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A SYSTEMATIC APPROACH TO EXACT SOLUTIONS OF NONLINEAR WAVE EQUATIONS VIA THE EXTENDED FAN SUB-EQUATION METHOD

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ABSTRACT

This paper explores the application of the Extended Fan Sub- Equation Method to solve the nonlinear Benjamin-Ono (BO) equation. The method involves two key steps: first, transforming the nonlinear BO equation into an ordinary differential equation (ODE), and then solving it using the Fan Sub-Equation approach. The effectiveness and practicality of this method are demonstrated, highlighting its ability to provide exact solutions in a sys- tematic and manageable way. The utility of this method lies in its simplicity flexibility, and its capacity to handle complex nonlinear equations, making it a powerful tool for solving problems in mathematical physics and engineer- ing. However, several challenges arise when applying this method to the BO equation, including the non-locality introduced by the Hilbert transform, the difficulty in selecting an appropriate sub-equation, and the complexity of vali- dating the obtained solutions. Despite these challenges, the method remains a valuable approach for deriving exact solutions and understanding the behavior of nonlinear systems.

Keywords: The extended Fan sub-equation Method; Nonlinear partial differential equations; Nonlinear partial integro-differential equation; The Benjamin Ono Equation.

1. INTRODUCTION

Mathematical bodily performs a vital position with inside to observe of many physical processes- hydro dynamics, elasticity and electrodynamics. A physical phenomenon cannot be reduced to the mathematical concepts it contains, but it trans- lates using these. This point of view would join that

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of consider mathematics as models, of physical phenomena there is a concept mathematics that has lived for centuries and that remarkably assume the role of model for physical phenomena that of differential equation Non-linear differential equation have an important role in the study of non-linear physical sciences. Exact solution of the non-linear (PDEs) give a good express of the physical problems. In the past several decades many significant methods have been established such as the inverse scattering method [1] Hirotas bilinear transformation [8], the tanh method [16], sine-cosine method [2] homogeneous balance method [11], exp-function method [4], trial equation method [7]. Recently, due to its mathematical simplicity, the Fan sub-equation method has become an elective tool for solving nonlinear evolution equations, leading to the derivation of several solutions for many such equations. This method, known as the Fan method [5, 9], is particularly effective in addressing nonlinear differential equations, especially in the context of partial differential equations (PDEs). This method is an extension of Fan's method, and allows exact solutions to be found

in the form of special functions or series, some equations where this method has been applied include the nonlinear Schrödinger equation [12]. The method has enabled the discovery of exact solutions in the form of hyperbolic, trigonometric, or exponential functions. One notable application is the Korteweg-de Vries (KdV) equation, which is a nonlinear partial differential equation (PDE) that models solitary waves in a fluid medium.

The Extended Fan Sub-Equation Method was employed to obtain analytical solutions in the form of solitons for various non-linear equations. The Boussinesq equation [14], which describes wave propagation in shallow water, was solved using this method to find exact solutions. Similarly, the non-linear Klein-Gordon equation [15], used in theoretical physics to describe scalar fields, was addressed, yielding solutions in the form of elliptical or hyperbolic functions. The Sine-Gordon equation [17], which arises in mathematical physics problems such as field theory, was also tackled, enabling the discovery of exact soliton solutions.

Reaction-diffusion equations, which model biological, chemical, or physical phenomena, were another area of application for this method. The Extended Fan Sub-Equation Method proved effective in solving some of these non-linear equations. Additionally, Burgers-Huxley-type equations [18], which combine diffusion, convection, and reaction terms, were analyzed, leading to the derivation of analytical solutions. The Ginzburg-Landau equations [19], used in materials

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physics and nonlinear optics, were also solved using this method, resulting in exact solutions.

More recently, the method was discussed in the context of the time fractional Burgers-Fisher equation [20], further demonstrating its versatility and applicability across a range of complex equations.

We focus our study on the Benjamin-Ono (BO) equation, which has been widely researched since its introduction in the 1960s. This equation is a significant model in mathematical physics, particularly for describing internal waves in deep water. It is used to model various physical phenomena, including internal waves in oceanography, where it describes the propagation of internal waves in a stratified fluid, such as layers of water with different densities in the ocean. Additionally, the BO equation appears in plasma and particle physics, where it is applied to study waves in plasmas and charged particle systems. In solid-state physics, it is used to model phenomena in crystals and low-dimensional materials.

Over time, the BO equation has been generalized and modified to study more complex phenomena. For instance, the generalized Benjamin-Ono equation incorporates additional terms to account for dispersion or increased non-linearity.

Due to its non-linear and non-local structure, solving the BO equation requires specialized methods. Both analytical and numerical approaches have been employed, including the Inverse Scattering Transform [6], the Hirota Method [22], and the Symmetry Reduction Method (Lie Group Theory) [21]. These techniques have been instrumental in advancing the understanding and application of the BO equation across various fields of physics.

Current research on the Benjamin-Ono (BO) equation focuses on two main ar- eas the stability of solutions and N-soliton solutions. The stability of solutions involves studying the behavior of solitons and periodic waves under perturbations, while N-soliton solutions focus on the construction and analysis of interactions be- tween multiple solitons. Additionally, there is growing interest in its applications in physics and engineering, where the equation is used to model phenomena in real-world systems.

In this paper, we concentrate on applying the extended Fan sub-equation method to solve the Benjamin-Ono equation. This equation is a nonlinear partial integro- differential equation.

It was introduced by Benjamin (1967) [3] and Ono (1975) [10]. The standard form of the Benjamin-Ono equation is given by

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$$u_t + u_x u + \mathcal{H} u_{xx} = 0 \qquad (1.1)$$

where:

- u is the wave function,
- $\mathcal{H}$ is the Hilbert operator, which accounts for dispersion effects,

to the equation

$$u_{tt} + \beta((v^2)_{xx} + 2(v\epsilon\ )_{xx}) + \gamma u_{xxx} = 0 \qquad (1.2)$$

we proceed as follows. First, we differentiate the Benjamin-Ono equation with respect to t yielding:

$$u_{tt} + (u_x u)_t + \mathcal{H} u_{xxt} = 0 \qquad (1.3)$$

Next, we expand the terms using the chain rule and introduce the variables v and

ϵ,assumed to be proportional to u : $v = ku$ and $\epsilon\ = mu$, where k and m are constants.

