A New Fractional Integral Transform: Katugampola Aboodh Transform and Its Applications

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Abstract. The Aboodh transform is a powerful tool in the analysis of signals and functions, particularly in the context of fractional calculus and signal processing. In this article, we introduce a new form of the Aboodh transform, which we refer to as the Katugampola Aboodh transform. This new definition is based on the Katugampola derivative, a generalization of the classical derivative that retains many of the essential properties of traditional calculus. We provide examples to reinforce the applicability of this new transform.

Keywords: Aboodh transform; fractional differential equation; Katugampola fractional derivative; Adomian polynomial.

Subject Classification(MSC): 26A33, 35A22, 44A15.

1 Introduction

It is well understood that fractional calculus is an extension of the classical integer calculus. In recent years, it has garnered significant attention due to its wide-ranging applications across various scientific and engineering fields. On 1695, fractional calculus was questioned by Hopital, in a letter to Leibniz. On that day, The Hopital made an investigation into the outcome of $\frac{d^n x}{dx^n}$ in the case $n = \frac{1}{2}$. Leibniz replied that it would be "an apparent paradox, from which we will one day draw useful consequences. Since then, fractional calculus is the main field of interest to mathematicians. Over the years, a number of well-known mathematicians, including Liouville, Riemann, Abel, Weyl, Fourier, Lacroix, Leibniz, and Letnikov, Grunwald, contributed to the theory of fractional calculus. In 1819, Lacroix was the first mathematician published a paper mentioning a fractional derivative, it was discovered that a variety of applications, particularly multidisciplinary applications, may be neatly described.In recent years, numerous analytical and approximation methods have been introduced to solve fractional differential equations.

Recently, Qasim Khan and et. al [7] explored linear and nonlinear dynamic systems of FIDEs through the application of the Aboodh transform decomposition method. In [21], the authors solve linear local fractional differential equations and explore their applications using the local fractional Aboodh Transform. Srivastava and et. al [19] investigate the existence of positive solutions for the fractional boundary value problem at resonance, incorporating the Katugampola fractional derivative. In [2], the authors broaden the application of the Aboodh transform by utilizing it to solve fractional-order partial differential equations, incorporating Caputo's fractional derivative. Ali Khalouta [9] proposed a novel general integral transform, which was employed to address differential equations of fractional order in the Caputo sense. Jadhav C.P. and et. al utilized the Laplace, Elzaki and Sumudu transforms to solve ordinary fractional differential equations, as detailed in [3, 4, 5]. Several theorems associated with this method have been established. Khalil, Horani, and Youcef [8] presented the conformable fractional derivative, while Katugampola [6] proposed and analyzed a distinct derivative known as the Katugampola derivative.In [13], Elkhatib et al. expanded the concept of the Laplace transform to include the Katugampola fractional order and established a set of notable rules and properties. Our objective is to use this idea for developing the definition of the Aboodh transform with Katugampola fractional derivative. Also, a very nice relationships and properties are derived and proved. Moreover, we give two important and attractive applications for fractional Katugampola Aboodh transform. We apply together with Adomain decomposition method for presenting the general analytical solution of linear and nonlinear fractional equations. Finally, the results show that our proposed method is an efficient method and applied successfully to find the general solutions.
This paper is structured as follows: In Section 2, some basic preliminaries of the Katugampola derivative and Aboodh transform. In Section 3, some properties of the Katugampola Aboodh transform are demonized, the next section 4 we introduce the Katugampola-Aboodh decomposition method (KADM), and the last section 5 is constructed for illustrative examples.

2 Preliminary

This section presents a compilation of key definitions and theorems from fractional calculus, serving as a foundation for our subsequent developments.

Definition 2.1. *[6] let $t>0; 0<\alpha\leq 1$ and $f:[0,\infty[\to\mathbb{R}$. The Katugampola derivative of f of order α is given as:*

$$D^{\alpha}f(t)=lim_{\varepsilon\to 0}\frac{f(te^{\varepsilon t^{-\alpha}})-f(t)}{\varepsilon}. \tag{1}$$

If f is an α-differentiable function in some $(0,a); a>0$ and $\lim_{t\to 0}D^{\alpha}f(t)$ exist, then

$$\lim_{\varepsilon\to 0}D^{\alpha}f(t)=D^{\alpha}f(0).$$

Definition 2.2. *[6] Let $k<\alpha<k+1$, for $k\in\mathbb{N}$ and $f:[0,\infty[\to\mathbb{R}$ be an n-differentiable at $t>0$ then the α-fractional derivative of f is defined by:*

$$D^{\alpha}f(t)=lim_{\varepsilon\to 0}\frac{f^{(n)}(te^{\varepsilon t^{-\alpha}})-f^{(n)}(t)}{\varepsilon}. \tag{2}$$

if the limit exists.

