eta_1 + \eta_2 = \frac{1}{|\Omega|} \int_\Omega (r_0(x) + z_0(x))\, dx.$

The following lemma is a useful tool in the proof of the Theorem

Lemma4.1

Let (r, z) be a solution of (1.1)-(1.3). we have

(i) $\int_{Q_T} |\nabla r|^2 dxdt < \infty$

(ii) $\int_{Q_T} |\nabla z|^2 dxdt < \infty$

here $Q_T = \Omega \times [0, T]$ and $0 < T < \infty$

Proof

We have $\frac{\partial r}{\partial t} - d_1 \Delta r = Z(t, x, r, z)$,by integrating over $(0, T)$ is obtained

$$\int_0^T \frac{\partial r}{\partial t}(x, t)dt = d_1 \int_0^T \Delta r dt + \int_0^T Z(t, x, r(x, t), z(x, t))dt$$

$$r(x, T) - r(x, 0) = d_1 \int_0^T \Delta r dt + \int_0^T Z(t, x, r(x, t), z(x, t))dt$$

and integrating a second time is obtained over Ω

$$\int_\Omega r(x, T)dx - \int_\Omega r(x, 0)dx = d_1 \int_\Omega \int_0^T \Delta r dtdx + \int_\Omega \int_0^T Z(t, x, r(x, t), z(x, t))dtdx$$

Green's formula is applied to $\int_\Omega \Delta r dx$ and $\int_\Omega \Delta z dx$, we gain

$$\int_\Omega \Delta r dx = \int_{\partial\Omega} \frac{\partial r}{\partial \eta} d\sigma - \int_\Omega \nabla r \nabla 1 dx \qquad \text{hereinafter} \int_\Omega \Delta r dx = 0$$

$$\int_\Omega \Delta z dx = \int_{\partial\Omega} \frac{\partial z}{\partial \eta} d\sigma - \int_\Omega \nabla z \nabla 1 dx \qquad \text{onward} \int_\Omega \Delta z dx = 0$$

thus

$$\int_\Omega \int_0^T Z(t, x, r(x, t), z(x, t))dtdx = \int_\Omega r_0(x)dx - \int_\Omega r(x, T)dx < \infty$$

because $r(T) \in C(\overline{\Omega})$

Implying

$$\int_{Q_T} Z(t, x, r(x, t), z(x, t))dtdx < \infty$$

In the same way, we get that

$$\int_{Q_T} \mu(t, x, r(x, t), z(x, t))dtdx < \infty$$

Multiply now the equation $\frac{\partial r}{\partial t} - d_1 \Delta r = Z(t, x, r, z)$ by r and integrating over Q_T

We obtain

$$\int_\Omega \int_0^T r \frac{\partial r}{\partial t}(x,t)dtdx = d_1 \int_\Omega \int_0^T r \Delta r dtdx + \int_\Omega \int_0^T rZ(t,x,r(x,t),z(x,t))dtdx$$

by using the Green formula

$$\int_\Omega r\Delta r dx = \int_{\partial\Omega} \frac{\partial r}{\partial \eta} d\sigma - \int_\Omega |\nabla r|^2 dx, \qquad \int_\Omega r\Delta r dx = -\int_\Omega |\nabla r|^2 dx$$

and a simple calculation , it becomes

$$\frac{1}{2}\int_\Omega [r^2(x,t)]_0^T dx = -d_1 \int_0^T \int_\Omega |\nabla r|^2 dxdt + \int_0^T \int_\Omega r(x,t)Z(t,x,r(x,t),z(x,t))dxdt$$

thus

$$\int_\Omega r^2(x,T) + 2d_1 \int_{Q_T} |\nabla r|^2 dxdt = \int_\Omega r_0^2(x)dx + 2\int_{Q_T} r(x,t)Z(t,x,r(x,t),z(x,t))dxdt$$

we conclude

$$2d_1 \int_{Q_T} |\nabla r|^2 dxdt \leq \int_\Omega r_0^2(x)dx + 2\int_{Q_T} r(x,t)Z(t,x,r(x,t),z(x,t))dxdt \tag{4.1}$$

