Fractional order predator-prey model

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Abstract

In this paper we considered a fractional order differential system with a derivative in the sense of Caputo. We start with results of existence, uniqueness and positivity of solutions. Next, the local and global asymptotic stability of equilibria are obtained according to some parameters values of our system. Finally, we present some numerical simulations to illustrate the theoretical results.

Key words and phrases: Fractional order derivative, predator prey model, local and global stability, numerical simulations.

1 Introduction

Many researchers have traditionally relied on integer-order differential equations to construct mathematical models, which have proven essential for understanding the dynamics of biological systems. However, fractional-order differential equations have recently gained significant attention due to their ability to accurately describe complex nonlinear phenomena. These equations offer greater degrees of freedom compared to their integer-order counterparts and are particularly suited to systems with memory, a characteristic prevalent in most biological systems. In this study, we consider a fractional-order ecological model of prey and predator populations, aiming to explore how these memory effects influence their dynamic interactions. Some important work have been done with fractional order differential equations in biological systems [1], [21],[10] and in the other field of science and engineering [3], [6],[7].

The model investigate is defined by the following system of fractional-order differential equations, employing the Caputo fractional derivative of order $q \in (0,1)$

$$\begin{cases} D^q x = x(r_b - \frac{r_b x}{K} - \alpha y), \\ D^q y = y(r_c - r_b + \frac{r_b x}{K} - \alpha y(r_c - 1)), \end{cases} \tag{1}$$

where $x(t)$ and $y(t)$ represent the prey and predator population densities, respectively, at time t. Here, r_b denotes the intrinsic growth rate of the prey, K is the carrying capacity of the environment, α measures the predation rate, and r_c reflects the predator's death rate in the absence of prey, adjusted by interaction terms. The fractional order q introduces a memory effect, allowing the system's evolution to depend on its historical states; a feature that aligns with the long-term dependencies often observed in ecological interactions, such as delayed responses in population growth or decline.

Motivated by the need to understand how fractional dynamics influence ecological systems, we embarked on a comprehensive analysis of this model. We start with establishing the fundamental properties of the system: we rigorously proved the existence and uniqueness of solutions. Then We demonstrated the boundedness of solutions, confirming that populations remain within realistic ecological limits. Next, we identified the equilibrium points, focusing on the coexistence equilibrium where both species persist, and analyzed its local and global stability using linearization techniques and Lyapunov functions, respectively. To explore the model's behavior across different parameter regimes, we derived stability conditions for various cases, such as when $r_c > r_b > 1, r_b > r_c > 1$, and several scenarios where $0 < r_c < 1 < r_b$, adjusting constraints on $r_c + r_b$ and q. Finally, we conducted numerical simulations to validate our theoretical findings.

2 Existence, Uniqueness and boundedness

2.1 Existence, uniqueness and positivity of solutions

The model in Equation (1) can be written as

$$\begin{cases} D^q x &= f(x,y), \\ D^q y &= g(x,y), \end{cases} \tag{2}$$

where f and g represent the right hand side of (1). A unique solution of system (2) exists if the mapping $F(x,y) = [f,g]^T(x,y)$ satisfies the local Lipschitz condition with respect to x and y. Clearly, both functions f and g are continuous smooth on $\mathbb{R}_+^2$. Then, the existence and uniqueness of solutions of such system follow by the Theorems 6.1 and 6.5 in "[5]".

In this paper, we focus only on the positivity and boundedness properties. In the next proposition, we show that the solutions of system are nonnegative.

Proposition 2.1. *All solutions of system (1) with nonnegative initial conditions are nonnegative.*

Proof. Let (x,y) be a solution of (1) associated to the initial condition $(x(0),y(0)) \in \mathbb{R}_+^2$. We prove the nonnegativity by applying the Theorem 1 in [16] . We have the following implications

$$x(t) = 0 \;\Rightarrow\; D^q x(t) \geq 0,$$

and

$$y(t) = 0 \;\; \Rightarrow \;\; D^q y(t) \geq 0.$$

Clearly, the solution of (1) remains nonnegative for all $t > 0$. This yields $x(t) \geq 0$ and $y(t) \geq 0$, for $t > 0$. □

Next, we state the theorem corresponding to the boundedness of solution of the system. Before that, we need to define the Mittag-Leffler function. The one-parameter Mittag-Leffler function is defined as

$$E_q(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(qk+1)}.$$

Clearly, $E_1(z) = e^z$. It is known that the solution of

$$\begin{cases} D^q x(t) = -\nu x(t), \\ x(t_0) = x_0 \in \mathbb{R}, \end{cases} \tag{3}$$

is given by the term of the Mittag-Leffler function and we have

$$x(t) = x_0 E_q[-\nu(t-t_0)^q].$$

We give the boundedness property for the solutions of our system.