Remark 2.1. *The Katugampola derivative satisfies key principles, including the product rule, quotient rule, addition rule, and chain rule, making its properties consistent with those of classical calculus for integer orders.*

Theorem 2.1. *Let $k < \alpha \leq k+1$, for $k \in \mathbb{N}$ and $f : [0,\infty[\to \mathbb{R}$ be an $(k+1)$-differentiable at $t > 0$; so*

$$D^{\alpha} f(t) = t^{k+1-\alpha} f^{(k+1)}(t). \tag{3}$$

Proof 2.1. *The detail proof is given in [13]*

Definition 2.3. *[12] Given a function $f : [t_0,\infty[\to \mathbb{R}, t \geq 0$ and $\alpha in(0,1)$. Conformable derivative of f with respect to t of order α is defined by:*

$$D^{\alpha} f(t) = lim_{\varepsilon\to 0} \frac{f(t+et^{1-\alpha}) - f(t)}{\varepsilon}.$$

3 Katugampola Aboodh fractional transform

In this section, we introduce the Katugampola Aboodh transform and explore its connection to the Aboodh transform.

The Katugampola Aboodh transform for certain functions is derived.

Definition 3.1. *[10] An Aboodh transform is defined for functions of exponential order. We consider functions in the set F defined by:*

$$F = \left\{ f(t) : |f(t)| < Me^{-vt}, if t \in [0,\infty[, M, k_1, k_2 > 0; k_1 \leq v \leq k_2 \right\}.$$

For a given function in the set F, the constant M must be finite number and k_1, k_2 may be finite or infinite with variable v define as: $k_1 \leq v \leq k_2$.Then, the Aboodh integral transform denoted by the operator $A(:)$ is defined by the integral equation:

$$T(v) = A(f(t)) = \frac{1}{v}\int_0^{\infty} e^{-vt} f(t)dt; t \geq 0; k_1 \leq v \leq k_2. \tag{4}$$

Definition 3.2. *Let $k < \alpha \leq k+1$, for $k \in \mathbb{N}$ and $f : [0,\infty[$ The Katugampola fractional Abooth transform of order is defined as:*

$$A_{\alpha}(f(t))(v) = \frac{1}{v}\int_0^{\infty} e^{-v\frac{t^{\alpha-k}}{(\alpha-k)}} f(t) t^{\alpha-k-1} dt, k < \alpha \leq k+1. \tag{5}$$

Theorem 3.1. *Let $k < \alpha \leq k+1$, for $k \in \mathbb{N}$, and $f : [0,\infty[\to \mathbb{R}$ then,*

$$A_{\alpha}(D^{\alpha} f(t))(v) = vA_{\alpha}(f^{(k)}(t))(v) - \frac{1}{v} f^{(k)}(0). \tag{6}$$

for a function $f(t)$ and is piecewise continuous with exponential order.

Proof 3.1. *By using definition (3.2) and theorem (2.1), we have:*

$$\begin{aligned}
A_{\alpha}(D^{\alpha} f(t))(v) &= A_{\alpha}(t^{k+1-\alpha} f^{(k+1)}(t))(v) \\
&= \frac{1}{v}\int_0^{\infty} e^{-v\frac{t^{\alpha-k}}{\alpha-k}} t^{k+1-\alpha} f^{(k+1)} t^{\alpha-k-1} dt \\
&= \frac{1}{v}\int_0^{\infty} e^{-v\frac{t^{\alpha-k}}{\alpha-k}} f^{(k+1)} dt.
\end{aligned}$$

Using Integration by parts, we obtain:

$$\begin{aligned}
A_{\alpha}(D^{\alpha} f(t))(v) &= \frac{1}{v}[e^{-v\frac{t^{\alpha-k}}{\alpha-k}} f^{(k)}(t)]_0^{\infty} + v\int_0^{\infty} e^{-v\frac{t^{\alpha-k}}{\alpha-k}} f^{(k)}(t)dt \\
&= vA_{\alpha}(f^{(k)}(t))(v) - \frac{1}{v} f^{(k)}(0).
\end{aligned}$$

Corollary 3.1. *Let* $0<\alpha\leq 1$*, and* $f:[0,\infty[\to\mathbb{R}$ *then:*

$$A_\alpha(D^\alpha f(t))(v)=vA_\alpha(f(t))(v)-\frac{1}{v}f(0). \tag{7}$$

We deduce the proof from the theorem (3.1) by putting $k=0$.

