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The Study of the Existence and Uniqueness of Solutions to the Separated Fractional Boundary Value Problem Using the Banach Fixed Point Theorem

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ABSTRACT

In this work, we investigate a novel class of boundary value problems involving nonlinear differential equations with fractional derivatives. The nonlinearity f depends on a fractional derivative of lower order, and the boundary conditions are given in a separated form:

1
$$\begin{cases} {}^{c}D_{0^+}^{\alpha} x(t) = f\left(t, x(t), {}^{c}D_{0^+}^{\beta} x(t)\right) \quad, t\epsilon[0,T],\ 1 < \alpha \le 2\,, 0 < \beta \le 1 \\ a_1 x(0) + b_1\left({}^{c}D_{0^+}^{\gamma} x(0)\right) = c_1 \\ a_2 x(T) + b_2\left({}^{c}D_{0^+}^{\gamma} x(T)\right) = c_2 \qquad 0 < \gamma < 1 \end{cases}$$

Here, ${}^{c}D_{0^+}^{\alpha}$ denotes the Caputo fractional derivative. The function $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, and a_i, b_i, c_i, i = 1, 2 are real constants with $a_1 \neq 0$ and $T > 0$.

The primary goal of this study is to establish the existence and uniqueness of solutions for the proposed fractional boundary value problem using the Banach fixed point theorem (contraction mapping principle). This fundamental result ensures the existence of a unique fixed point for any contraction mapping defined on a complete metric space, and serves as a powerful tool in various areas of mathematical analysis. An illustrative example is also provided to demonstrate the application of this method to the given class of equations.

Keywords: Fractional differential equations, separated fractional boundary conditions, Banach fixed point theorem, existence.

1. INTRODUCTION

The theory of fractional calculus has been around almost as long as differential calculus, with its foundations having been laid by Leibniz, Gauss, and Newton **[1-6].**

Nonlinear fractional differential equations are a natural extension of ordinary differential equations, and have become a crucial subfield of mathematics due to their application in many scientific and engineering disciplines **[7-9,2,4]** such as physics, chemistry, biology, and so on.

The current work aims to study the existence and uniqueness of solutions for a specific class of nonlinear fractional differential equations. These equations are of the form where the function f depends on the lower-order derivative of an unknown function x(t), subject to separated fractional boundary conditions, as given by:

$$\begin{cases} {}^{c}D_{0^+}^{\alpha} x(t) = f\left(t, x(t), {}^{c}D_{0^+}^{\beta} x(t)\right), \; t \in [0,T], 1 < \alpha \leq 2, 0 < \beta \leq 1 \\ a_1 x(0) + b_1 \left({}^{c}D_{0^+}^{\gamma} x(0)\right) = c_1 \\ a_2 x(T) + b_2 \left({}^{c}D_{0^+}^{\gamma} x(T)\right) = c_2 \qquad 0 < \gamma < 1 \end{cases} \quad (1)$$

Here ${}^{c}D_{0^+}^{\alpha}$ is the Caputo fractional derivative, $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, and a_i, b_i, c_i, i = 1, 2 are real constants, with : $a_1 \neq 0$ and T > 0.
The **Banach** fixed point theorem will be employed to achieve this goal.

2. Some background material

Definition 2.1. (Gamma function)
For any complex number $\mathbf{z}$ such that Re $(\mathbf{z}) > 0$, we define the following function called Gamma and denoted by the Greek letter "$\boldsymbol{\Gamma}$".

$$\boldsymbol{\Gamma} : \mathbb{R}^{*+} \to \mathbb{R}$$

$$\mathbf{z} \to \boldsymbol{\Gamma}(\mathbf{z}) = \int_{0}^{+\infty} \mathbf{t}^{\mathbf{z}-1} \exp^{-\mathbf{t}} \mathbf{dt}.$$