Theorem 2.2. *All solutions of system (1) with initial conditions $(x(0), y(0))$ in $\mathbb{R}_+^2$ are uniformly bounded.*

Proof. Let us define the function

$$\begin{aligned} W : \mathbb{R}_+ &\rightarrow \mathbb{R}_+, \\ t &\mapsto W(t) = x(t) + \gamma y(t). \end{aligned}$$

The Caputo fractional derivative of order q of W along the solution trajectory of (1) is given by

$$D^q W(t) = D^q x(t) + \gamma D^q y(t) = x(r_b - \frac{r_b x}{K} - \alpha y) + \gamma y(r_c - r_b + \frac{r_b x}{K} - \alpha y(r_c - 1)).$$

Therefore,

$$D^q W(t) + \beta W = x(r_b - \frac{r_b x}{K} - \alpha y) + \gamma y(r_c - r_b + \frac{r_b x}{K} - \alpha y(r_c - 1)) + \beta(x + \gamma y).$$

Therefore,

$$D^qW(t) + \beta W = x(\beta + r_b - \frac{r_b x}{K} - \alpha y + \frac{r_b\gamma}{K}y) + \gamma y(r_c - r_b + \beta - \alpha y(r_c - 1)).$$

If we choose $0 < \gamma \leq \frac{K\alpha}{r_b}$, then

$$D^qW(t) + \beta W \leq x(\beta + r_b - \frac{r_b x}{K}) + \gamma y(r_c - r_b + \beta - \alpha(r_c - 1)y).$$

This leads to,

$$D^qW(t) + \beta W \leq \frac{K(\beta + r_b)^2}{4r_b} + \frac{2\gamma(r_c - r_b + \beta)^2}{4\alpha(r_c - 1)} =: \nu.$$

By applying the standard comparison theorem for fractional order, we obtain

$$0 \leq W(x(t), y(t)) \leq \left[W(0) - \frac{\nu}{\beta}\right] E_q(-\beta t^q) + \frac{\nu}{\beta},$$

where E_q is the Mittag-Leffler function. Since, $E_q(-\beta t^q) \to 0$ as $t \to +\infty$ This implies that,

$$0 \leq \lim_{t\to+\infty} W(x(t), y(t)) \leq \frac{\nu}{\beta},$$

Hence, all the solution of (1) are bounded. □

3 Equilibria

The aim of this section is to study the existence of equilibrium points for the fractional order system(1). To calculate the equilibrium points we use the following equation:

$$\begin{cases} x(r_b - \frac{r_b x}{K} - \alpha y) = 0, \\ y(r_c - r_b + \frac{r_b x}{K} - \alpha y(r_c - 1)) = 0. \end{cases} \tag{4}$$

Therefore, the system (1) admits a trivial equilibrium $E_0 = (0, 0)$, predator-free equilibrium $E_1 = (K, 0)$ and prey-free equilibrium $E_2 = (0, \frac{r_c - r_b}{(r_c - 1)\alpha})$ if and only if $(r_c - r_b)(r_c - 1) > 0$. In addition, for $r_b > 1$ an endemic equilibrium $E^* = (\frac{r_b - 1}{r_b}K, 1/\alpha)$ exist.

Theorem 3.1 (Existence)**.**

1. *The trivial equilibrium $E_0 = (0, 0)$ and predator-free equilibrium $E_1 = (K, 0)$ consistently exists.*

2. *The prey-free equilibrium $E_2 = (0, \frac{r_c - r_b}{(r_c - 1)\alpha})$ existe if and only if $r_c > max(r_b, 1)$ or $r_c < min(r_b, 1)$.*

3. *The endemic equilibrium $E^* = (\frac{r_b - 1}{r_b}K, 1/\alpha)$ if and only if $r_b > 1$.*

4 Local stability analysis

To examine the local asymptotic stability of these equilibria we compute their linearization.