Lemma 3.1. *Let* $\alpha\in]\frac{k-1}{k},1];k\in\mathbb{N}^*$ *and* $u(x,t)$ *be* $k\alpha$*-differentiable real value function. Then:*

$$A_\alpha(\frac{\partial^{k\alpha}u(x,t)}{\partial t^{k\alpha}})(v)=v^kA_\alpha(u(x,t))(v)-\sum_{i=0}^{k-1}\frac{1}{v^{i-k+2}}\frac{\partial^{i\alpha}u(x,0)}{\partial t^{i\alpha}}. \tag{8}$$

Proof 3.2. *We use te proof by induction, For* $k=1$ *we have:*

$$A_\alpha(\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}})(v)=vA_\alpha(u(x,t))(v)-\frac{1}{v}u(x,0)$$

The property is true by the corollary(3.1) Assume that the property is true for a particular value of k and we prove it correct from $k+1$*. So;*

$$\begin{aligned}A_\alpha(\frac{\partial^{(k+1)\alpha}u(x,t)}{\partial t^{(k+1)\alpha}})(v)&=A_\alpha(\frac{\partial^{\alpha}}{\partial t^{\alpha}}(\frac{\partial^{k\alpha}u(x,t)}{\partial t^{k\alpha}}))(v)\\&=vA_\alpha(\frac{\partial^{k\alpha}u(x,t)}{\partial t^{k\alpha}})(v)-\frac{1}{v}\frac{\partial^{k\alpha}u(x,0)}{\partial t^{k\alpha}}\\&=v\left[v^kA_\alpha(u(x,t))(v)-\sum_{i=0}^{k-1}\frac{1}{v^{i-k+2}}u(x,0)\right]-\frac{1}{v}\frac{\partial^{k\alpha}u(x,0)}{\partial t^{k\alpha}}\\&=v^{k+1}A_\alpha(u(x,t))(v)-\sum_{i=0}^{k-1}\frac{1}{v^{i-k+2}}\frac{\partial^{i\alpha}u(x,0)}{\partial t^{i\alpha}}.\end{aligned}$$

Hence; the property is true for every natural number k.

We present in the next lemma the relation between the Katugampola Aboodh transform and the Aboodh transform.

Lemma 3.2. *Let* $f:[0,\infty[\to\mathbb{R}$ *be a function where* $\alpha\in]k,k+1]$ *so,*

$$\begin{aligned}A_\alpha(f(t))(v)&=\frac{1}{v}\int_0^\infty e^{\left(-v\frac{t^{\alpha-k}}{\alpha-k}\right)f(t)t^{\alpha-k-1}dt}\\&=A\left(f\left(((\alpha-k)t)^{\frac{1}{\alpha-k}}\right)\right)(v),\end{aligned} \tag{9}$$

where

$$A(f(t))(v)=\frac{1}{v}\int_0^\infty e^{-vt}f(t)dt.$$

Proof 3.3. *By taking* $z=\frac{t^{\alpha-k}}{\alpha-k}$ *then* $t=((\alpha-k)z)^{\frac{1}{\alpha-k}}$ *and* $\mathrm{d}z=t^{\alpha-k-1}dt$ *let injecting this change in the last formula we get:*

$$\begin{aligned}A_\alpha(f(t))(v)&=\frac{1}{v}\int_0^\infty e^{-vz}f\left(((\alpha-k)z)^{\frac{1}{\alpha-k}}\right)dz\\&=\frac{1}{v}\int_0^\infty e^{-vt}f\left(((\alpha-k)t)^{\frac{1}{\alpha-k}}\right)dt\\&=A\left(f\left(((\alpha-k)t)^{\frac{1}{\alpha-k}}\right)\right)(v).\end{aligned}$$

Let us given the Katugampola Aboodh transform of some functions.

Theorem 3.2. *Let $\alpha \in]k,k+1]$ where $k \in \mathbb{N}$ we have the following transformations:*

1. $A_\alpha\left(e^{\pm\frac{t^{\alpha-k}}{\alpha-k}s^2}\right)(v) = \frac{1}{v^2 \pm s^2 v}$.

2. $A_\alpha\left(\sin\left(s\frac{t^{\alpha-k}}{\alpha-k}\right)\right)(v) = \frac{s}{v^3+s^2v}$.

3. $A_\alpha\left(\cos\left(s\frac{t^{\alpha-k}}{\alpha-k}\right)\right)(v) = \frac{1}{v^3+s^2v}$.

4. $A_\alpha\left(\sinh\left(s\frac{t^{\alpha-k}}{\alpha-k}\right)\right)(v) = \frac{s}{v^3-s^2v}$.

5. $A_\alpha\left(\cosh\left(s\frac{t^{\alpha-k}}{\alpha-k}\right)\right)(v) = \frac{1}{v^2-s^2}$.