Definition 2.2. ([12]) Fractional derivative in Caputo's sense
Let $\boldsymbol{f} : [\mathbf{0}, +\infty[\to \mathbb{R}, \boldsymbol{\alpha} > \mathbf{0}$ $\boldsymbol{and\, n} = [\boldsymbol{\alpha}] + \mathbf{1}$, fractional derivative in Caputo's sense to the right of 0 of order $\boldsymbol{\alpha} > \mathbf{0}$ is defined by :

$$\boldsymbol{{}^{c}D_{0^+}^{\alpha} f(t)} = \frac{\mathbf{1}}{\boldsymbol{\Gamma(n-\alpha)}} \int_{\mathbf{0}}^{\boldsymbol{t}} \boldsymbol{(t-s)^{n-\alpha-1} f^{(n)}(s) ds}$$

$$= \boldsymbol{I_{0^+}^{n-\alpha}} \left(\boldsymbol{\frac{d^n}{dt^n} f(t)} \right).$$

Definition 2.3. (Contracting application)
The operator $\mathbf{T}$ is said to be a contracting operator (or simply a contraction) on $\mathbf{Q} \subset \mathbf{D}(\mathbf{T})$, if there exists $\mathbf{L} \in]\mathbf{0}, \mathbf{1}[$ such that:

$$\forall\, \mathbf{x}, \mathbf{y} \in \mathbf{Q} : \; \|\mathbf{T}(\mathbf{x}) - \mathbf{T}(\mathbf{y})\| \leq \mathbf{L}\|\mathbf{x} - \mathbf{y}\|$$

L: is called the contraction ratio.

Theorem 2.1 ([10]) "**Banach fixed point theorem** (or contraction mapping theorem)"
Suppose that the operator $\mathbf{T}$ applies a bounded fermata $\mathbf{Q} \subset \mathbf{X}$ ($\mathbf{X}$ is Banach) in itself, i.e.: $\mathbf{T}(\mathbf{Q}) \subset \mathbf{Q}$ is a contraction of ratio $\mathbf{L}$ on $\mathbf{Q}$.
Then, $\boldsymbol{T}$ admits in $\boldsymbol{Q}$ one fixed point $\mathbf{x}^*$ and only one.

3. Presentation of the problem

We'll be studying the existence and uniqueness of a solution to a boundary value problem for fractional differential equations.

$$(1)\quad \begin{cases} {}^{c}D_{0^+}^{\alpha}x(t) = f\left(t, x(t), {}^{c}D_{0^+}^{\beta}x(t)\right), \; t \in [0,T], 1 < \alpha \leq 2, 0 < \beta \leq \\ a_1 x(0) + b_1\left({}^{c}D_{0^+}^{\gamma}x(0)\right) = c_1 \\ a_2 x(T) + b_2\left({}^{c}D_{0^+}^{\gamma}x(T)\right) = c_2 \qquad 0 < \gamma < 1 \end{cases}$$

Here ${}^{c}D_{0^+}^{\alpha}$ represents Caputo's fractional derivative, f : $[0, T] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and a_i, b_i, c_i , i = 1, 2 are real constants with : $a_1 \neq 0$ et $T > 0$.

We'll present the outcome of the existence, based on Banach's fixed point theorem.
We'll start off studying the linear boundary value problem.

Lemma 3.1 ([11]) Let $y(t) \in C([0,T])$ is given function and $\alpha \in]1,2]$, we consider the linear equation:

$${}^{c}D_{0^+}^{\alpha}x(t) = y(t)$$

Then the linear boundary value problem:

$$(2)\quad \begin{cases} {}^{c}D_{0^+}^{\alpha}x(t) = y(t), \quad t \in [0,T], \qquad 1 < \alpha \leq 2, 0 < \beta \leq 1 \\ a_1 x(0) + b_1\left({}^{c}D_{0^+}^{\gamma}x(0)\right) = c_1 \\ a_2 x(T) + b_2\left({}^{c}D_{0^+}^{\gamma}x(T)\right) = c_2 \qquad 0 < \gamma < 1 \end{cases}$$