In like manner, multiply the equation $\frac{\partial z}{\partial t} - d_2\Delta z = \mu(t,x,r,z)$ by z and integrating over Q_T is obtained

$$\int_\Omega \int_0^T z \frac{\partial z}{\partial t}(x,t)dtdx = d_2 \int_\Omega \int_0^T z\Delta z dtdx + \int_\Omega \int_0^T z\mu(t,x,r,z)\,dtdx$$

Adopting the Green formula:

$$\int_\Omega z\Delta z dx = \int_{\partial\Omega} \frac{\partial z}{\partial \eta} d\sigma - \int_\Omega |\nabla z|^2 dx, \quad \text{ergo} \quad \int_\Omega z\Delta z dx = -\int_\Omega |\nabla z|^2 dx$$

thereupon,

$$\frac{1}{2}\int_\Omega [z^2(x,t)]_0^T dx = d_2 \int_0^T \int_\Omega |\nabla z|^2 dxdt + \int_0^T \int_\Omega z(x,t)\mu(t,x,r,z))dxdt$$

$$\int_\Omega z^2(x,T) + 2d_2 \int_{Q_T} |\nabla z|^2 dxdt = \int_\Omega z_0^2(x)dx + 2\int_{Q_T} z(x,t)\mu(t,x,r(t,x),z(t,x))\,dxdt$$

henceforward ,

$$2d_2 \int_{Q_T} |\nabla z|^2 dxdt \leq \int_\Omega z_0^2(x)dx + 2\int_{Q_T} z(x,t)\mu(t,x,r(t,x),z(t,x))\,dxdt \tag{4.2}$$

since

$$\int_\Omega r_0^2(x)dx < \infty, \quad \int z_0^2(x)dx < \infty$$

Ω

and

$$\int_{Q_T} z(x,t)\mu(t,x,r(x,t),z(x,t))dxdt \le |z|_{K^\infty(Q_T)} \int_{Q_T} \mu(t,x,r(x,t),z(x,t))dxdt < \infty$$

and

$$\int_{Q_T} r(x,t)Z(t,x,r(x,t),z(x,t))dxdt \le |r|_{K^\infty(Q_T)} \int_{Q_T} Z(t,x,r(x,t),z(x,t))dxdt < \infty$$

hence

$$\int_{Q_T} |ðu|^2 dxdt < \infty; 6T > 0$$

$$\int_{Q_T} |ðz|^2 dxdt < \infty$$

Then r, z is globally bounded

Proof. of theorem ■

first we note that

$$\int_\Omega \frac{6r}{6t}(x,t)dx = d_1 \int_\Omega \text{A}r dx + \int_\Omega Z(t,x,r(x,t),z(x,t))dx$$

thus

$$\int_\Omega \frac{6r}{6t}(x,t)dx = \int_\Omega Z(t,x,r(x,t),z(x,t))dx \quad \text{since} \int_\Omega \text{A}r dx = 0$$

and

$$\int_\Omega \frac{6z}{6t}(x,t)dx = d_2 \int_\Omega \text{A}z dx + \int_\Omega \mu(t,x,r(x,t),z(x,t))dx$$

then

$$\int_\Omega \frac{6z}{6t}(x,t)dx = \int_\Omega \mu(t,x,r(x,t),z(x,t))dx \quad \text{since} \int_\Omega \text{A}z dx = 0$$

imliying

$$\int_\Omega (\frac{6r}{6t}(x,t) + \frac{6z}{6t}(x,t))dx = 0 \quad \text{as } \mu = —Z$$

so

$$\begin{aligned}
\int_0^t \int_\Omega (\frac{6r}{6t}(x,t) + \frac{6z}{6t}(x,t))dx &= \int_\Omega \int_0^t (\frac{6r}{6t}(x,t) + \frac{6z}{6t}(x,t))dtdx \\
&= \int_\Omega [r(x,t) + z(x,t)]_0^t dx \\
&= \int_\Omega [r(x,t) + z(x,t)]dx — \int_\Omega [r_0(x) + z_0(x)]dx = 0
\end{aligned}$$

we deduce that

$$\int_\Omega [r(x,t) + z(x,t)]dx = \int_\Omega [r_0(x) + z_0(x)]dx = 0 \tag{4.3}$$

Integrating the first equation of system in Ω we have

$$\int_\Omega (\frac{6r}{6t}(x,t)dx = \int_\Omega Z(t,x,r(x,t),z(x,t))dx > 0$$

this means that $\frac{d}{dt}\int_\Omega r(x,t)dx > 0$, ie the fonction $t \to \int_\Omega r(x,t)dx$ is increasing and Ω is bounded. Then $t \to \frac{1}{|\Omega|}\int_\Omega r(x,t)dx$ is increasing and according to the positivity of r was $\frac{1}{|\Omega|}\int_\Omega r(x,t)dx \geq 0$.