Proof 3.4. *We give the proofs of transformations; where we use the substitution: $z = \frac{t^{\alpha-k}}{\alpha-k}$;*

1.

$$\begin{aligned} A_\alpha\left(e^{\frac{t^{\alpha-k}}{\alpha-k}s^2}\right)(v) &= \frac{1}{v}\int_0^\infty e^{-v\frac{t^{\alpha-k}}{\alpha-k}} \cdot e^{\pm\frac{t^{\alpha-k}}{\alpha-k}s^2} t^{\alpha-k-1} dt \\ &= \frac{1}{v}\int_0^\infty e^{-\left(v\mp s^2\right)z} dz \\ &= A\left(e^{\pm s^2 z}\right)(v) \\ &= \frac{1}{v^2 \mp s^2 v}. \end{aligned}$$

2.

$$\begin{aligned} A_\alpha\left(\sin\left(s\frac{t^{\alpha-k}}{\alpha-k}\right)\right)(v) &= \frac{1}{v}\int_0^\infty e^{-v\frac{t^{\alpha-k}}{\alpha-k}} \left[\sin\left(s\frac{t^{\alpha-k}}{\alpha-k}\right)\right] t^{\alpha-k-1} dt \\ &= \frac{1}{v}\int_0^\infty e^{-vz}[\sin(sz)]dz \\ &= A(\sin(sz))(v) \\ &= \frac{s}{v^3+s^2v}. \end{aligned}$$

3.

$$\begin{aligned} A_\alpha\left(\cos\left(s\frac{t^{\alpha-k}}{\alpha-k}\right)\right)(v) &= \frac{1}{v}\int_0^\infty e^{-v\frac{t^{\alpha-k}}{\alpha-k}} \left[\cos\left(s\frac{t^{\alpha-k}}{\alpha-k}\right)\right] t^{\alpha-k-1} dt \\ &= \frac{1}{v}\int_0^\infty e^{-vz}[\cos(sz)]dz = A(\cos(sz))(v) \\ &= \frac{1}{v^3+s^2}. \end{aligned}$$

4.

$$A_\alpha\left(\sinh\left(s\frac{t^{\alpha-k}}{\alpha-k}\right)\right)(v)=\frac{1}{v}\int_0^\infty e^{-v\frac{t^{\alpha-k}}{\alpha-k}}\left[\sinh\left(s\frac{t^{\alpha-k}}{\alpha-k}\right)\right]t^{\alpha-k-1}dt$$
$$=\frac{1}{v}\int_0^\infty e^{-vz}[\sinh(sz)]dz$$
$$=A(\sinh(sz))(v)$$
$$=\frac{s}{v^3-s^2v}.$$

5.

$$A_\alpha\left(\cosh\left(s\frac{t^{\alpha-k}}{\alpha-k}\right)\right)(v)=\frac{1}{v}\int_0^\infty e^{-v\frac{t^{\alpha-k}}{\alpha-k}}\left[\cosh\left(s\frac{t^{\alpha-k}}{\alpha-k}\right)\right]t^{\alpha-k-1}dt$$
$$=\frac{1}{v}\int_0^\infty e^{-vz}[\cosh(sz)]dz$$
$$=A(\cosh(sz))(v)$$
$$=\frac{1}{v^2-s^2}.$$

Theorem 3.3. *Let $\alpha\in]k;k+1]$ where $k\in\mathbb{N}$ we have*

$$A_\alpha\left(t^m\right)(v)=\frac{(\alpha-k)^{\frac{m}{\alpha-k}}}{v^{2+\frac{m}{\alpha-k}}}\Gamma\left(1+\frac{m}{\alpha-k}\right). \tag{10}$$

Proof 3.5.

$$(A_\alpha(t^m)(v)=A(f(((\alpha-k)t)^{\frac{1}{\alpha-k}}))(v)$$
$$=\frac{1}{v}\int_0^\infty e^{-vt}((\alpha-k)t)^{\frac{1}{\alpha-k}}))^m dt$$
$$=\frac{1}{v}\int_0^\infty e^{-vt}(\alpha-k)^{\frac{m}{\alpha-k}}t^{\frac{m}{\alpha-k}}dt$$
$$=\frac{(\alpha-k)^{\frac{m}{\alpha-k}}}{v}\int_0^\infty e^{-vt}t^{\frac{m}{\alpha-k}}dt;$$