Substituting these into the equation and combining terms, we obtain:

$$u_{tt} + \beta(k^2 + 2km)(u^2)_{xx} + \gamma u_{xxx} = 0 \qquad (1.4)$$

By defining $\beta' = \beta(k^2 + 2km)$ and setting $\beta' = \beta$ (which implies $k^2 + 2km = 1$) the equation simplifies to:

$$u_{tt} + \beta(u^2)_{xx} + \gamma u_{xxx} = 0 \qquad (1.5)$$

where β, γ are parameters. Note that the Hilbert operator $\mathcal{H}$ replaced by additional terms modeling higher-order nonlinear and dipersive effects. This generalized equation is nonlinear and incorporates dispersion effects, making it particularly suitable for modeling wave propagation in environments such as deep water.

The paper is organized as follows. In Section 1, the extended Fan sub-equation method is described. In Section 2, the Benjamin-Ono equation is introduced. In Section 3, the application of the extended Fan sub-equation method to solve the Benjamin-Ono equation is presented. Finally, conclusions and perspectives are provided in Section 4.

2. METHODOLOGY

The Fan Sub-Equation Method distinguishes itself from other nonlinear equation solving techniques

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by several key advantages, making it a powerful and versatile tool for the study of nonlinear partial differential equations (PDEs). Unlike methods such as the inverse scattering transform or the Hirota method, which require advanced spectral theory or complex bilinear transformations, the Fan Sub-Equation Method relies on a direct, systematic approach to obtaining exact solutions in the form of special functions. Here are the main advantages of this method:

Reduction to a Simple Sub-Equation:

Suppose that a non-linear partial differential equation in (x, t) is given by

$$P(u, u_t, u_x, u_{xt}, u_{tt}, u_{xx}\ldots\ldots) = 0 \quad (2.1)$$

Where u is an unknown function and P is a polynomial in $u(x, t)$ and its partial derivatives then by means of an appropriate transformation, it can be reduced to a nonlinear ordinary differential equation (ODE) as follow

$$P(u, u', u'', u'''\ldots\ldots) = 0 \quad (2.2)$$

Assumed the solution as a finite series involving a function Φ satisfying

$$u(\xi) = \sum_{i=0}^{n} a_i \Phi^i(\xi) \quad (2.3)$$

where a_i $(i = 0, 1, 2, \ldots\ldots\ldots\ldots\ldots\ldots n)$ are constants to be determined later and $\Phi = \Phi(\xi)$ satisfies the following ODE

$$\frac{d\Phi}{d\xi} = \epsilon(b_0 + b_1\Phi + b_2\Phi^2 + b_3\Phi^3 + b_4\Phi^4)^{\frac{1}{2}} = \epsilon\sqrt{\sum_{i=1}^{4} b_i\Phi^i} \quad (2.4)$$

Where $\epsilon = \pm 1$ and b_i are constants

The derivatives with respect to ξ

$$\frac{du}{d\xi} = \frac{d\Phi}{d\xi}\frac{du}{d\Phi} = \epsilon\sqrt{\sum_{i=1}^{4} b_i\Phi^i}\,\frac{du}{d\Phi} \quad (2.5)$$

$$\frac{d^2u}{d\xi^2} = \frac{1}{2}\sum_{i=1}^{4} ib_i\Phi^{i-1}\frac{du}{d\Phi} + \sum_{i=1}^{4} b_i\Phi^i\frac{d^2u}{d\Phi^2}, \ldots \quad (2.6)$$

$$\frac{d^3u}{d\xi^3} = \left(\frac{1}{2}\sum_{i=0}^{4}\left(i(i-1)b_i\Phi^{i-2}\right)\frac{d\Phi}{d\xi}\right)\frac{du}{d\Phi} + \left(\frac{3}{2}\sum_{i=0}^{4} ib_i\Phi^{i-1}\frac{d\Phi}{d\xi}\right)\frac{d^2u}{d\Phi^2} + \left(\sum_{i=0}^{4} b_i\Phi\frac{d\Phi}{d\xi}\right)\frac{d^3u}{d\Phi^3}$$

(2.7)

The solutions of Eq (2.4)

Case 1:

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When $b_0 = b_1 = b_3 = 0$, $b_2 < 0$ and $b_4 > 0$ the solution is

$$\Phi(\xi) = \sqrt{\frac{-b_2}{b_4}}\sec\left(\sqrt{-b_2}\xi\right) \tag{2.8}$$

When $b_0 = b_1 = b_3 = 0$, $b_2 > 0$ and $b_4 < 0$ the solution is

$$\Phi(\xi) = \sqrt{\frac{-b_2}{b_4}}\operatorname{sech}\left(\sqrt{b_2}\xi\right) \tag{2.9}$$

When $b_0 = b_1 = b_3 = 0$, $b_2 = 0$ and $b_4 > 0$ the solution is

$$\Phi(\xi) = -\frac{\epsilon}{\sqrt{b_4}\xi} \tag{2.10}$$

Case 2:

When $b_1 = b_3 = 0$, $b_0 = \frac{b_2^2}{4b_4}$, $b_2 < 0$ and $b_4 > 0$ the solution is

$$\Phi(\xi) = \epsilon\sqrt{\frac{-b_2}{2b_4}}\tanh\left(\sqrt{\frac{-b_2}{2}}\ \xi\right) \tag{2.11}$$

When $b_1 = b_3 = 0$, $b_0 = \frac{b_2^2}{4b_4}$, $b_2 > 0$ and $b_4 < 0$ the solution is

$$\Phi(\xi) = \epsilon\sqrt{\frac{b_2}{2b_4}}\tanh\left(\sqrt{\frac{b_2}{2}}\xi\right) \tag{2.12}$$

Case 3:

When $b_1 = b_3 = 0$, $b_2 > 0$ and $b_4 < 0$,$b_0 = \frac{1-m^2}{(2m^2-1)^2}$ the solution is

$$\Phi(\xi) = \sqrt{\frac{b_2 m^2}{b_4(2m^2-1)}}cn\left(\sqrt{\frac{b_2}{2m^2-1}}\xi, m\right) \tag{2.13}$$

When $b_1 = b_3 = 0$, $b_2 < 0$ and $b_4 > 0$,$b_0 = \frac{b_2^2 m^2}{2b_4(m^2+1)}$ the solution is

$$\Phi(\xi) = \epsilon\sqrt{\frac{b_2 m^2}{b_4(m^2+1)}}sn\left(\sqrt{-\frac{b_2}{m^2+1}}\xi, m\right) \tag{2.14}$$