Therefore $\frac{1}{|\Omega|}\int_\Omega r(x,t)dx$ is bounded below and increasing then

$\exists \lim_{t\to\infty} \frac{1}{|\Omega|}\int_\Omega r(x,t)dx = e_1$.

and the same way for last equation of system was

$$\int_\Omega \left(\frac{\partial z}{\partial t}(x,t)dx = -\int_\Omega Z(t,x,r(x,t),z(x,t))dx < 0\right.$$

this means that $\frac{d}{dt}\int_\Omega z(x,t)dx < 0$, ie $t \to \int_\Omega z(x,t)dx$ is decreasing and Ω is bounded then $t \to \frac{1}{|\Omega|}\int_\Omega z(x,t)dx$ is decreasing.

We know that the solution z is bounded, so $\frac{1}{|\Omega|}\int_\Omega z(x,t)dx < \infty$. Thus $\exists q > 0$ such as $\frac{1}{|\Omega|}\int_\Omega z(x,t)dx \leq q$.

Thereupon $\frac{1}{|\Omega|}\int_\Omega z(x,t)dx$ is bounded above and decreasing. From here

$\exists \lim_{t\to\infty} \frac{1}{|\Omega|}\int_\Omega v(x,t)dx = e_2$.

On the other hand , since sets $\{r(t), t \geq 0\}$ and $\{z(t), t \geq 0\}$ are precompacts in $C(\overline{\Omega})$.

There exists a sequence $(t_n)_{n\geq 0}$,$t_n \to \infty$ such that

$$\lim_{n\to\infty} r(t_n) = r_s \quad \text{in} \quad C(\Omega)$$
$$\lim_{n\to\infty} z(t_n) = z_s \quad \text{in} \quad C(\Omega)$$

Now , denote by $v(r_0, z_0)$ the w-limite set for (r_0, z_0) and Γ the set of the solution of the elliptic system

$$\begin{cases} -d_1 A r_s = Z(t,x,r_s,z_s) & in\ \mathbb{R}^+ \times \Omega \\ -d_2 A z_s = -Z(t,x,r_s,z_s) & in\ \mathbb{R}^+ \times \Omega \\ \frac{\partial r_s}{\partial \eta} = \frac{\partial z_s}{\partial \eta} = 0 & in\ \mathbb{R}^+ \times \partial\Omega \end{cases} \quad ((s))$$

and prove $\Gamma = \{((\gamma_1, \eta_2))\}$ where γ_1, η_2 are two constants; In fact .

Multipliying this equation $-d_1 A r_s = Z(t,x,r_s,z_s)$ by r_s and integrating over Ω yields

$$-d_1 \int_\Omega r_s A r_s dx = \int_\Omega r_s Z(t, x, r_s, z_s)dx$$

Apply Green formular :

$$d_1 \int_\Omega |\nabla r_s|^2 dx = \int_\Omega r_s Z(t,x,r_s,z_s)dx$$

we deduce that

$$\int_\Omega |\nabla r_s|^2 dx = 0 \Rightarrow \nabla r_s = 0 \Rightarrow r_s = \eta_1 \quad (4.4)$$

Similarly, we obtain

$$\int_\Omega |\nabla v_s|^2 dx = 0 \Rightarrow \nabla v_s = 0 \Rightarrow v_s = \eta_2 \quad (4.5)$$

Replacing $r = \eta_1, z = \eta_2$ in first equation of (s). It is clear that $Z(t,x,\eta_1,\eta_2) = 0$

Hereafter, we are going to show that $v(r_0,z_0) = \Gamma = \{(\eta_1,\eta_2)\}$

We observe that $v(u_0, v_0) \neq \emptyset$, because $(r_s, z_s) \in v(r_0, z_0)$.