By integration par party we get:

$$A_\alpha(t^m)(v)=\frac{(\alpha-k)^{\frac{m}{\alpha-k}}}{v^2}\int_0^\infty e^{-vt}(\frac{m}{\alpha-k})t^{\frac{m}{\alpha-k}-1}dt$$
$$=\frac{(\alpha-k)^{\frac{m}{\alpha-k}}}{v^3}(\frac{m}{\alpha-k})(\frac{m}{\alpha-k}-1)\int_0^\infty e^{-vt}t^{(\frac{m}{\alpha-k}-2)}dt$$
$$=\frac{(\alpha-k)^{\frac{m}{\alpha-k}}}{v^4}(\frac{m}{\alpha-k})(\frac{m}{\alpha-k}-1)(\frac{m}{\alpha-k}-2)\int_0^\infty e^{-vt}t^{(\frac{m}{\alpha-k}-3)}dt$$
$$=\ldots$$
$$=\frac{(\alpha-k)^{\frac{m}{\alpha-k}}}{v^{1+\frac{m}{\alpha-k}}}(\frac{m}{\alpha-k})(\frac{m}{\alpha-k}-1)\ldots(\frac{m}{\alpha-k}-[\frac{m}{\alpha-k}-1])$$
$$=\frac{(\alpha-k)^{\frac{m}{\alpha-k}}}{v^{2+\frac{m}{\alpha-k}}}\Gamma(1+\frac{m}{\alpha-k}).$$

Some important property is of the Katugampola Aboodh transform.

Lemma 3.3. *(The Katugampola Aboodh linear). Let f and g be two functions, a and b two scaler so,*

$$A_\alpha(af+bg) = aA_\alpha(f) + bA_\alpha(g).$$

The proof is trivial.

Theorem 3.4. *(The shifting property) By introducing the Katugampola Aboodh transform:*

$$\begin{aligned} A_\alpha(e^{-a\frac{t^{\alpha-k}}{\alpha-k}} f(t))(v) &= A_\alpha(e^{-at} f(t))(v) \\ &= A(f(((\alpha-k)t)^{\frac{1}{\alpha-k}}))(v+a). \end{aligned} \tag{11}$$

The following results are readily follow:

1. $A_\alpha(e^{-a\frac{t^{\alpha-k}}{\alpha-k}} t^m)(v) = \frac{(\alpha-k)^{\frac{m}{\alpha-k}}}{(v+a)^{1+\frac{m}{\alpha-k}}}\Gamma(1+\frac{m}{\alpha-k})$.

2. $A_\alpha(e^{-a\frac{t^{\alpha-k}}{\alpha-k}} e^{-b\frac{t^{\alpha-k}}{\alpha-k}})(v) = \frac{1}{(v-a)^2+b(v+a)}$

3. $A_\alpha(e^{-a\frac{t^{\alpha-k}}{\alpha-k}} \sin(b\frac{t^{\alpha-k}}{\alpha-k}))(v) = \frac{b}{(v-a)^3+b^2(v+a)}$.

4. $A_\alpha(e^{-a\frac{t^{\alpha-k}}{\alpha-k}} \cos(b\frac{t^{\alpha-k}}{\alpha-k}))(v) = \frac{1}{(v-a)^2+b^2}$.

Theorem 3.5. *(Change of scale property) Let $k < \alpha \leq k+1$ for $k \in \mathbb{N}$ and $f : [0,\infty[\to \mathbb{R}$ than we have:*

$$A_\alpha(f(at))(v) = \frac{1}{a^{\alpha-k}} A_\alpha(f(t))(v). \tag{12}$$

The proof follow easily by the aid of definition(3.2).

Theorem 3.6. *Let $k < \alpha \leq k+1$ for $k \in \mathbb{N}$ and $f : [0,\infty[\to \mathbb{R}$ than;*

$$A_\alpha\left(\left(\frac{t^{\alpha-k}}{\alpha-k}\right)^m f(t)\right)(v) = (-1)^m \frac{d^m}{dv^m} A_\alpha(f(t))(v), where; m = 1,2,3,... \tag{13}$$

Proof 3.6. *We have:*

$$\begin{aligned} A_\alpha\left(\left(\frac{t^{\alpha-k}}{\alpha-k}\right)^m f(t)\right)(v) &= \frac{1}{v}\int_0^\infty e^{-v\frac{t^{\alpha-k}}{\alpha-k}} \left[\left(\frac{t^{\alpha-k}}{\alpha-k}\right)^m f(t)\right] t^{\alpha-k-1} dt \\ &= \frac{1}{v}\int_0^\infty \left(\frac{t^{\alpha-k}}{\alpha-k}\right)^m e^{-v\frac{t^{\alpha-k}}{\alpha-k}} f(t) t^{\alpha-k-1} dt \\ &= (-1)^m \frac{d^m}{dv^m} A_\alpha(f(t))(v). \end{aligned}$$