Case 4:

When $b_0 = b_1 = b_4 = 0$, $b_2 > 0$ the solution is

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$$\Phi(\xi) = \frac{-b_2}{b_3} \sec h^2\left(\frac{\sqrt{b_2}}{2}\xi\right) \tag{2.15}$$

When $b_0 = b_1 = b_4 = 0,\ b_2 < 0$ the solution is

$$\Phi(\xi) = \frac{-b_2}{b_3} \sec^2\left(\frac{\sqrt{-b_2}}{2}\xi\right) \tag{2.16}$$

When $b_0 = b_1 = b_4 = 0,\ b_2 = 0$ the solution is

$$\Phi(\xi) = \frac{1}{b_3\xi^2} \tag{2.17}$$

The search for solutions to Eq (2.2) is described by the following steps 1-5. [5].

Step 1 - To reduce the NPDE Eq. (2.1) to the ODE Eq. (2.2), we use the travelling wave transformation $u(x,t) = u(\xi)$, where $\xi = kx + \omega t$.

Step 2- For the suggested method, we assume that the solution of Eq. (2.2) can be presented in the following form:

$$u(\xi) = \sum_{i=0}^{n} a_i \Phi^i(\xi) \tag{2.18}$$

where a_i are constants to be determined, and $\Phi(\xi)$ satisfies Eq. (2.4)

Step 3- To determine the positive integer n, we balance the linear term of the highest order with the nonlinear term by substituting Eq. (2.18) and Eq. (2.4) into Eq. (2.2).

Step 4- By substituting Eq. (2.4) and Eq. (2.18) into Eq. (2.2) and collecting all terms with the same power of Φ, the left-hand side of Eq. (2.2) is transformed into a polynomial. Subsequently, by setting each coefficient of this polynomial to zero, we derive a system of algebraic equations in terms of a_i where (i = 0, 1, 2) and b_j where (j = 0, 1, 2, 3, 4).

Step 5- Obtain the solution in the form (2.18) of the NPDE (2.1) by solving the sub-equation (2.4) for all possible parameters a_i where (i = 0, 1, 2).

3. APPLICATION TO BENJAMIN-ONO EQUATION

In this section, we will apply the Fan sub-equation method to the BO equation. Let us first consider Eq. (1.4) from the previous section, written in the following form:

$$u_{tt} + \beta(u^2)_{xx} + \gamma u_{xxx} = 0 \tag{3.1}$$

Taking the traveling wave transformation $u = u(\xi)$, $\xi = kx + \omega t$, reduces Eq.(1.4) to the following ODE:

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$$(\omega^2)u'' + 2k^2\beta u'' + 2k\beta(u')^2 + \gamma k^2 u'''' = 0. \quad (3.2)$$

Balancing u'''' with uu'' (or $(u')^2$), we obtain $n = 2$ and assume that the solution of Eq.(3.2) has the following formal solutions:

$$u(\xi) = a_0 + a_1\Phi(\xi) + a_2\Phi^2(\xi) \quad (3.3)$$

where a_0, a_1 and a_2 to be determinate later and $\Phi(\xi)$ satisfies Eq.(3.3)

By substituting Eq. (2.18) and Eq. (2.4) into Eq. (2.2), we obtain a system of algebraic equations.

$$\begin{cases} 2k\beta\epsilon a_1^2 b_0 + \frac{3}{2}\gamma k^2\epsilon^2 a_2 b_1^2 + \frac{1}{2}\omega^2\epsilon a_1 b_1 + 2\omega^2\epsilon a_2 b_0 + k^2\beta\epsilon a_0 a_1 b_1 + \\ 4k^2\beta\epsilon a_0 a_2 b_0 + \frac{1}{2}\gamma k^2\epsilon^2 a_1 b_1 b_2 + 3\gamma k^2\epsilon^2 a_1 b_0 b_3 + 8\gamma k^2\epsilon^2 a_2 b_0 b_2 = 0 \\ 6k^2\beta\epsilon a_0 a_2 b_1 + 4k^2\beta\epsilon a_1 a_2 b_0 + 8k\beta\epsilon a_1 a_2 b_0 + \frac{9}{2}\gamma k^2\epsilon^2 a_1 b_1 b_3 + 15\gamma k^2\epsilon^2 a_2 b_1 b_2 + \\ 12\gamma k^2\epsilon^2 a_1 b_0 b_4 + +30\gamma k^2\epsilon^2 a_2 b_0 b_3 + 2k^2\beta\epsilon a_0 a_1 b_2 + \omega^2\epsilon a_1 b_2 + 3\omega^2\epsilon a_2 b_1 + \\ k^2\beta\epsilon a_1^2 b_1 + \gamma k^2\epsilon^2 a_1 b_2^2 = 0 \\ 2k^2\beta\epsilon a_1^2 b_2 + 4k^2\beta\epsilon a_2^2 b_0 + 8k\beta\epsilon a_2^2 b_0 + 2k\beta\epsilon a_1^2 b_2 + 16\gamma k^2\epsilon^2 a_2 b_2^2 + \\ 3k^2\beta\epsilon a_0 a_1 b_3 + 7k^2\beta\epsilon a_1 a_2 b_1 + 8k^2\beta\epsilon a_0 a_2 b_2 + 8k\beta\epsilon a_1 a_2 b_1 + 15\gamma k^2\epsilon^2 a_1 b_1 b_4 \\ + \frac{15}{2}\gamma k^2\epsilon^2 a_1 b_2 b_3 + 42\gamma k^2\epsilon^2 a_2 b_1 b_3 + 72\gamma k^2\epsilon^2 a_2 b_0 b_4 + 4\omega^2\epsilon a_2 b_2 + \frac{3}{2}\omega^2\epsilon a_1 b_3 = 0 \\ 6k^2\beta\varepsilon a_2^2 b_1 + 8k\beta\epsilon\, a_2^2 b_1 + 2k\beta\epsilon a_1^2 b_3 + \frac{15}{2}\gamma k^2\epsilon^2 a_1 b_3^2 + 3k^2\beta\epsilon a_1^2 b_3 + \\ +10k^2\beta\epsilon a_0 a_2 b_3 + 8k\beta\epsilon a_1 a_2 b_2 + 20\gamma k^2\epsilon^2 a_1 b_2 b_4 + 90\gamma k^2\epsilon^2 a_2 b_1 b_4 \\ +65\gamma k^2\epsilon^2 a_2 b_2 b_3 + 4k^2\beta\epsilon a_0 a_1 b_4 + 10k^2\beta\epsilon a_1 a_2 b_2 + 2\omega^2\epsilon a_2 b_4 + 5\omega^2\epsilon a_2 b_3 = 0 \\ 8k^2\beta\epsilon a_2^2 b_2 + 8k\beta\epsilon a_1^2 b_4 + 2k\beta\epsilon a_1^2 b_4 + \frac{105}{2}\gamma k^2\epsilon^2 a_2 b_3^2 + 4k^2\beta\epsilon a_1^2 b_4 + 30\gamma k^2\epsilon^2 a_1 b_3 b_4 + \\ 120\gamma k^2\epsilon^2 a_2 b_1 b_4 + 13k^2\beta\epsilon a_1 a_2 b_3 + 12k^2\beta\epsilon a_0 a_2 b_4 + 8k\beta\epsilon a_1 a_2 b_3 + 6\omega^2\epsilon a_2 b_4 = 0 \\ 10k^2\beta\epsilon a_2^2 b_3 + 8k\beta\epsilon a_2^2 b_3 + 24\gamma k^2\epsilon^2 a_1 b_4^2 + 8k\beta\epsilon a_1 a_2 b_4 + 168\gamma k^2\epsilon^2 a_2 b_3 b_4 + \\ +16k^2\beta\epsilon a_1 a_2 b_4 = 0 \\ 120\gamma k^2\epsilon^2 a_2 b_4^2 + 12k^2\beta\epsilon a_2^2 b_4 + 8k\beta\epsilon a_2^2 b_4 = 0 \end{cases}$$