Has a unique solution given by:

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds - \frac{t}{v_1}\left\{a_2 \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + b_2 \int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)ds\right\} + v_2 t + \frac{c_1}{a_1}$$

Such that:

$$v_1 = \frac{a_2\, T\Gamma(2-\gamma) + b_2 T^{1-\gamma}}{\Gamma(2-\gamma)} \qquad , \qquad v_2 = \frac{a_1 c_2 - a_2 c_1}{a_1 v_1}$$

2. RESULTS AND DISCUSSIONS

In light of Banach's fixed point theorem, we are considering the existence and uniqueness of the boundary value problem (1)
Let : $\mathbf{I} = [\mathbf{0}, \mathbf{T}]$ and $\mathbf{C}(\mathbf{I})$ the space of all continuous real functions defined on **I** and defining the space: $\mathbf{X} = \left\{\mathbf{x}(\mathbf{t}) : \mathbf{x}(\mathbf{t}) \in \mathbf{C}([\mathbf{0},\mathbf{T}]) \;\; \textbf{and} \;\; {}^{\mathbf{c}}\mathbf{D}_{\mathbf{0}^+}^{\boldsymbol{\beta}}\mathbf{x}(\mathbf{t}) \in \mathbf{C}([\mathbf{0},\mathbf{T}]), \mathbf{0} < \boldsymbol{\beta} \leq \mathbf{1}\right\}$

Muni norm: $\|\mathbf{x}\|_{\mathbf{X}} = \|\mathbf{x}\|_\infty + \left\|{}^{\mathbf{c}}\mathbf{D}_{\mathbf{0}^+}^{\boldsymbol{\beta}}\mathbf{x}(\mathbf{t})\right\|_\infty$

Here : $\|\mathbf{x}\|_\infty = \max_{\mathbf{t}\in[\mathbf{0},\mathbf{T}]}|\mathbf{x}(\mathbf{t})|$ **and** $\left\|{}^{\mathbf{c}}\mathbf{D}_{\mathbf{0}^+}^{\boldsymbol{\beta}}\right\|_\infty = \max_{\mathbf{t}\in[\mathbf{0},\mathbf{T}]}\left|{}^{\mathbf{c}}\mathbf{D}_{\mathbf{0}^+}^{\boldsymbol{\beta}}\mathbf{x}(\mathbf{t})\right|$

$(\mathbf{X}, \|.\|)$ is a Banach space.
Thus, the solution of the boundary value problem (1) is equivalent to (2).
The operator F : X → X is defined as:

$$\mathbf{Fx(t)} = \int_0^t \frac{(\mathbf{t}-\mathbf{s})^{\alpha-1}}{\Gamma(\alpha)} \mathbf{f}\left(\mathbf{s}, \mathbf{x}(\mathbf{s}), {}^{c}D_{0^+}^{\beta}\mathbf{x}(\mathbf{s})\right)\mathbf{ds} - \frac{\mathbf{t}}{\mathbf{v_1}}\left\{\mathbf{a_2}\int_0^T \frac{(\mathbf{T}-\mathbf{s})^{\alpha-1}}{\Gamma(\alpha)} \mathbf{f}\left(\mathbf{s}, \mathbf{x}(\mathbf{s}), {}^{c}D_{0^+}^{\beta}\mathbf{x}(\mathbf{s})\right)\mathbf{ds} + \mathbf{b_2}\int_0^T \frac{(\mathbf{T}-\mathbf{s})^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} \mathbf{f}\left(\mathbf{s}, \mathbf{x}(\mathbf{s}), {}^{c}D_{0^+}^{\beta}\mathbf{x}(\mathbf{s})\right)\mathbf{ds}\right\} + \mathbf{v_2 t} + \frac{\mathbf{c_1}}{\mathbf{a_1}} \qquad (3)$$

Thus, every solution of the boundary value problem is also a solution of (3), and the opposite is true.