Theorem 4.1. *For system (1)*

1. *The equilibrium point* $E_0 = (0,0)$ *is unstable.*

2. *The equilibrium point* $E_1 = (K,0)$ *is unstable.*

3. *If* $r_b < 1 < r_c$, *the equilibrium point* $E_2 = (0, \frac{r_c - r_b}{(r_c-1)\alpha})$ *is locally asymptotically stable. And if we have the following conditions:*

   (a) $0 < r_c < r_b < 1$,

   (b) $r_c > r_b > 1$,

   (c) $0 < r_c < 1 < r_b$.

   The equilibrium point $E_2 = (0, \frac{r_c - r_b}{(r_c-1)\alpha})$ *is unstable.*

4. *For the following conditions:*

   (a) $r_c > r_b > 1$,

   (b) $r_b > r_c > 1$,

   (c) $0 < r_c < 1 < r_b$ *and* $r_b + r_c > 2$,

   (d) $0 < r_c < 1 < r_b$ *and* $r_b + r_c = 2$,

   (e) $0 < r_c < 1 < r_b$, $r_b + r_c < 2$ *and* $0 < q < \frac{2}{\pi}\tan^{-1}\left(\frac{\sqrt{4(r_b-1)-(r_b-r_c)^2}}{2-r_b-r_c}\right)$.

   We have, the equilibrium point $E^* = \left(\frac{r_b-1}{r_b}K, \frac{1}{\alpha}\right)$ *is locally asymptotically stable. And unstable if* $0 < r_c < 1 < r_b$, $r_b + r_c < 2$ *and* $1 > q > \frac{2}{\pi}\tan^{-1}\left(\frac{\sqrt{4(r_b-1)-(r_b-r_c)^2}}{2-r_b-r_c}\right)$.

Proof. Let $f_1(x,y) = x(r_b - \frac{r_b x}{K} - \alpha y)$, $f_2(x,y) = y(r_c - r_b + \frac{r_b x}{K} - \alpha y(r_c - 1))$, and $f(x,y) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$

The linearization matrix of the system (1) is

$$J(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r_b - \frac{2r_b}{K}x - \alpha y & -\alpha x \\ r_b \frac{y}{K} & r_c - r_b + \frac{r_b}{K}x - 2\alpha y(r_c - 1) \end{pmatrix},$$

1. At the equilibrium $(0,0)$, we obtain the linearization

$$J(0,0) = \begin{pmatrix} r_b & 0 \\ 0 & r_c - r_b \end{pmatrix}.$$

The eigenvalues of this matrix are $\lambda_1 = r_b$ and $\lambda_2 = r_c - r_b$.

Using the condition of Matignon, we have

$$|arg(\lambda_1)| = 0 < q\frac{\pi}{2}, \qquad for \quad 0 < q < 1.$$

Therefore the equilibrium point $(0,0)$ is unstable.

2. At the equilibrium $(K,0)$, we obtain the linearization

$$J(K,0) = \begin{pmatrix} -r_b & -\alpha K \\ 0 & r_c \end{pmatrix}.$$

The eigenvalues of this matrix are $\lambda_1 = -r_b$ and $\lambda_2 = r_c$, we have, $|arg(\lambda_2)| = 0 < q\frac{\pi}{2}$, for $0 < q < 1$. Therefore the equilibrium point $(K,0)$ is unstable.

3. At the equilibrium $(0, \frac{r_c - r_b}{(r_c-1)\alpha})$, we obtain the linearization

$$J(0, \frac{r_c - r_b}{(r_c-1)\alpha}) = \begin{pmatrix} r_c\frac{r_b-1}{r_c-1} & 0 \\ \frac{r_b(r_b-r_c)}{K(r_c-1)} & r_b - r_c \end{pmatrix}.$$

The eigenvalues of this matrix are $\lambda_1 = r_c\frac{r_b-1}{r_c-1}$ and $\lambda_2 = r_b - r_c$.

Using the condition of Matignon, if $0 < r_b < 1 < r_c$, we have

$$|arg(\lambda_{1,2})| = \pi > q\frac{\pi}{2}, \qquad for \quad 0 < q < 1.$$

Therefore the equilibrium point $(0, \frac{r_c - r_b}{(r_c-1)\alpha})$ is locally asymptotically stable.