(3.4)

by solving the algebraic system (3.4) with the assistance of Maple we obtain the following solutions

$$a_0 = a_0, a_1 = 0, a_2 = 0, b_0 = b_0, b_1 = b_1, b_2 = b_2, b_3 = b_3, b_4 = b_4 \quad (3.5)$$

$$a_0 = \frac{-1}{2}\frac{5k^3\gamma\epsilon b_2 + 3k\omega^2 + 2\omega^2}{k^2\beta(3k+2)} \quad , a_1 = a_1, a_2 = 0, b_0$$

$$= \frac{25}{12}\frac{k^2\gamma^2\epsilon^2 b_2^3}{\beta^2 a_1^2(9k^2 + 12k + 4)}$$

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$$b_1 = \frac{-5}{2}\frac{\gamma k\epsilon b_2^2}{a_1\beta(3k+2)}, b_2 = b_2, b_3 = \frac{-2}{15}\frac{a_1\beta(3k+2)}{k\gamma\epsilon}, b_4 = 0 \qquad (3.6)$$

$$a_0 = a_0, a_1 = a_1, a_2 = a_2, b_0 = 0, b_1 = 0, b_2 = 0, b_3 = 0, b_4 = 0 \qquad (3.7)$$

$$a_0 = \frac{1}{4}\frac{k^2\beta a_1^2 - 2\omega^2 a_2}{k^2\beta a_2}, a_1 = a_1, a_2 = a_2, b_0 = \frac{-1}{480}\frac{4\beta a_1^4(3k+2)}{ka_2^3\gamma\epsilon}, b_1 = \frac{-1}{60}\frac{a_1^3\beta(3k+2)}{\gamma k\epsilon a_2^2},$$

$$b_2 = \frac{-1}{20}\frac{a_1^2\beta(3k+2)}{\gamma k\epsilon a_2}, b_3 = \frac{-1}{15}\frac{a_1\beta(3k+2)}{k\gamma\epsilon}, b_4 = \frac{-1}{30}\frac{\beta a_2(3k+2)}{k\gamma\epsilon} \qquad (3.8)$$

Case 1: if $b_0 = b_1 = b_3 = 0, b_2 < 0 \; and \; b_4 > 0$, Eq (2.4) admits a triangular solution

$$\Phi(\xi) = \sqrt{\frac{-b_2}{b_4}}\sec\sqrt{-b_2}\xi = \sqrt{\frac{-3a_1^2}{2a_2^2}}\sec\left(\frac{1}{10}\sqrt{5}\sqrt{\frac{1}{k}\frac{\beta}{\gamma\epsilon}\frac{a_1^2}{a_2}(3k+2)}\xi\right) \qquad (3.9)$$

Case 2: if $b_0 = \frac{b_2^2}{4b_4}, b_1 = b_3 = 0, b_2 > 0 \; and \; b_4 > 0$, Eq (2.4) admits two periodic solutions

$$\Phi(\xi) = \pm\sqrt{\frac{b_2}{2b_4}}\tan\sqrt{\frac{b_2}{2}}\xi = \pm\frac{\sqrt{3}}{2}\sqrt{\frac{a_1^2}{a_2^2}}\tan\left(\frac{1}{20}\sqrt{10}\sqrt{\frac{-1}{k}\frac{\beta}{\gamma\epsilon}\frac{a_1^2}{a_2}(3k+2)}\,\xi\right) \qquad (3.10)$$

Case 3: if if $b_0 = b_1 = b_3 = 0, b_2 > 0 \; and \; b_4 < 0$, Eq (2.4) admits an hyperbolic function solution

$$\Phi(\xi) = \sqrt{-\frac{3}{2}\frac{a_1^2}{a_2^2}}\sec h\left(\frac{1}{10}\sqrt{5}\sqrt{-\frac{1}{k}\frac{\beta}{\gamma\epsilon}\frac{a_1^2}{a_2}(3k+2)}\,\xi\right) \qquad (3.11)$$

Case 4: for $b_0 = \frac{b_2^2}{4b_4}, b_1 = b_3 = 0, b_2 < 0 \; and \; b_4 > 0$, Eq (2.4) admits two hyperbolic function solutions

$$\Phi(\xi) = \pm\sqrt{-\frac{3}{2}\frac{a_1^2}{a_2^2}}\tanh\left(\sqrt{\frac{-b_2}{2}}\xi\right) \qquad (3.12)$$