Therefore, we need to look for the fixed point of the operator F, which is also a solution of (1).

Theorem 4.1:
We suppose $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the condition:

$$|f(t, \mathbf{x_1}, \mathbf{y_1}) - f(t, \mathbf{x_2}, \mathbf{y_2})| \le m(t)(|\mathbf{x_1} - \mathbf{x_2}| + |\mathbf{y_1} - \mathbf{y_2}|),$$

For all $\in [\mathbf{0}, \mathbf{T}]$, $\mathbf{x_i}, \mathbf{y_i} \in \mathbb{R}$, i=1,2 and $\mathbf{m} \in \mathbf{L}^{\frac{1}{\tau}}([0,T], \mathbb{R}^+)$, $\tau \in (\mathbf{0}, \alpha - \mathbf{1})$ if :

$$\frac{\|\mathbf{m}\|\mathbf{T}^{\alpha-\tau}}{\Gamma(\alpha)}\left(\frac{\mathbf{1}-\tau}{\alpha-\tau}\right)^{1-\tau}\left(\mathbf{1} + \frac{\mathbf{a_2 T}}{\mathbf{v_1}} + \frac{|\mathbf{a_2}|\mathbf{T}^{1-\beta}}{|\mathbf{v_1}|\Gamma(2-\beta)}\right) + \left(\mathbf{1} + \frac{\mathbf{T}^{\beta}}{\Gamma(2-\beta)}\right)\frac{\|\mathbf{m}\||\mathbf{b_2}|\mathbf{T}^{\alpha-\gamma-\tau+1}}{|\mathbf{v_1}|\Gamma(\alpha-\gamma)}\left(\frac{\mathbf{1}-\tau}{\alpha-\gamma-\tau}\right)^{1-\tau} + \frac{\|\mathbf{m}\|\mathbf{T}^{\alpha-\beta-\gamma}}{\Gamma(\alpha-\beta)}\left(\frac{\mathbf{1}-\tau}{\alpha-\beta-\gamma}\right)^{1-\tau} < \mathbf{1}$$

Then the problem (1) has a unique solution.
proof:
We know that the operator **F** maps the closed, bounded $\mathbf{B_r}$ set into itself.
Let : $\mathbf{Fx} = \mathbf{F_1 x} + \mathbf{F_2 x}$, such that : $(\mathbf{F_1 x})(\mathbf{t}) = \int_0^t \frac{(\mathbf{t}-\mathbf{s})^{\alpha-1}}{\Gamma(\alpha)}(\mathbf{Nx})(\mathbf{s})\mathbf{ds}$, $(\mathbf{F_2 x})(\mathbf{t}) = -\mathbf{K_x t} + \frac{\mathbf{c_1}}{\mathbf{a_1}}$

With : $\mathbf{K_x} = \frac{\mathbf{1}}{\mathbf{v_1}}\left\{\mathbf{a_2}\int_0^T \frac{(\mathbf{T}-\mathbf{s})^{\alpha-1}}{\Gamma(\alpha)}(\mathbf{Nx})(\mathbf{s})\mathbf{ds} + \mathbf{b_2}\int_0^T \frac{(\mathbf{T}-\mathbf{s})^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}(\mathbf{Nx})(\mathbf{s})\mathbf{ds}\right\}\mathbf{v_2}$

And $(\mathbf{Nx})(\mathbf{t}) = f\left(\mathbf{t}, \mathbf{x}(\mathbf{t}), {}^{c}D_{0^+}^{\beta}\mathbf{x}(\mathbf{t})\right)$, $\mathbf{t} \in [\mathbf{0}, \mathbf{T}]$.