(a) if $0 < r_c < r_b < 1$, we have

$$|arg(\lambda_{1,2})| = 0 < q\frac{\pi}{2}, \qquad for \quad 0 < q < 1.$$

Therefore the equilibrium point $(0, \frac{r_c - r_b}{(r_c-1)\alpha})$ is unstable.

(b) if $r_c > r_b > 1$, then we have

$$|arg(\lambda_1)| = 0 < q\frac{\pi}{2}, \qquad for \quad 0 < q < 1.$$

Therefore the equilibrium point $(0, \frac{r_c - r_b}{(r_c-1)\alpha})$ is unstable.

(c) if $0 < r_c < 1 < r_b$, we have

$$|arg(\lambda_2)| = 0 < q\frac{\pi}{2}, \qquad for \quad 0 < q < 1.$$

Therefore the equilibrium point $(0, \frac{r_c - r_b}{(r_c-1)\alpha})$ is unstable.

4. At the equilibrium $(\frac{r_b-1}{r_b}K, 1/\alpha)$, we obtain the linearization

$$J(\frac{r_b-1}{r_b}K, 1/\alpha) = \begin{pmatrix} -r_b+1 & \frac{-\alpha K(r_b-1)}{r_b} \\ \frac{r_b}{\alpha K} & 1-r_c \end{pmatrix} = A.$$

The eigenvalues of $J(\frac{r_b-1}{r_b}K, 1/\alpha)$ are the roots of the following equation:

$$\lambda^2 - Tr(A)\lambda + det(A) = 0,$$

with $Tr(A) = (2 - r_b - r_c)$ and $det(A) = (r_b - 1)r_c$. We obtain $\Delta = Tr(A)^2 - 4det(A) = (r_b - r_c)^2 - 4(r_b - 1)$, and

If $r_c > r_b + 2\sqrt{r_b - 1}$ or $0 < r_c < r_b - 2\sqrt{r_b - 1}$, then we obtain $\Delta > 0$,

If $r_b - 2\sqrt{r_b - 1} < r_c < r_b + 2\sqrt{r_b - 1}$, then we obtain $\Delta < 0$,

If $r_c = r_b + 2\sqrt{r_b - 1}$ or $r_c = b - 2\sqrt{r_b - 1}$, then we obtain $\Delta = 0$

(a) If $r_b > r_c > 1$

i. If $r_c > r_b - 2\sqrt{r_b - 1}$, then $\Delta < 0$, therefore we have $\lambda_1 = \frac{Tr(A) - i\sqrt{-\Delta}}{2}$ and $\lambda_2 = \frac{Tr(A) + i\sqrt{-\Delta}}{2}$. Then $Re\lambda_{1,2} = Tr(A) < 0$, we have

$$|arg(\lambda_{1,2})| > q\frac{\pi}{2}, \qquad for \quad 0 < q < 1.$$

Therefore the equilibrium point $(\frac{r_b-1}{r_b}K, 1/\alpha)$ is locally asymptotically stable.

ii. If $0 < r_c < r_b - 2\sqrt{r_b - 1}$ then $\Delta > 0$, therefore $\lambda_{1,2} < 0$, we have

$$|arg(\lambda_{1,2})| = \pi > q\frac{\pi}{2}, \qquad for \quad 0 < q < 1.$$

Therefore the equilibrium point $(\frac{r_b-1}{r_b}K, 1/\alpha)$ is locally asymptotically stable.

iii. If $r_c = r_b - 2\sqrt{r_b - 1}$ then $\Delta = 0$, therefore we have $\lambda_{1,2} = \frac{Tr(A)}{2} < 0$. Then we have,

$$|arg(\lambda_{1,2})| = \pi > q\frac{\pi}{2}, \qquad for \quad 0 < q < 1.$$

Therefore the equilibrium point $(\frac{r_b-1}{r_b}K, 1/\alpha)$ is locally asymptotically stable.