Case 5: for $b_0 = b_1 = 0, b_3 = \pm 2\sqrt{b_2 b_4}, b_2 > 0 \; and \; b_4 > 0$, Eq (2.4) admits the hyperbolic solutions

$$\Phi(\xi) = \pm\frac{1}{2}\sqrt{\frac{3}{2}\frac{a_1^2}{a_2^2}}\left[1 + \tanh\left(\frac{1}{20}\sqrt{5}\sqrt{\frac{-1}{k}\frac{\beta}{\gamma\epsilon}\frac{a_1^2}{a_2}(3k+2)}\,\xi\right)\right] \qquad (3.13)$$

Case 6: for $b_0 = \frac{b_2^2 k_1^2(k_1^2-1)}{b_4 p_1}, b_1 = b_3 = 0,, b_2 > 0 \; and \; b_4 < 0$, Eq (2.4) admits the following Jacobian function solutions

$$\Phi(\xi) = \sqrt{-\frac{3}{2}\frac{a_1^2}{a_2^2}\frac{k_1^2}{p_1}}\, cn\left(\sqrt{-\frac{1}{20k}\frac{\beta}{\gamma\epsilon}\frac{a_1^2}{a_2 p_1}(3k+2)}\,\xi, k_1\right), \qquad (3.14)$$

$p_1 = 2k_1^2 - 1, k_1 \in \left(\frac{\sqrt{2}}{2}, 1\right)$

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Case 7: for $b_0=\frac{b_2^2(1-k_2^2)}{b_4p_2^2}, b_1=b_3=0,, b_2>0 \; and \; b_4<0$, Eq (2.4) admits the following Jacobian elliptic function solution

$$\Phi(\xi)=\sqrt{-\frac{3}{2}\frac{a_1^2}{a_2^2p_2}}dn\left(\frac{1}{10}\sqrt{5}\sqrt{-\frac{1}{k}\frac{\beta}{\gamma\epsilon}\frac{a_1^2}{a_2p_1}(3k+2)}\,\xi,k_2\right), \tag{3.15}$$

$p_2=2-k_2^2, k_2\in(0,1)$

Case 8: For $b_0=\frac{b_2k_3^2}{b_4p_3^2}, b_1=b_3=0, b_2>0 \; and \; b_4<0$, Eq (2.4) admits two kinds of Jacobian elliptic doubly periodic wave solutions:

$$\Phi(\xi)=\pm\sqrt{-\frac{3}{2}\frac{a_1^2k_3^2}{a_2^2p_3}}sn\left(\frac{1}{10}\sqrt{5}\sqrt{\frac{1}{k}\frac{\beta}{\gamma\epsilon}\frac{a_1^2}{a_2p_3}(3k+2)}\,\xi,k_3\right), \tag{3.16}$$

$p_3=1+k_3^2, k_3\in(0,1)$

Solution u_1

if $b_0=b_1=b_3=0, b_2<0 \; and \; b_4>0, and \; \Phi(\xi)=\sqrt{\frac{-3a_1^2}{2a_2^2}}\sec\left(\frac{1}{10}\sqrt{5}\sqrt{\frac{1}{k}\frac{\beta}{\gamma\epsilon}\frac{a_1^2}{a_2}(3k+2)}\xi\right)$

Substituting (3.9) and (3.5) -(3.8) into (2.18) we obtain

$$u_1(\xi)=a_0+\sqrt{\frac{-3a_1^2}{2a_2^2}}a_1\sec\left(\frac{1}{10}\sqrt{5}\sqrt{\frac{1}{k}\frac{\beta}{\gamma\epsilon}\frac{a_1^2}{a_2}(3k+2)}\xi\right)-\frac{-3a_1^2}{2a_2}\sec^2\left(\frac{1}{10}\sqrt{5}\sqrt{\frac{1}{k}\frac{\beta}{\gamma\epsilon}\frac{a_1^2}{a_2}(3k+2)}\xi\right) \tag{3.17}$$

Solution u_2

If $b_0=\frac{b_2^2}{4b_4}, b_1=b_3=0, b_2>0 \; and \; b_4>0$, Eq (2.4) admits two periodic solutions

$$\Phi(\xi)=\pm\sqrt{\frac{b_2}{2b_4}}\tan\sqrt{\frac{b_2}{2}}\xi=\pm\frac{\sqrt{3}}{2}\sqrt{\frac{a_1^2}{a_2^2}}\tan\left(\frac{1}{20}\sqrt{10}\sqrt{\frac{-1}{k}\frac{\beta}{\gamma\epsilon}\frac{a_1^2}{a_2}(3k+2)}\,\xi\right) \tag{3.18}$$

Substituting (3.10) and (3.5) -(3.8) into (2.18) we obtain a solution

$$u_2(\xi)=\frac{1}{4}\frac{k^2\beta a_1^2-2\omega^2a_2}{k^2\beta a_2}+a_1\left(\frac{\sqrt{3}}{2}\sqrt{\frac{a_1^2}{a_2^2}}\tan\left(\frac{1}{20}\sqrt{10}\sqrt{\frac{-1}{k}\frac{\beta}{\gamma\epsilon}\frac{a_1^2}{a_2}(3k+2)}\,\xi\right)\right)+ a_2\frac{3}{4}\frac{a_1^2}{a_2^2}\tan^2\left(\frac{1}{20}\sqrt{10}\xi\sqrt{\frac{-1}{k}\frac{\beta}{\gamma\epsilon}\frac{a_1^2}{a_2}(3k+2)}\,\xi\right) \tag{3.19}$$

$$u_2(\xi)=\frac{1}{4}\frac{k^2\beta a_1^2-2\omega^2a_2}{k^2\beta a_2}+a_1\left(-\frac{\sqrt{3}}{2}\sqrt{\frac{a_1^2}{a_2^2}}\tan\left(\frac{1}{20}\sqrt{10}\sqrt{\frac{-1}{k}\frac{\beta}{\gamma\epsilon}\frac{a_1^2}{a_2}(3k+2)}\,\xi\right)\right)+ a_2\frac{3}{4}\frac{a_1^2}{a_2^2}\tan^2\left(\frac{1}{20}\sqrt{10}\xi\sqrt{\frac{-1}{k}\frac{\beta}{\gamma\epsilon}\frac{a_1^2}{a_2}(3k+2)}\,\xi\right) \tag{3.20}$$