We now show that **F** is a contraction:
We note : $\|\mathbf{m}\| = \left(\int_0^T |\mathbf{m}(\mathbf{s})|^{\frac{1}{\tau}}\mathbf{ds}\right)^{\tau}$, $\forall \mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\forall \mathbf{t} \in [\mathbf{0}, \mathbf{T}]$, according inequality of H**öl**der , we got :

$$|(\mathbf{F_1 x})(\mathbf{t}) - (\mathbf{F_1 y})(\mathbf{t})| = \left|\int_0^t \frac{(\mathbf{t}-\mathbf{s})^{\alpha-1}}{\Gamma(\alpha)}\left((\mathbf{Nx})(\mathbf{s}) - (\mathbf{Ny})(\mathbf{s})\right)\mathbf{ds}\right|$$

$$\leq \int_0^t \frac{(t-s)^{\alpha-1}|m(s)|}{\Gamma(\alpha)}\left(|x(s)-y(s)| + \left|{}^cD_{0^+}^{\beta}\, x(s) - {}^cD_{0^+}^{\beta}\, y(s)\right|\right)ds$$
$$\leq \frac{\|m\|\|x-y\|}{\Gamma(\alpha)}\left(\frac{1-\tau}{\alpha-\tau}\right)^{1-\tau} T^{\alpha-\tau}$$

And $$|(F_2x)(t) - (F_2y)(t)| = \left|t\left(K_x - K_y\right)\right|$$
$$\leq T\left\{\left|\frac{a_2}{v_1}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\left((Nx)(s) - (Ny)(s)\right)ds + \frac{b_2}{v_1}\int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}\left((Nx)(s) - (Ny)(s)\right)ds\right|\right\}$$

Expanding (Nx) and (Ny) and using $\|m\|$ as in the preceding proof, we obtain:

$$|(F_2x)(t) - (F_2y)(t)| \leq \frac{\|m\|}{|v_1|}\left\{\frac{|a_2|T^{\alpha-\tau-1}}{\Gamma(\alpha)}\left(\frac{1-\tau}{\alpha-\tau}\right)^{1-\tau} + \frac{|b_2|T^{\alpha-\gamma-\tau-1}}{\Gamma(\alpha-\gamma)}\left(\frac{1-\tau}{\alpha-\gamma-\tau}\right)^{1-\tau}\right\}\|x-y\|_X$$

Using the addition rule gives us:

$$\left\|F_x - F_y\right\|_\infty \leq \|m\|\left\{\frac{T^{\alpha-\tau}}{\Gamma(\alpha)}\left(\frac{1-\tau}{\alpha-\tau}\right)^{1-\tau} + \frac{|a_2|T^{\alpha-\tau+1}}{|v_1|\Gamma(\alpha)}\left(\frac{1-\tau}{\alpha-\tau}\right)^{1-\tau} + \frac{|b_2|T^{\alpha-\gamma-\tau+1}}{|v_1|\Gamma(\alpha-\gamma)}\left(\frac{1-\tau}{\alpha-\gamma-\tau}\right)^{1-\tau}\right\}\|x-y\|_X$$

Now, let us calculate: $\left\|{}^cD_{0^+}^{\beta}\, F_x - {}^cD_{0^+}^{\beta}\, F_y\right\|_\infty$

$$\left|{}^CD_{0^+}^{\beta}\, F_x(t) - {}^cD_{0^+}^{\beta}\, F_y(t)\right|$$
$$= \left|(I_{0^+}^{\alpha-\beta}Nx)(t) - K_x\frac{t^{1-\beta}}{\Gamma(2-\beta)} - (I_{0^+}^{\alpha-\beta}Ny)(t) - K_y\frac{t^{1-\beta}}{\Gamma(2-\beta)}\right|$$
$$\leq \int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}|m(s)|\left(|x(s)-y(s)| + \left|{}^cD_{0^+}^{\beta}\, x(s) - {}^cD_{0^+}^{\beta}\, y(s)\right|\right)ds$$
$$+ \frac{T^{1-\beta}}{\Gamma(2-\beta)}\left|\frac{a_2}{v_1}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}|m(s)|\left(|x(s)-y(s)| + \left|{}^cD_{0^+}^{\beta}\, x(s) - {}^cD_{0^+}^{\beta}\, y(s)\right|\right)ds\right|$$
$$\leq \frac{\|m\|}{\Gamma(\alpha-\beta)}\left(\frac{1-\tau}{\alpha-\beta-\tau}\right)^{1-\tau} T^{\alpha-\beta-\tau}\|x-y\|_X + \frac{\|m\|T^{1-\beta}}{|v_1|\Gamma(2-\beta)}$$
$$\times\left\{\frac{|a_2|T^{\alpha-\tau}}{\Gamma(\alpha)}\left(\frac{1-\tau}{\alpha-\tau}\right)^{1-\tau} + \frac{|b_2|T^{\alpha-\gamma-\tau}}{\Gamma(\alpha-\gamma)}\left(\frac{1-\tau}{\alpha-\gamma-\tau}\right)^{1-\tau}\right\}\|x-y\|_X$$