4 Examples

Example (1). Consider the fractional linear differential equation

$$D^{\alpha}y(t)=y(t)+1,\quad 0\leqslant t\leqslant 1,\quad 0<\alpha\leqslant 1. \tag{4.1}$$

with initial condition

$$y(0)=0, \tag{4.2}$$

In this case, we have $k=0$. Apply Katugampola Aboodh transform to both side of (4.1)

$$A_{\alpha}\left[D^{\alpha}y(t)\right]=A_{\alpha}\left[y(t)+1\right].$$

So,

$$vA_{\alpha}\left[y(t)\right]-\tfrac{1}{v}y(0)=A_{\alpha}\left[y(t)\right]+A_{\alpha}\left[1\right]$$

$$\Leftrightarrow A_{\alpha}\left[y(t)\right](v-1)=\frac{1}{v^{2}},$$

$$\Leftrightarrow A_{\alpha}\left[y(t)\right]=\frac{1}{v^{2}(v-1)}=-\frac{1}{v^{2}}+\frac{1}{v(v-1)}. \tag{4.3}$$

Applying the inverse Katugampola Aboodh transform to(4.3), we get

$$y(t)=-A_{\alpha}^{-1}\left[\frac{1}{v^{2}}\right]+A_{\alpha}^{-1}\left[\frac{1}{v(v-1)}\right]=\mathrm{e}\left(\frac{t^{\alpha}}{\alpha}\right)-1.$$

for $\alpha=1$, we obtain the exact solution

$$y(t)=-1+\mathrm{e}t.$$

which plots in Figure 1.

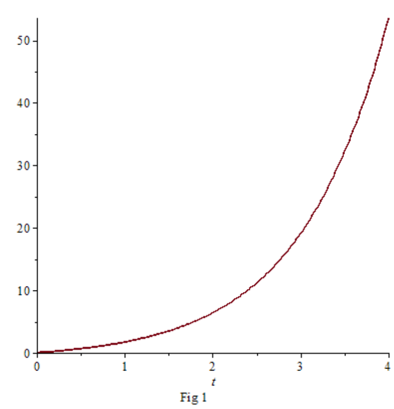

Figure 1: Represents the solution of Example 1

Example (2). Consider the problem

$$D^{\frac{1}{2}}y(t)+y(t)=\frac{8}{3\sqrt{\pi}}t^{\frac{3}{2}}+t^{2},\quad 0\leqslant t\leqslant 1,\quad y(0)=0, \tag{4.4}$$

We applying the Katugampola Aboodh transform to(4.4), we have

$$A_{\frac{1}{2}}\left[D^{\frac{1}{2}}y(t)\right]+A_{\frac{1}{2}}[y(t)]=A_{\frac{1}{2}}\left[\frac{8}{3\sqrt{\pi}}t^{\frac{3}{2}}\right]+A_{\frac{1}{2}}\left[t^{2}\right], \tag{4.5}$$

so,

$$vA_{\frac{1}{2}}[y(t)]+A_{\frac{1}{2}}[y(t)]=\frac{8}{3\sqrt{\pi}}\frac{1}{8}\frac{3!}{v^{5}}+\frac{1}{16}\frac{4!}{v^{6}} \tag{4.6}$$

imply that,

$$A_{\frac{1}{2}}[y(t)](v+1)=\frac{2}{\sqrt{\pi}}\frac{1}{v^{5}}+\frac{3}{2v^{6}},$$

then,

$$A_{\frac{1}{2}}[y(t)]=\frac{2}{\sqrt{\pi}}\frac{1}{v^{5}(v+1)}+\frac{3}{2v^{6}(v+1)}, \tag{4.7}$$

Applying the inverse Katugampola Aboodh transform to(4.7), we get

$$y(t)=\frac{2}{\sqrt{\pi}}A_{\frac{1}{2}}^{-1}\left[\frac{1}{v^{5}(v+1)}\right]+\frac{3}{2}A_{\frac{1}{2}}^{-1}\left[\frac{1}{v^{6}(v+1)}\right]. \tag{4.8}$$

since,

$$\frac{1}{v^{6}(v+1)}=\frac{1}{v^{6}}-\frac{1}{v^{5}}+\frac{1}{v^{4}}-\frac{1}{v^{3}}+\frac{1}{v^{2}}-\frac{1}{v(v+1)},$$

$$\frac{1}{v^{5}(v+1)}=\frac{1}{v^{5}}-\frac{1}{v^{4}}+\frac{1}{v^{3}}-\frac{1}{v^{2}}+\frac{1}{v(v+1)}.$$