10.48047/jocaaa.2024.34.04.2

Solution u_3

If $b_0 = b_1 = b_3 = 0, b_2 > 0 \; and \; b_4 < 0, and \; \Phi(\xi) = \sqrt{-\frac{3}{2}\frac{a_1^2}{a_2^2}} \sec h\left(\frac{1}{10}\sqrt{5}\sqrt{-\frac{1}{k}\frac{\beta}{\gamma\epsilon}\frac{a_1^2}{a_2}(3k+2)}\,\xi\right)$

Substituting (3.11) and (3.5) -(3.8) into (2.18) we obtain a solution

$$u_3(\xi) = a_0 + \sqrt{-\frac{3}{2}\frac{a_1^2}{a_2^2}}\, a_1 \sec h\left(\frac{1}{10}\sqrt{5}\sqrt{-\frac{1}{k}\frac{\beta}{\gamma\epsilon}\frac{a_1^2}{a_2}(3k+2)}\,\xi\right) - - \frac{3}{2}\frac{a_1^2}{a_2^2} sech^2\left(\frac{1}{10}\sqrt{5}\sqrt{-\frac{1}{k}\frac{\beta}{\gamma\epsilon}\frac{a_1^2}{a_2}(3k+2)}\,\xi\right) \quad (3\ .21)$$

Solution u_4

For $b_0 = \frac{b_2^2}{4b_4}, b_1 = b_3 = 0, b_2 < 0 \; and \; b_4 > 0, with \; \Phi(\xi) = \pm\sqrt{-\frac{3}{2}\frac{a_1^2}{a_2^2}} \tanh\left(\sqrt{\frac{-b_2}{2}}\,\xi\right)$

Substituting (3.12) and (3.5) -(3.8) into (2.18) we obtain two hyperbolic function solutions

$$u_4(\xi) = a_0 \pm \sqrt{-\frac{3}{2}\frac{a_1^2}{a_2^2}} \tanh\left(\sqrt{\frac{-b_2}{2}}\,\xi\right) - \frac{3}{2}\tanh^2\left(\sqrt{\frac{-b_2}{2}}\,\xi\right) \quad (3.22)$$

Solution u_5

For $b_0 = b_1 = 0, b_3 = \pm 2\sqrt{b_2 b_4}, b_2 > 0 \; and \; b_4 > 0, and$

$$\Phi(\xi) = \pm\frac{1}{2}\sqrt{\frac{3}{2}\frac{a_1^2}{a_2^2}}\left[1 + \tanh\left(\frac{1}{20}\sqrt{10}\sqrt{\frac{-1}{k}\frac{\beta}{\gamma\epsilon}\frac{a_1^2}{a_2}(3k+2)}\,\xi\right)\right]$$

Substituting (3.13) and (3.5) -(3.8) into (2.18) we obtain

$$u_5(\xi) = \frac{1}{4}\frac{k^2\beta a_1^2 - 2\omega^2 a_2}{k^2\beta a_2} + a_1\left(\frac{1}{2}\sqrt{\frac{3}{2}\frac{a_1^2}{a_2^2}}\left[1 + \tanh\left(\frac{1}{20}\sqrt{5}\sqrt{\frac{-1}{k}\frac{\beta}{\gamma\epsilon}\frac{a_1^2}{a_2}(3k+2)}\,\xi\right)\right]\right) + \; a_2\left(\left(\frac{1}{2}\sqrt{\frac{3}{2}\frac{a_1^2}{a_2^2}}\left[1 + \tanh^2\left(\frac{1}{20}\sqrt{5}\sqrt{\frac{-1}{k}\frac{\beta}{\gamma\epsilon}\frac{a_1^2}{a_2}(3k+2)}\,\xi\right)\right]\right)\right) \quad (3.23)$$

$$u_5(\xi) = \frac{1}{4}\frac{k^2\beta a_1^2 - 2\omega^2 a_2}{k^2\beta a_2} + a_1\left(-\frac{1}{2}\sqrt{\frac{3}{2}\frac{a_1^2}{a_2^2}}\left[1 + \tanh\left(\frac{1}{20}\sqrt{5}\sqrt{\frac{-1}{k}\frac{\beta}{\gamma\epsilon}\frac{a_1^2}{a_2}(3k+2)}\,\xi\right)\right]\right) +$$

$$a_2\left(\frac{1}{2}\sqrt{\frac{3}{2}\frac{a_1^2}{a_2^2}}\left[1+\tanh^2\left(\frac{1}{20}\sqrt{5}\sqrt{\frac{-1}{k}\frac{\beta}{\gamma\epsilon}\frac{a_1^2}{a_2}(3k+2)}\;\xi\right)\right]\right) \quad (3.24)$$

Solution u_6

For $b_0=\frac{b_2^2k_1^2(k_1^2-1)}{b_4p_1}, b_1=b_3=0,, b_2>0\ and\ b_4<0$, and

$$\Phi(\xi)=\sqrt{-\frac{3}{2}\frac{a_1^2}{a_2^2}\frac{k_1^2}{p_1}}\ cn\left(\sqrt{-\frac{1}{20k}\frac{\beta}{\gamma\epsilon}\frac{a_1^2}{a_2p_1}(3k+2)}\;\xi,k_1\right), p_1=2k_1^2-1, k_1\in\left(\frac{\sqrt{2}}{2},1\right)$$

Substituting (3.14) and (3.5) -(3.8) into (2.18) we obtain the following Jacobian function solutions

$$u_6(\xi)=a_0+\sqrt{-\frac{3}{2}\frac{a_1^2}{a_2^2}\frac{k_1^2}{p_1}}a_1\ cn\left(\sqrt{-\frac{1}{20k}\frac{\beta}{\gamma\epsilon}\frac{a_1^2}{a_2p_1}(3k+2)}\;\xi,k_1\right)-$$
$$\frac{3}{2}\frac{a_1^2}{a_2^2}\frac{k_1^2}{p_1}cn^2\left(\sqrt{-\frac{1}{20k}\frac{\beta}{\gamma\epsilon}\frac{a_1^2}{a_2p_1}(3k+2)}\;\xi,k_1\right) \quad (3.25)$$