So, we finally have:

$$\|F_x - F_y\|_X \le \left\{\frac{\|m\|T^{\alpha-\tau}}{\Gamma(\alpha)}\left(\frac{1-\tau}{\alpha-\tau}\right)^{1-\tau}\left(1+\frac{|a_2|T}{|v_1|}+\frac{|a_2|T^{1-\beta}}{|v_1|\Gamma(2-\beta)}\right) + \left(1+\frac{T^{-\beta}}{\Gamma(2-\beta)}\right)\frac{\|m\|\,|b_2|T^{\alpha-\gamma-\tau+1}}{|v_1|\Gamma(\alpha-\gamma)}\left(\frac{1-\tau}{\alpha-\gamma-\tau}\right)^{1-\tau} + \frac{\|m\|\,|b_2|T^{\alpha-\beta-\tau}}{\Gamma(\alpha-\beta)}\left(\frac{1-\tau}{\alpha-\beta-\tau}\right)^{1-\tau}\right\}\|x-y\|_X$$

But since,

$$\frac{\|m\|T^{\alpha-\tau}}{\Gamma(\alpha)}\left(\frac{1-\tau}{\alpha-\tau}\right)^{1-\tau}\left(1+\frac{|a_2|T}{|v_1|}+\frac{|a_2|T^{1-\beta}}{|v_1|\Gamma(2-\beta)}\right) + \left(1+\frac{T^{-\beta}}{\Gamma(2-\beta)}\right)\frac{\|m\|\,|b_2|T^{\alpha-\gamma-\tau+1}}{|v_1|\Gamma(\alpha-\gamma)}\left(\frac{1-\tau}{\alpha-\gamma-\tau}\right)^{1-\tau} + \frac{\|m\|\,|b_2|T^{\alpha-\beta-\tau}}{\Gamma(\alpha-\beta)}\left(\frac{1-\tau}{\alpha-\beta-\tau}\right)^{1-\tau} < 1.$$

We can thus show that F is a contraction.
Banach's fixed-point theorem, problem (1) has a unique solution which is the fixed point of F.

Remark 4.2.([12]) $I_{0^+}^{\alpha} f(t)$is the Riemann-Liouville inequality of order α such that :

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)\,ds$$

Corollary 4.3. Suppose that the continuous function f satisfying :

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \le L\,(|x_1 - x_2| + |y_1 - y_2|)$$

Given $t\in[0,T]$, $x_i, y_i \in \mathbb{R}$, i=1,2 and $L > 0$ is a constant, if :

$$\frac{LT^{\alpha}}{\Gamma(\alpha+1)}\left(1+\frac{|a_2|T^{1-\beta}}{|v_1|\Gamma(2-\beta)}\right)+\frac{LT^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \left(1+\frac{T^{-\beta}}{\Gamma(2-\beta)}\right)\frac{L|b_2|T^{\alpha-\gamma+1}}{|v_1|\Gamma(\alpha-\gamma+1)} < 1,$$

Then problem (1) has a unique solution.