Now, $\forall x \in \Omega$, $\sigma \in]-1,1[$ and let

$$\varsigma_n(x,\sigma) = r(x,t_n+\sigma), \tau_n(x,\sigma) = z(x,t_n+\sigma)$$

Multiply the first equation of (1.1) by $\frac{6m}{6t}$

$$\frac{6r}{6t}\frac{6r}{6t} - d_1\frac{6r}{6t}\mathrm{A}r = \frac{6r}{6t}Z(t,x,r,z)$$

and integrate over Ω , we obtain

$$\int_\Omega (\frac{6r}{6t})^2 dx - d_1\int_\Omega \frac{6r}{6t}\mathrm{A}r dx = \int_\Omega \frac{6r}{6t}Z(t,x,r,z)dx$$

as

$$\left\|\frac{6r}{6t}\right\|^2_{K^2(\Omega)} = d_1\int_\Omega \frac{6r}{6t}\mathrm{A}r dx + \int_\Omega \frac{6r}{6t}Z(t,x,r,z)dx$$

intégrating result over $(t_0,+\infty)$, we get

$$\int_{t_0}^{+\infty}\left\|\frac{6r}{6t}\right\|^2_{K^2(\Omega)} dt = d_1\int_{t_0}^{+\infty}\int_\Omega \frac{6r}{6t}\mathrm{A}r dxdt + \int_{t_0}^{+\infty}\int_\Omega \frac{6r}{6t}Z(t,x,r,z)dxdt < \infty$$

Consequently, $\frac{6m}{6t} \in G^2\ \ t_0,+\infty, G^2(\Omega)$

6 $\in]-1,1[$,we have:

$$\begin{aligned} \varsigma_n(x,\sigma) - r(x,t_n) &= r(x,t_n+\ \ ) - r(x,t_n) \\ &= \int_{t_n}^{t_n+} \frac{6r}{6t}(x,t)dt \leq \int_{t_{n-1}}^{t_n+1}\frac{6r}{6t}(x,t)dt \quad \text{(because } t_{n-1} < t_n\ ,\ \ < 1\ ,\ t_n + \ \ < t_n+1) \\ &\leq \left(\int_{t_{n-1}}^{t_n+1}(1)^2dt\right)^{\frac{1}{2}}\left(\int_{t_{n-1}}^{t_n+1}(\frac{6r}{6t}(x,t))^2dt\right)^{\frac{1}{2}} \quad \text{(thanks to inequality of Cauchy Schwartz)} \\ &\leq \sqrt{2}\left(\int_{t_{n-1}}^{t_n+1}(\frac{6r}{6t}(x,t))^2dt\right)^{\frac{1}{2}} \end{aligned}$$

Thus

$$|\varsigma_n(x,\ \ ) - r(x,t_n)|^2 = 2\int_{t_{n-1}}^{t_n+1}(\frac{6r}{6t}(x,t))^2dt$$

Integrating the latter inequality Ω yields

$$\int_\Omega |\varsigma_n(x,\ \ ) - r(x,t_n)|^2 dx \leq 2\int_\Omega\int_{t_{n-1}}^{t_n+1}(\frac{6r}{6t}(x,t))^2dtdx$$

We pass to the limit as $n\to\infty$, we have

$$|\varsigma_n(x,\ \ ) - r_s|^2_{K^2(\Omega)} \leq 2\lim_{n\to\infty}\int_\Omega\int_{t_{n-1}}^{t_n+1}(\frac{6r}{6t}(x,t))^2dtdx = 0$$

Also,

$$|\varsigma_n(x,\ \ ) - r_s|^2_{K^2(\Omega)} \underset{n\to\infty}{\to} 0$$

As a result, we will all $\ \in]-1,1[$,

$$|\varsigma_n(x,\sigma) - r_s|^2_{K^2(\Omega)} \underset{n\to\infty}{\to} 0, \text{ hence } \sup_{-1\ \ 1}|\varsigma_n(x,\ \ ) - r_s|^2_{K^2(\Omega)} \underset{n\to\infty}{\to} 0$$

and by the same mode are obtained

$$\sup_{-1\ \ 1}|\tau_n(x,\ \ ) - z_s|^2_{K^2(\Omega)} \underset{n\to\infty}{\to} 0$$

So, we can gain

$$\sup_{-1\ 1} |ðç_n(x,\ ) - ðr_s|^2_{K^2(\Omega)} \underset{n\to\infty}{\to} 0 \quad \text{and} \quad \sup_{-1\ 1} |ð\tau_n(x,\sigma) - ðz_s|^2_{K^2(\Omega)} \underset{n\to\infty}{\to} 0$$