Solution u_7

For $b_0=\frac{b_2^2(1-k_2^2)}{b_4p_2^2}, b_1=b_3=0,, b_2>0\ , b_4<0$ and

$$\Phi(\xi)=\sqrt{-\frac{3}{2}\frac{a_1^2}{a_2^2p_2}}dn\left(\frac{1}{10}\sqrt{5}\sqrt{-\frac{1}{k}\frac{\beta}{\gamma\epsilon}\frac{a_1^2}{a_2p_1}(3k+2)}\;\xi,k_2\right),$$
$$p_2=2-k_2^2, k_2\in(0,1)$$

Substituting (3.15) and (3.5) -(3.8) into (2.18) we obtain the following Jacobian function solutions

$$u_7(\xi)=a_0+\sqrt{-\frac{3}{2}\frac{a_1^2}{a_2^2p_2}}dn\left(\frac{1}{10}\sqrt{5}\sqrt{-\frac{1}{k}\frac{\beta}{\gamma\epsilon}\frac{a_1^2}{a_2p_1}(3k+2)}\;\xi,k_2\right)-$$
$$\frac{3}{2}\frac{a_1^2}{a_2^2p_2}dn^2\left(\frac{1}{10}\sqrt{5}\sqrt{-\frac{1}{k}\frac{\beta}{\gamma\epsilon}\frac{a_1^2}{a_2p_1}(3k+2)}\;\xi,k_2\right) \quad (3.26)$$

Solution u_8

For $b_0=\frac{b_2k_3^2}{b_4p_3^2}, b_1=b_3=0, b_2>0\ and\ b_4<0$ and

$$\Phi(\xi)=\pm\sqrt{-\frac{3}{2}\frac{a_1^2k_3^2}{a_2^2p_3}}sn\left(\frac{1}{10}\sqrt{5}\sqrt{\frac{1}{k}\frac{\beta}{\gamma\epsilon}\frac{a_1^2}{a_2p_3}(3k+2)}\;\xi,k_3\right),\ p_3=1+k_3^2, k_3\in(0,1)$$

Substituting (3.16) and (3.5) -(3.8) into (2.18) we obtain two kinds of Jacobian elliptic doubly periodic wave solution

$$u_8(\xi) = a_0 \pm \sqrt{-\frac{3}{2}\frac{a_1^2 k_3^2}{a_2^2 p_3}} sn\left(\frac{1}{10}\sqrt{5}\sqrt{\frac{1}{k}\frac{\beta}{\gamma\epsilon}\frac{a_1^2}{a_2 p_3}(3k+2)}\,\xi, k_3\right) - \frac{3}{2}\frac{a_1^2 k_3^2}{a_2^2 p_3} sn^2\left(\frac{1}{10}\sqrt{5}\sqrt{\frac{1}{k}\frac{\beta}{\gamma\epsilon}\frac{a_1^2}{a_2 p_3}(3k+2)}\,\xi, k_3\right) \quad (3.27)$$

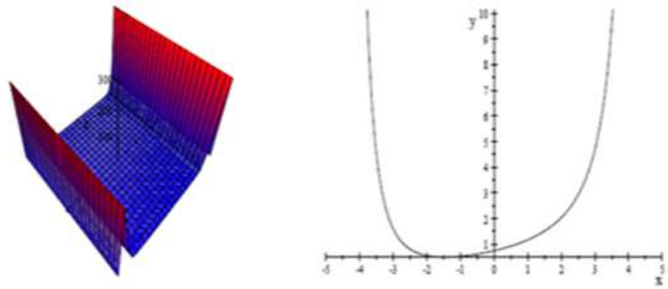

Fig. 1 The hyperbolic solutions of Eq. (1.5) by sutituting the values
$k = a_1 = a_2 = \omega = \gamma = \epsilon = 1, \beta = -1$ in Eq. (3.19)

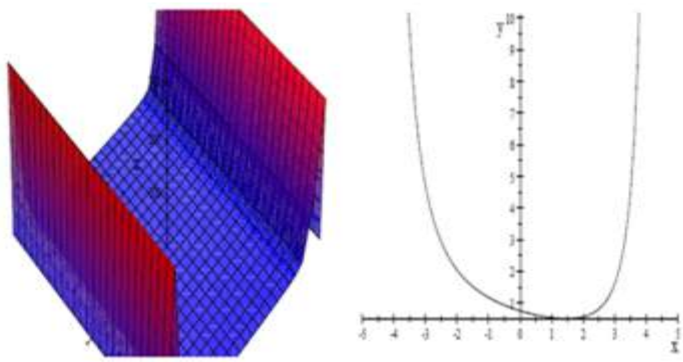

Fig. 2 The hyperbolic solutions of Eq. (1.5) by sutituting the values
$k = a_1 = a_2 = \omega = \gamma = \epsilon = 1, \beta = -1$ in Eq. (3.20)

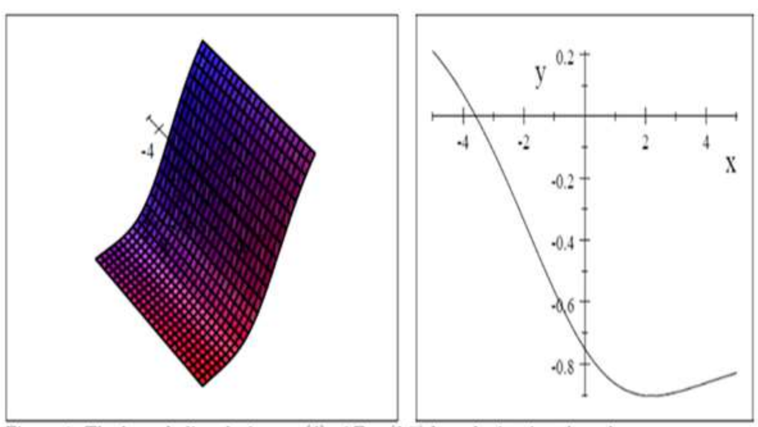

Figure 4. The hyperbolic solutions $u_5(\xi)$ of Eq. (1.5) by substituning the values
$k = 1, \beta = -1, a_1 = 1, a_2 = 1, \omega = 1, \gamma = 1, \epsilon = 1$ in Eq (3.23)

