Groups with Nilpotent Factor-group by (polycyclic-by-Černikov)

Azra Souad [a] and Bouchelaghem Mounia[b]

[a] Department of Mathematics, Faculty of Mathematics and Computer Science, University Mohamed El-Bachir El-Ibrahimi of Bordj Bou Arreridj, Bordj Bou Arreridj, 34030 El Anasser Bordj- Bou-Arreridj, Algeria

[b] Laboratory of Fundamental and Numerical Mathematics, Departments of Mathematics, University Setif 1, 19000 Setif, Algeria

Correspondence : Azra Souad « souad.azra@univ-bba.dz»

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Abstract

The aim of this article is study of groups with nilpotent Factor-group by (polycyclic-by-Černikov).

A relevant role this investigation is played by the structure of groups in wich all propre subgroups are nilpotent-by-polycyclic-by-Černikov or polycyclic-by-Černikov -by-nilpotent , and such groups are discribed in the first part of the paper

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1. Introduction

A class of groups X is said to be **countably recognizable** if, whenever all countable subgroups of a group G belong to X, then G itself is an X-group. Countably recognizable classes of groups were introduced by R. Baer [2]. In his paper, Baer produced many interesting examples of countably recognizable group classes, and later many other relevant classes of groups with such a property were discovered and the more recent papers; in particular, a detailed account of countable recognizability for generalized soluble and nilpotent group classes.

Let X and Y be group classes. We shall denote by XY the product of X and Y, i.e. the class consisting of all groups G containing a normal X-subgroup N such that the factor group G/N belongs to Y. It seems to be unknown under which hypotheses the product of two countably recognizable classes is likewise countably recognizable. On the other hand, this problem has a positive solution, in the case of varieties, since it is well-known that the product of two varieties is again a variety. Moreover, if $\{B_n\}_{n\in N}$ and $\{C_n\}_{n\in N}$ are sequences of group varieties, then ([6],Lemma 2) implies that the class of groups

$$(\cup_m B_m)(\cup_n C_n) = \cup_{n,m}(B_n C_m)$$

is countably recognizable

In this direction the aim of our article is to consider groups with the minimal condition non- « polycyclic-by-Černikov-by-nilpotent» subgroups. $\overline{B\check{C}\aleph_k}$ a corresponding result for $B\check{C}\aleph_k$ groups, where B is the class of all polycyclic groups (Č denotes the class of all Černikov groups, $\aleph_k$ de notes the class of all nipotent groups of class at most k)

2. Nilpoten-by-polyciclic-by- Černikov-groups

The aim of this section is study locally (polycyclic-by-Černikov) groups whose proper subgroups are nilpotent-by-polycyclic-by-Černikov. Our results should be seen in relation with a recent article by F. Napolitani and E. Pegoraro [8] dealing with locally graded groups whose proper subgroups are nilpotent-by-Černikov, it is obvious that such groups are locally (nilpotent-by-Černikov) and so even locally (polycyclic-by-Černikov)

Proposition 2. 1. (2.6, [5]) Let X be a subgroup closed class of groups such that the product of two normal X-subgroups of an arbitrary groups is likewise an X-group. If X is countably recognizable, then also the class $X\mathscr{F}$ of all X-by-finite groups is countably recognizable.

Proposition 2. 2. Let G be an uncountable group whose proper subgroups are nilpotent-by polycyclic-by-finite. Then also G is nilpotent-by-polycyclic-by-Černikov.

Proof. The group G is soluble-by-finite, and obviosly is nilpotent-by-polycyclic- by -Černikov provided that it contains a proper subgroup of quotient is Černikov. Suppose now that G has no proper subgroups of quotient is -Černikov, so that in particular it is soluble and Lemma (2.7 [5]) allows to assume that G' is nilpotent and G/G' is a group of type p^∞ for some prime p.

Let X be any countable subgroup of G containing a transversal to G' in G, and let Y be the Fitting subgroup of X. Then

$$G=XG' \text{ and } X/Y \text{ is polycyclic,}$$

so that

$$G=YG'$$

and

$$X=YG'\cap X = Y(G' \cap X)$$

Is a nilpotent group. Therefore every countable subgroup of G is nilpotent, and so G it self is nilpotent, the theoremn is proved.

Theorem 2.1 Let G be a locally (polycyclic-by-finite) group whose proper subgroups are nilpotent-by-polycyclic-by-finite. Then either G is nilpotent-by-polycyclic-by- Černikov or it is a countable locally finite group.

Proof. By theorem(2.13 [5]) G is nilpotent-by-polycyclic-by-finite or it is a countable locally finite group.

3. Polyciclic-by- Černikov-by-Nilpoten-groups

The aim of this section is to prove that a locally graded minimal non-(polycyclic-by-Chernikov-by-nilpotent) group is locally finite . this result will be accomplished by a series of lemma

Lemma 3.1 Let G be a polycyclic-by-finite-by-nilpotent group. Then G is also nilpotent-by-polycyclic-by-Černikov.

Proof. Let G be a polycyclic-by-finite-by-nilpotent group, then there existe N polycyclic-by-finite normal subgroup of G such that the factor group G/N is nilpotent.

Then the centralizer C_G (N) is nilpotent and G/C_G (N) is polycyclic-by-finite (see [12], Theorem 3.27) so that G/C_G (N) is polycyclic-by-Černikov, we have K=C_G (N) nilpotent normal subgroup of G such that G/C_G (N) polycyclic-by-Černikov. Therefore, G is nilpotent-by-polycyclic-by-Černikov.

Theorem 3.1 Let G be a group whose proper subgroups are polycyclic-by-Černikov-by-nilpotent. If G is the product of two proper normal subgroups, then G is polycyclic-by-Černikov-by-nilpotent.

Proof. Let H and K be proper normal subgroups of G, so H and K are polycyclic- by-Černikov-by-nilpotent normal subgroup, such that G=HK.

There exist polycyclic-by-Černikov characteristic subgroups H_0 of H and K_0 of K such that H/H_0 and K/K_0 are nilpotent groups. Clearly H_0K_0 is a polycyclic-by-Černikov normal subgroup of G, and the factor group G/H_0K_0 is nilpotent.

Therefore, G is polycyclic-by-Černikov-by-nilpotent.

Theorem 3.2 Let G be a locally graded group whose proper subgroups are polycyclic-by-Černikov-by-nilpotent. Then either G is polycyclic-by-Černikov-by- nilpotent or it is locally finite.

Proof. Suppose that the group G is not locally finite, so that it is nilpotent-by-polycyclic-by-Černikov by lemma (3.2, [5]). Clearly G is not periodic and lemma (3.8 [5]) vields that G is polycyclic-by-Černikov-by-nilpotent

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