Through positivity and boundedness of the solution was

$$0 \le r(x, t_n + \ ) \le \Gamma$$
$$0 \le z(x, t_n + \ ) \le \mathrm{A}$$

as $Z \in C^\infty(\mathbb{R}^+)$, we can conclude , using Lebesgue's theorem, that

$$Z(t, x, ç_n(x,\ ), \tau_n(x,\ )) \to Z(t, x, r_s, z_s) \text{ in } G^2(\Omega \times (-1, 1)) \text{ weak}$$

Now, let $8_i \in C^1\ \overline{\Omega}$ such that $8_i = 0$ on 6Ω where $i = 1, 2$
and let $e \in C^1\ \overline{\Omega}$ such that supp$e \subset [-1, 1]$, $\int_{-1}^{1} e(s)ds = 1$
and $e(-1) = e(1)$
We multiply the first equation (1.1) by $e(t - t_n)8_1$ and integrate
over $(t_n - 1, t_n + 1) \times \Omega$, we hold

$$\int_{t_n-1}^{t_n+1}\int_\Omega v(t-t_n)8_1\frac{6r}{6t}dxdt - d_1\int_{t_n-1}^{t_n+1}\int_\Omega v(t-t_n)8_1\mathrm{A}rdxdt = \int_{t_n-1}^{t_n+1}\int_\Omega v(t-t_n)8_1 Z(t,x,r,z)dxdt \tag{4.6}$$

Calculate the integral $\int_{t_n-1}^{t_n+1} v(t-t_n)8_1\frac{6m}{6t}dt$ by part, we find

$$\int_{t_n-1}^{t_n+1} v(t-t_n)8_1\frac{6r}{6t}dt = -\int_{t_n-1}^{t_n+1} v'(t-t_n)8_1 r(x,t)dt \tag{4.7}$$

To calculate $\int_\Omega v(t-t_n)8_1\mathrm{A}rdx$, Practising Green's formula

$$\int_\Omega v(t-t_n)8_1\mathrm{A}rdx = \int_{6\Omega} v(t-t_n)8_1\frac{6r}{6\eta}d\sigma - \int_\Omega ðv(t-t_n)8_1ðrdx = -\int_\Omega ðv(t-t_n)8_1ðrdx$$

Thence

$$\int_\Omega v(t-t_n)8_1\mathrm{A}rdx = -\int_\Omega ðv(t-t_n)8_1ðrdx \tag{4.8}$$

Using Green's formula for manipulate $\int_\Omega v(t-t_n)8_1\mathrm{A}zdx$

$$\int_\Omega v(t-t_n)8_1\mathrm{A}zdx = \int_{6\Omega} v(t-t_n)8_1\frac{6z}{6\eta}d\sigma - \int_\Omega ðv(t-t_n)8_1ðzdx = -\int_\Omega ðv(t-t_n)8_1ðzdx$$

So

$$\int_\Omega v(t-t_n)8_1\mathrm{A}zdx = -\int_\Omega ðv(t-t_n)8_1ðvdx \tag{4.9}$$

Injected (4.7) and (4.8) and (4.9) in (4.6) is obtained

$$-\int_{t_n-1}^{t_n+1}\int_\Omega v'(t-t_n)8_1 r(x,t)dxdt + d_1\int_{t_n-1}^{t_n+1}\int_\Omega ðv(t-t_n)8_1ðrdxdt - \int_{t_n-1}^{t_n+1}\int_\Omega v(t-t_n)8_1 Z(t,x,r,z)dxdt = 0 \tag{4.10}$$

We multiply last equation of (1.1) by $\upsilon(t-t_n)8_2$ and integrating over $(t_n-1, t_n+1)\times\Omega$, we get

$$\int_{t_n-1}^{t_n+1}\int_\Omega \upsilon(t-t_n)8_2\frac{6z}{6t}dxdt - d_3\int_{t_n-1}^{t_n+1}\int_\Omega \upsilon(t-t_n)8_2\mathrm{A}rdxdt - d_4\int_{t_n-1}^{t_n+1}\int_\Omega \upsilon(t-t_n)8_2\mathrm{A}zdxdt \quad (4.11)$$
$$= -\int_{t_n-1}^{t_n+1}\int_\Omega \upsilon(t-t_n)8_2Z(t,x,r,z)dxdt$$