(b) If $1 < r_b < r_c$

i. If $r_c < r_b + 2\sqrt{r_b - 1}$ then $\Delta < 0$ we have $\lambda_1 = \frac{Tr(A) - i\sqrt{-\Delta}}{2}$ and $\lambda_2 = \frac{Tr(A) + i\sqrt{-\Delta}}{2}$. Then $Re\lambda_{1,2} = Tr(A) < 0$, we have

$$|arg(\lambda_{1,2})| > q\frac{\pi}{2}, \qquad for \quad 0 < q < 1.$$

Therefore the equilibrium point $(\frac{r_b-1}{r_b}K, 1/\alpha)$ is locally asymptotically stable.

ii. If $r_c > r_b + 2\sqrt{r_b - 1}$ then $\Delta > 0$, therefore $\lambda_{1,2} < 0$, we have

$$|arg(\lambda_{1,2})| = \pi > q\frac{\pi}{2}, \qquad for \quad 0 < q < 1.$$

Therefore the equilibrium point $(\frac{r_b-1}{r_b}K, 1/\alpha)$ is locally asymptotically stable.

iii. If $r_c = r_b + 2\sqrt{r_b - 1}$ then $\Delta = 0$ we have $\lambda_{1,2} = \frac{Tr(A)}{2} < 0$. Then we have

$$|arg(\lambda_{1,2})| = \pi > q\frac{\pi}{2}, \qquad for \quad 0 < q < 1.$$

Therefore the equilibrium point $(\frac{r_b-1}{r_b}K, 1/\alpha)$ is locally asymptotically stable.

(c) If $0 < r_c < 1 < r_b$ and $r_b + r_c > 2$

i. If $r_c > r_b - 2\sqrt{r_b - 1}$ then $\Delta < 0$ we have $\lambda_1 = \frac{Tr(A) - i\sqrt{-\Delta}}{2}$ and $\lambda_2 = \frac{Tr(A) + i\sqrt{-\Delta}}{2}$. Then $Re\lambda_{1,2} = Tr(A) < 0$, we have

$$|arg(\lambda_{1,2})| > q\frac{\pi}{2}, \qquad for \quad 0 < q < 1.$$

Therefore the equilibrium point $(\frac{r_b-1}{r_b}K, 1/\alpha)$ is locally asymptotically stable.

ii. If $0 < r_c < r_b - 2\sqrt{r_b - 1}$ then $\Delta > 0$, therefore $\lambda_{1,2} < 0$, we have

$$|arg(\lambda_{1,2})| = \pi > q\frac{\pi}{2}, \qquad for \quad 0 < q < 1.$$

Therefore the equilibrium point $(\frac{r_b-1}{r_b}K, 1/\alpha)$ is locally asymptotically stable.

iii. If $r_c = r_b - 2\sqrt{r_b - 1}$ then $\Delta = 0$ we have $\lambda_{1,2} = \frac{Tr(A)}{2} < 0$ Then we have

$$|arg(\lambda_{1,2})| = \pi > q\frac{\pi}{2}, \qquad for \quad 0 < q < 1.$$

Therefore the equilibrium point $(\frac{r_b-1}{r_b}K, 1/\alpha)$ is locally asymptotically stable.

(d) If $0 < r_c < 1 < r_b$ and $r_b + r_c = 2$, the eigenvalues of this matrix are $\lambda_1 = i\sqrt{(r_b-1)r_c}, \lambda_2 = -i\sqrt{(r_b-1)r_c}$, hence

$$|arg(\lambda_1)| = \frac{\pi}{2} > q\frac{\pi}{2}, \qquad for \quad 0 < q < 1.$$

And

$$|arg(\lambda_2)| = \frac{3\pi}{2} > q\frac{\pi}{2}, \qquad for \quad 0 < q < 1.$$

Therefore the equilibrium point $(\frac{r_b-1}{r_b}K, 1/\alpha)$ is locally asymptotically stable.

(e) If $0 < r_c < 1 < r_b$ and $r_b + r_c < 2$

i. If $r_c < r_b - 2\sqrt{r_b - 1}$ then $\Delta > 0$ we have $\lambda_1 = \frac{Tr(A)-\sqrt{\Delta}}{2} > 0$ and $\lambda_2 = \frac{Tr(A)+\sqrt{\Delta}}{2} > 0$ therefore,

$$|arg(\lambda_2)| = 0 < q\frac{\pi}{2}, \qquad for \quad 0 < q < 1.$$

hence the equilibrium point $(\frac{r_b-1}{r_b}K, 1/\alpha)$ is unstable.