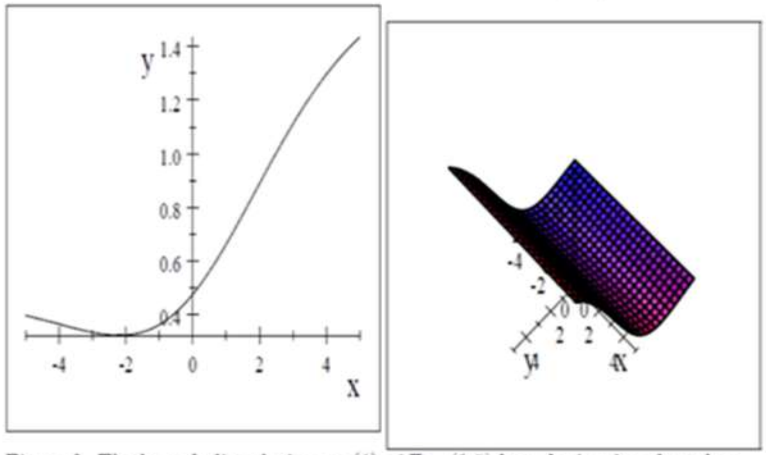

Figure 3. The hyperbolic solutions $u_5(\xi)$ of Eq. (1.5) by substituning the values $k = 1, \beta = -1, a_1 = 1, a_2 = 1, \omega = 1, \gamma = 1, \epsilon = 1$ in Eq (3.22)

4. CONCLUSION

The application of Fan's sub-equation method to the Benjamin-Ono equation has demonstrated its effectiveness in obtaining relevant analytical solutions. The method, which is simple and easy to implement, generated regular and stable solu- tions in line with the physical and mathematical properties of the problem.

However, some complex solutions were rejected due to their lack of realistic physical interpretation, underlining the importance of rigorous selection criteria to validate solutions. On the other hand, this equation, describing internal waves in deep water, remains a rich and complex integrable model, offering many research opportunities.

For future perspectives, it would be interesting to extend this method to the fractional case of this equation, which could model long-memory processes or non- local phenomena. In addition, the study of perturbations or additional terms in the equation could lead to a better understanding of the stability of solutions and their behavior under more realistic conditions. Finally, practical applications in fields such as oceanography, plasma physics or low-dimensional materials could benefit from these theoretical advances, opening up new avenues for the modeling and prediction of complex physical phenomena.

REFERENCES

[1] M. J. Ablowitz, P.A. Clarkson. Solitons, nonlinear evolution equations and inverse scatting,

Cambridge University Press, New York, 1991

[2] M.Alquran, A. Qawasmeh. Classification of solutions to some nonlinear evolution equation , CSP Cambridge, UK;I&S-Florida. USA 2013, VOL20, No.2, pp263-272.

[3] Benjamin, T Brooke. Internal waves of permanent form in fluids of great depth", Journal of Fluid Mechanics, 29 (03) (1967): 559, doi:10.1017/s002211206700103x.

[4] A. Borhanifar, M.M. Kabir. New periodic and soliton solutions by application of Exp-function method for nonlinear evolution equations, Journal of Computational and Applied Mathematics 229 (2009) 158–167

[5] E.G. Fan. Uniformly constructing a series of explicit exact solutions to nonlinear equations in mathematical physics, Chaos Soliton Fract. 16 (2003) 819–839.

[6] Fokas, A. S., & Ablowitz, M. J. (1983). "On the inverse scattering transform of multidimensional nonlinear equations related to first-order systems in the plane." Journal of Mathematical Physics, 24(11), 2494–2503.

[7] Y. Gurefe, A. Sonmezoglu, E. Misirli. Application of the trial equation method for solving some nonlinear evolution equations arising in mathematical physics. Pramana journal of physics. Indian academy of sciences Vol. 77, No.6.December 2011. pp.1023-1023-1029.

[8] J. Hietarinta. Hirota's bilinear method and soliton solutions, Physics AUC.VOL15(part 1)31-37(2005).

[9] W. Malfliet, W. Hermann. The Tanh method Exact solutions of nonlinear evolution and wave equation, physica scripta. Vol 54. (563-568)1996.

[10] Ono Hioraki. Algebraic solitary waves in stratified fluids", Journal of the Physical Society of Japan, 39 (4) (1975): 1082–1091, doi:10.1143/JPSJ.39.1082, MR 0398275.

[11] E.M.E Zayed, K.A.E Alurfi. The homogenous balance method and it's applications for finding the exact solution for nonlinear evolution equation, Italian journal of pure and applied mathematics-N.33-2014(307-318).

[12] E. Fan. Extended tanh-function method and its applications to nonlinear equations, Physics Letters A, vol. 277, no. 4-5, pp. 212–218, 2000.

[13] E. Fan, H. Zhang. A note on the homogeneous balance method," Physics Letters A, vol. 246, no. 5, pp. 403–406, 1998.

[14] E. Fan. Traveling wave solutions for nonlinear equations using symbolic computation. Computers & Mathematics with Applications, vol. 43, no. 8-9, pp. 1151–1162, 2002.

[15] E. Fan. Multiple travelling wave solutions of nonlinear evolution equations using a unified algebraic method. Journal of Physics A: Mathematical and General, vol. 35, no. 32, pp. 6853–6872, 2002.

[16] E. Fan. Multiple travelling wave solutions of nonlinear evolution equations using a unified algebraic method. Journal of Physics A: Mathematical and General, vol. 35, no. 32, pp. 6853–6872, 2002.

[17] E. Fan, J. Zhang. Applications of the Jacobi elliptic function method to special-type nonlinear equations. Physics Letters A, vol. 305, no. 6, pp. 383–392, 2002.

[18] E. Fan, "A generalized extended Fan sub-equation method and its application to the Burgers-Huxley equation," Applied Mathematics and Computation, vol. 217, no. 1, pp. 1–7, 2010.

[19] E. Fan, Y. Hon, "Applications of the extended Fan sub-equation method to the Ginzburg-Landau equation," Journal of Mathematical Physics, vol. 48, no. 11, pp. 113501, 2007.

[20] D. Abbas, A. Kadem," Application of the extended Fan sub-equation method to time fractional Burgers-Fisher equation" Tatra Mountains Mathematical publications, DOI: 10.2478/tmmp-2021-0016 Tatra Mt. Math. Publ. 79 (2021), 1–12.

[21] Ovsiannikov, L. V. (1982). Group Analysis of Differential Equations. Academic Press.

[22] Hirota, R. (1971). "Exact Solution of the Korteweg-de Vries Equation for Multiple Collisions of Solitons." Physical Review Letters, 27(18), 1192–119.