we have,

$$A_{\frac{1}{2}}^{-1}\left[\frac{1}{v^{6}(v+1)}\right]=\frac{2}{3}t^{2}-\frac{4}{3}t\sqrt{t}+2t-2\sqrt{t}-\mathrm{e}\left(-2\sqrt{t}\right)+1, \tag{4.9}$$

and

$$A_{\frac{1}{2}}^{-1}\left[\frac{1}{v^{5}(v+1)}\right]=\frac{4}{3}t\sqrt{t}-2t+2\sqrt{t}+\mathrm{e}\left(-2\sqrt{t}\right)-1, \tag{4.10}$$

by (4.10) and (4.9), we obtain

$$y(t)=t^{2}+\left(\frac{8}{3\sqrt{\pi}}-2\right)t\sqrt{t}+\left(3-\frac{4}{\sqrt{\pi}}\right)\left(t-\sqrt{t}\right)+\left(\frac{2}{\sqrt{\pi}}-\frac{3}{2}\right)\left(\mathrm{e}\left(-2\sqrt{t}\right)-1\right).$$

which plots in Figure 2.

Example (3)**.** Consider the nonlinear delay differential equation of first order:

$$D^{\alpha}y(t)=1+2y^{2}\left(\frac{t}{2}\right),\quad 0\leqslant t\leqslant 1,\quad 0<\alpha\leqslant 1. \tag{4.11}$$

with initial condition

$$y(0)=0, \tag{4.12}$$

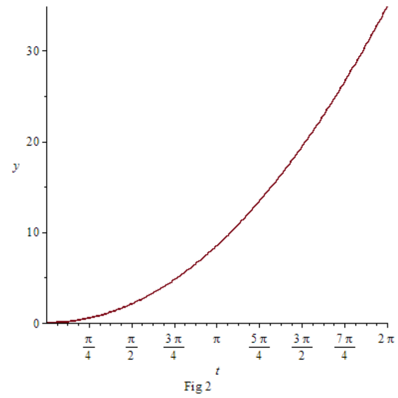

Figure 2: Represents the solution of Example 2

in this case, we have $k=0$. Apply Katugampola Aboodh transform to both side of equation(4.11):

$$A_\alpha\left[D^\alpha y(t)\right]=A_\alpha\left[1+2y^2\left(\frac{t}{2}\right)\right] \tag{4.13}$$

by using the definition and initial condition(4.12) we have:

$$vA_\alpha\left[y(t)\right]=\frac{1}{v^2}+A_\alpha\left[2y^2\left(\frac{t}{2}\right)\right] \tag{4.14}$$

$$A\left[y(t)\right]=\frac{1}{v^3}+\frac{1}{v}A_\alpha\left[2y^2\left(\frac{t}{2}\right)\right]. \tag{4.15}$$

Applying the inverse Katugampola Aboodh transform to (4.15) we get:

$$y(t)=A_\alpha^{-1}\left[\frac{1}{v^3}\right]+A_\alpha^{-1}\left[\frac{1}{v}A_\alpha\left[2y^2\left(\frac{t}{2}\right)\right]\right]$$

where

$$y_0(t)=A_\alpha^{-1}\left[\frac{1}{v^3}\right]=\frac{t^\alpha}{\alpha}.$$

So, we have

$$y_0\left(\frac{t}{2}\right)=\frac{t^\alpha}{\alpha 2^\alpha}. \tag{4.16}$$

And

$$y_{n+1}(t)=A_\alpha^{-1}\left[\frac{1}{v}A_\alpha\left[2A_n\right]\right] \tag{4.17}$$

where A_n is the Adomain polynomial series of $y_0, y_1; y_2......., y_n$ wich is a convergent, that are given by:

$$\begin{gathered} A_0=y_0^2\left(\frac{t}{2}\right)\\ A_1=2y_0\left(\frac{t}{2}\right)y_1\left(\frac{t}{2}\right)\\ A_2=y_2\left(\frac{t}{2}\right)2y_0\left(\frac{t}{2}\right)+\frac{1}{2}y_1^2.\\ \vdots \end{gathered} \tag{4.18}$$

For $n=0$, the equation (4.17)become

$$\begin{cases} y_1(t) = A_\alpha^{-1}\left[\frac{1}{v}A_\alpha\left[2y_0^2\left(\frac{t}{2}\right)\right]\right] \\ = A_\alpha^{-1}\left[\frac{1}{v}A_\alpha\left[2\left(\frac{t^\alpha}{\alpha 2^\alpha}\right)^2\right]\right] = A^{-1}\left[\frac{1}{v}A_\alpha\left[\left(\frac{t^{2\alpha}}{\alpha^2\, 2^{2\alpha}}\right)\right]\right] \\ = \frac{1}{2^{2\alpha-2}}A_\alpha^{-1}\left[\frac{1}{v^5}\right] = \frac{1}{2^{2\alpha-2}}\frac{t^{3\alpha}}{\alpha^3\, 3!}, \end{cases} \tag{4.19}$$