ii. If $r_c = r_b - 2\sqrt{r_b - 1}$ then $\Delta = 0$ we have $\lambda_{1,2} = \frac{Tr(A)}{2} > 0$, therefore,

$$|arg(\lambda_2)| = 0 < q\frac{\pi}{2}, \qquad for \quad 0 < q < 1.$$

hence the equilibrium point $(\frac{r_b-1}{r_b}K, 1/\alpha)$ is unstable.

iii. If $r_c > r_b - 2\sqrt{r_b - 1}$ then $\Delta < 0$, we have $\lambda_1 = \frac{Tr(A)-i\sqrt{-\Delta}}{2}$ and $\lambda_2 = \frac{Tr(A)+i\sqrt{-\Delta}}{2}$. Then

$$\lambda_{1.2} = \frac{(2 - r_b - r_c) \pm i\sqrt{4(r_b - 1) - (r_b - r_c)^2}}{2},$$

and we have

$$\lambda_{1.2} = \rho(cos(\theta)) \pm isin(\theta)) = \rho cos(\theta)(1 \pm itan(\theta)),$$

with $\rho = |\lambda_{1,2}| = \frac{1}{2}\sqrt{(2 - r_b - r_c)^2 + (4(r_b - 1) - (r_b - r_c)^2)}$ and $cos(\theta)) = (\frac{2-r_b-r_c}{2})/\rho$, which gives $\lambda_{1.2} = \frac{2-r_b-r_c}{2}(1+\pm itan(\theta))$, therefore,

$$\frac{2-r_b-r_c}{2}(1\pm itan(\theta)) = \frac{(2-r_b-r_c)\pm i\sqrt{4(r_b-1)-(r_b-r_c)^2}}{2},$$

wich gives $tan(\theta) = \frac{\sqrt{4(r_b-1)-(r_b-r_c)^2}}{2-r_b-r_c}$, we obtain $|arg(\lambda)| = |arg(\bar{\lambda})| = tan^{-1}(\frac{\sqrt{4(r_b-1)-(r_b-r_c)^2}}{2-r_b-r_c})$, because $2-r_b-r_c > 0$.

The equilibrium point $(\frac{r_b-1}{r_b}K, 1/\alpha)$ is locally asymptotically stable if and only if

$$tan^{-1}(\frac{\sqrt{4(r_b-1)-(r_b-r_c)^2}}{2-r_b-r_c}) > q\frac{\pi}{2},$$

The maximum value of q occurs at

$$\alpha^* = \frac{2}{\pi}tan^{-1}(\frac{\sqrt{4(r_b-1)-(r_b-r_c)^2}}{2-r_b-r_c}).$$

Then the equilibrium point $(\frac{r_b-1}{r_b}K, 1/\alpha)$ is locally asymptotically stable if $0 < q < \alpha^*$ and unstable if $\alpha^* < q < 1$.

□

5 Global Stability analysis

In this section we study the global asymptotic stability of the equilibrium points E_2 and E^*.

Theorem 5.1. *Assume that* $r_b < 1 < r_c$, *if* $\frac{k\alpha(r_c-1)}{(r_c-r_b)} < c_1 < \frac{\alpha k}{r_b}$ *then the equilibrium* $E_2 = (0, \frac{r_c-r_b}{(r_c-1)\alpha})$ *is globally asymptotically stable in* $\mathbb{R}_+^2/\mathbb{R}_+ \times \{0\}$.

Proof. We define $V_1 : \mathbb{R}_+^2/\mathbb{R}_+ \times \{0\} \to \mathbb{R}_+$ by

$$V_1(x,y) = c_1(y - y^* - y^* ln(\frac{y}{y^*})) + x.$$

This function is positive, moreover

$D^qV_1(x,y) = 0 \qquad$ if only if $\quad (x,y) = (0,y^*)$.