Remark 4.4. This corollary's proof is almost identical to the previous theorems.
Now, let's proceed with the application.

Example:

$$\begin{cases} {}^{c}D_{0^+}^{\frac{7}{4}} x(t) = \frac{-3\ln(t+1)}{2\ln(t^2)+5} + \frac{1}{(t+3)^2}\left(\sin x(t) + \frac{\left|{}^{c}D_{0^+}^{\frac{3}{4}} x(t)\right|}{1+\left|{}^{c}D_{0^+}^{\frac{3}{4}} x(t)\right|}\right), \\ t\in[0,T] \\ x(0) + \left({}^{c}D_{0^+}^{\frac{1}{4}} x(0)\right) = \frac{1}{2} \end{cases}$$

(4)

$$\frac{1}{2}x(1)+\frac{1}{3}\left({}^{c}D_{0^+}^{\frac{1}{4}}x(1)\right)=2$$

Where: $T=1$, $\alpha=\frac{7}{4}$, $\beta=\frac{3}{4}$, $\gamma=\frac{1}{4}$, $a_1=1$, $c_{1=}\frac{1}{2}$, $a_2=\frac{1}{2}$, $b_2=\frac{1}{3}$ **and** $c_2=2$,

It is clear that: $f\left(t,x(t),{}^{c}D_{0^+}^{\frac{3}{4}}x(t)\right)=\frac{-3\ln(t+1)}{2\ln(t^2)+5}+\frac{1}{(t+3)^2}\left(\sin x(t)+\frac{\left|{}^{c}D_{0^+}^{\frac{3}{4}}x(t)\right|}{1+\left|{}^{c}D_{0^+}^{\frac{3}{4}}x(t)\right|}\right)$

Calculating: $|f(t,x_1,y_1)-f(t,x_2,y_2)|$,

$$\begin{aligned}|f(t,x_1,y_1)-f(t,x_2,y_2)|&=\left|\frac{-3\ln(t+1)}{2\ln(t^2)+5}+\frac{1}{(t+3)^2}\left(\sin x_1(t)+\frac{y_1}{1+y_1}\right)\right|\\&+\left|\frac{-3\ln(t+1)}{2\ln(t^2)+5}+\frac{1}{(t+3)^2}\left(\sin x_2(t)+\frac{y_2}{1+y_2}\right)\right|\\&\le\frac{1}{(t+3)^2}\left(2\sin\left(\frac{x_1-x_2}{2}\right)\cos\left(\frac{x_1+x_2}{2}\right)+\frac{|y_1|-|y_2|}{(1+|y_1|)(1+|y_2|)}\right)\\&\le\frac{1}{9}\left(2\left|\frac{x_1-x_2}{2}\right|+|y_1-y_2|\right)\quad\le\frac{1}{9}(|x_1-x_2|+|y_1-y_2|)\end{aligned}$$

Therefore, we derive: $L=\frac{1}{9}>0$.

Calculating now: $H=\frac{LT^{\alpha}}{\Gamma(\alpha+1)}\left(1+\frac{|a_2|T^{1-\beta}}{|v_1|\Gamma(2-\beta)}\right)+\frac{LT^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+\left(1+\frac{T^{-\beta}}{\Gamma(2-\beta)}\right)\frac{L|b_2|T^{\alpha-\gamma+1}}{|v_1|\Gamma(\alpha-\gamma+1)}$

$\Gamma(\alpha+1)=1.6084$, $\Gamma(2-\beta)=0.9064$, $\Gamma(\alpha-\beta+1)=1$, $\Gamma(\alpha-\gamma+1)=1.3293$, $|v_1|=0.5796$.

We obtain: H= 0.3409.

Hence: $H<1$.

The assumptions of the corollary are met; consequently, problem (4) has a unique solution.

3. CONCLUSION

Thanks to **Banach's** fixed point theorem (the principle of the contractive mapping), we were able to ascertain the existence and uniqueness of the solution to the problem of the separated limits.

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