Similarly, we obtain

$$-\int_{t_n-1}^{t_n+1}\int_\Omega \upsilon'(t-t_n)8_2z(x,t)dxdt + d_2\int_{t_n-1}^{t_n+1}\int_\Omega ð\upsilon(t-t_n)8_2ðzdxdt$$
$$\int_{t_n-1}^{t_n+1}\int_\Omega \upsilon(t-t_n)8_2Z(t,x,r,z)dxdt$$
$$= 0 \quad (4.12)$$

By making the following change of variable

$$\begin{cases} \quad = t-t_n \to d \quad = dt \\ si \begin{array}{l} t = t_n - 1 \\ t = t_n + 1 \end{array} \to \begin{array}{l} = -1 \\ = 1 \end{array} \end{cases},$$

therefore the integral (4.10) becomes

$$\int_{-1}^{+1}\int_\Omega e'(\ )8_1\varsigma_n(x,\ )dxd\ - d_1\int_{-1}^{+1}\int_\Omega ðe(\ )8_1ð\varsigma_n(x,\ )dxd \quad (4.13)$$
$$+\int_{-1}^{+1}\int_\Omega e(\ )8_1f\,(t,x,\varsigma_n(x,\ ),\tau_n(x,\ ))\,dxdt$$
$$= 0$$

The same applies to the integral (4.12)

$$\int_{-1}^{+1}\int_\Omega e'(\ )8_2\tau_n(x,\ )dxd$$
$$-d_2\int_{-1}^{+1}\int_\Omega ðe(\ )8_2ð\tau_n(x,\ )dxd\ - \int_{-1}^{+1}\int_\Omega e(\ )8_2f\,(t,x,\varsigma_n(x,\ ),\tau_n(x,\ ))\,dxdt$$
$$= 0 \quad (4.14)$$

Using Lesbegue's theorem , we gain

$$\lim_{n\to\infty}\int_{-1}^{+1}\int_\Omega e'(\ )8_1\varsigma_n(x,\ )dxd \ = \int_{-1}^{+1}\int_\Omega e'(\ )8_1r_sdxd \ = \int_{-1}^{+1}e'(\ )d\int_\Omega 8_1r_sdx$$
$$= [e(\ )]_{-1}^{+1}\int_\Omega 8_1r_sdx = 0 \text{ because } e(1) = e(-1)$$

and same mannier for

$$\lim_{n\to\infty}\int_{-1}^{+1}\int_\Omega e'(\ )8_2\tau_n(x,\ )dxd \ = \int_{-1}^{+1}\int_\Omega e'(\ )8_2z_sdxd \ = \int_{-1}^{+1}e'(\ )d\int_\Omega 8_2z_sdx$$
$$= [8(\ )]_{-1}^{+1}\int_\Omega 8_2z_sdx = 0 \text{ because } e(1) = e(-1)$$

from inequality (4.13), we have

$$-d_1\int ð8_1ðr_s + \int 8_1Z\,(t,x,r_s,z_s)\,dx = 0$$

Ω Ω

and inequality (4.14) yields

$$-d_2 \int_\Omega \eth 8_2 \eth z_s - \int_\Omega 8_2 Z(t, x, r_s, z_s)\, dx = 0$$

But this form it is the same when we multiply (4.9) by 8_1 and (4.10) by 8_2 and integrating over Ω.
hence $v = '$
Finally, Combining (4.4) and (4.5) with (4.3)yields

$$\int_\Omega (\eta_1 + \eta_2)\, dx = \int_\Omega (r_0 + z_0)\, dx$$
$$(\eta_1 + \eta_2)\, |\Omega| = \int_\Omega (r_0 + z_0)\, dx$$

As

$$\eta_1 + \eta_2 = \frac{1}{|\Omega|} \int_\Omega (r_0 + z_0)\, dx.$$

.

The proof of the theorem is complete.

4 Conclusion

Until recently,the gloal existence of solutions to coupled reaction diffusion equations with nonlinearities and non constant signs remained unsolved , with only partial results found under severe contraints. Traditional techniques, such as entropy inequality and duality arguments, have failed to explain the surprising out comes in these scenarios.

This study utilized a simple yet effective functional technique toadress this challenge.

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