so, we have

$$y_1\left(\frac{t}{2}\right) = \frac{t^{3\alpha}}{2^{5\alpha-2}\alpha^3\, 3!}, \tag{4.20}$$

for $n=1$, the equation (4.17)become

$$\begin{cases} y_2(t) = A_\alpha^{-1}\left[\frac{1}{v}A_\alpha\left[2A_1\right]\right] = A_\alpha^{-1}\left[\frac{1}{v}A_\alpha\left[4y_0\left(\frac{t}{2}\right)y_1\left(\frac{t}{2}\right)\right]\right] \\ = A_\alpha^{-1}\left[\frac{1}{v}A\left[4\left(\frac{t^\alpha}{\alpha 2^\alpha}\right)\left(\frac{t^{3\alpha}}{2^{5\alpha-2}\alpha^3\, 3!}\right),\right]\right] \\ = A_\alpha^{-1}\left[\frac{1}{v}A_\alpha\left[\left(\frac{t^{4\alpha}}{2^{6\alpha-4}\alpha^4\, 3!}\right)\right]\right] \\ = A_\alpha^{-1}\left[\frac{1}{v^7}\frac{1}{2^{6\alpha-6}}\right] = \frac{t^{5\alpha}}{2^{6\alpha-6}\alpha^5\, 5!} \end{cases} \tag{4.21}$$

The series solution is given by:

$$y(t) = y_0(t) + y_1(t) + y_2(t) + \ldots \qquad = \frac{t^\alpha}{\alpha} + \frac{1}{2^{2\alpha-2}}\frac{t^{3\alpha}}{\alpha^3\, 3!} + \frac{t^{5\alpha}}{2^{6\alpha-6}\alpha^5\, 5!} + \ldots\ldots..$$

The exact solution when $\alpha = 1$ is given by

$$y(t) = t + \frac{t^3}{3!} + \frac{t^5}{5!} + \ldots = sinh(t) \tag{4.22}$$

which plots in Figures 3, 4,5.

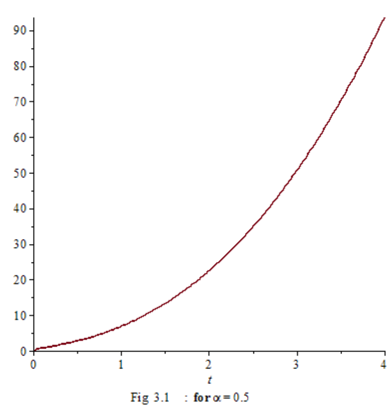

Figure 3: Solution of Example 3 for $\alpha = 0.5$

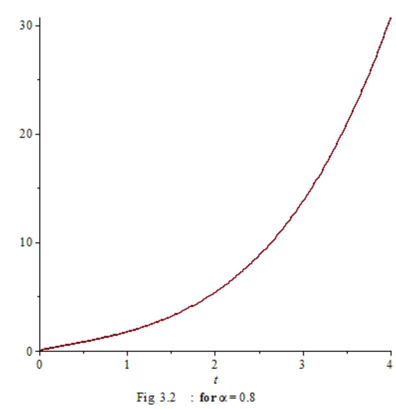

Figure 4: Solution of Example 3 for $\alpha = 0.8$

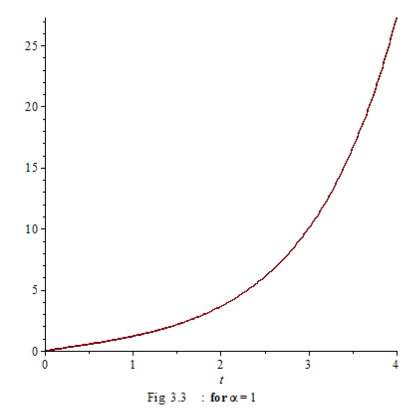

Figure 5: Solution of Example 3 for $\alpha = 1$

5 Conclusion

In this study, we explored the properties and applications of a newly defined fractional Aboodh transform, which extends the classical Aboodh transform framework. Our findings reveal several parallels between the results derived from this new transform and those established in classical calculus, suggesting a deeper connection between fractional calculus and traditional analysis. To demonstrate the applicability of this transform, we provide some specific applications to which our findings can be applied. This research can be extended to encompass delay systems and multidimensional partial differential equations (PDEs). Delay systems are crucial for modeling processes where past states impact future behavior, such as in control systems, biological models, and economic dynamics. Developing solutions for these systems could significantly broaden the scope and practical applications of the research.

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