We calculate the derivative of order q of $V_1(x,y)$, we obtain $D^qV_1(x,y) \leq c_1(y-y^*)(r_c - r_b + \frac{r_b x}{K} - \alpha y(r_c-1) - (r_c - r_b + \frac{r_b x^*}{K} - \alpha(r_c-1)y^*)) + r_b x - \frac{r_b x^2}{K} - \alpha xy$

$\leq \frac{r_b c_1 x}{K}(y-y^*) - \alpha c_1(y-y^*)^2(r_c-1) + r_b x - \frac{r_b x^2}{K} - \alpha xy$

$\leq \frac{r_b c_1 xy}{K} - \alpha xy - \frac{r_b c_1 xy^*}{K} + r_b x$

$\leq (\frac{r_bc_1}{K} - \alpha)xy + (\frac{-r_bc_1y^*}{K} + r_b)x$

$\leq (\frac{r_bc_1}{K} - \alpha)xy + (-\frac{r_bc_1(r_c-r_b)}{K(r_c-1)\alpha} + r_b)x.$

$\leq y(c_1(r_c - r_b) + x(\frac{r_bc_1}{K} - \alpha)) + x(-\frac{r_bc_1y^*}{K} + r_b)$

if $\frac{k\alpha(r_c-1)}{(r_c-r_b)} < c_1 < \frac{\alpha k}{r_b}$ then $(0, \frac{r_c-r_b}{(r_c-1)\alpha})$ is globally asymptotically stable in $\mathbb{R}^2_+/\mathbb{R}_+ \times \{0\}$. □

Theorem 5.2. *Assume that* $1 < r_b < r_c, r_b > r_c > 1$*, the equilibrium* $E^* = (\frac{r_b-1}{r_b}K, 1/\alpha)$ *is globally asymptotically stable in* $\mathbb{R}^2_+/\{0\} \times \mathbb{R}_+ \bigcup \mathbb{R}_+ \times \{0\}$.

Proof. We define $V_1 : \mathbb{R}^2_+/\{0\} \times \mathbb{R}_+ \bigcup \mathbb{R}_+ \times \{0\} \to \mathbb{R}_+$ by

$$V_2(x, y) = c_1(y - y^* - y^*ln(\frac{y}{y^*})) + \frac{r_bc_1}{k\alpha}(x - x^* - x^*ln(\frac{x}{x^*})).$$

This function is positive, moreover

$D^qV_2(x, y) = 0$ if only if $(x, y) = (x^*, y^*)$.

We calculate the derivative of order q of $V_2(x, y)$, we obtain $D^qV_2(x, y) \leq c_1(y - y^*)(r_c - r_b + \frac{r_bx}{K} - \alpha y(r_c - 1) - (r_c - r_b + \frac{r_bx^*}{K} - \alpha y^*(r_c - 1))) + \frac{r_bc_1}{k\alpha}(x - x^*)(r_b - \frac{r_bx}{K} - \alpha y - (r_b - \frac{r_bx^*}{K} - \alpha y^*))$

$\leq \frac{r_bc_1}{K}(x - x^*)(y - y^*) - \alpha c_1(r_c - 1)(y - y^*)^2 - \frac{(r_b)^2c_1}{k^2\alpha}(x - x^*)^2 - \frac{\alpha r_bc_1}{k\alpha}(x - x^*)(y - y^*)$

$= -\alpha c_1(r_c - 1)(y - y^*)^2 - \frac{(r_b)^2c_1}{k^2\alpha}(x - x^*)^2 < 0.$

The equilibrium $E^* = (\frac{r_b-1}{r_b}K, 1/\alpha)$ is globally asymptotically stable in $\mathbb{R}^2_+/\{0\} \times \mathbb{R}_+ \bigcup \mathbb{R}_+ \times \{0\}$. □

6 Numerical simulations

In this section, we provide some numerical simulations for the nonlinear fractional system (1) in the sense of Caputo.

In Figures 1,2,3,4 and 5, we perform different simulations using the following conditions

1. If $r_b < 1 < r_c$,
2. $r_c > r_b > 1$,
3. $r_b > r_c > 1$,
4. $0 < r_c < 1 < r_b$ and $r_b + r_c > 2$,
5. $0 < r_c < 1 < r_b$ and $r_b + r_c = 2$.

To illustrate the previous results, particularly Theorem 4.1. According to the results obtained, under each condition for each equilibrium, stability is numerically ensured. In Figure 6,7,8 and 9,for the condition $0 < r_c < 1 < r_b$, $r_b + r_c < 2$ we discuss the stability of E^* as a function of the parameter q (order of the fractional derivative). It can be seen that the fractional derivative can improve the stability of our system. Figure 6 and 7 shows the stability of the interior equilibrium point E^* for $q = 0.8 < q^*$ and Figure 8 and 9 shows the instability of the interior the interior equilibrium point E^* for $q = 0.88 > q^*$.

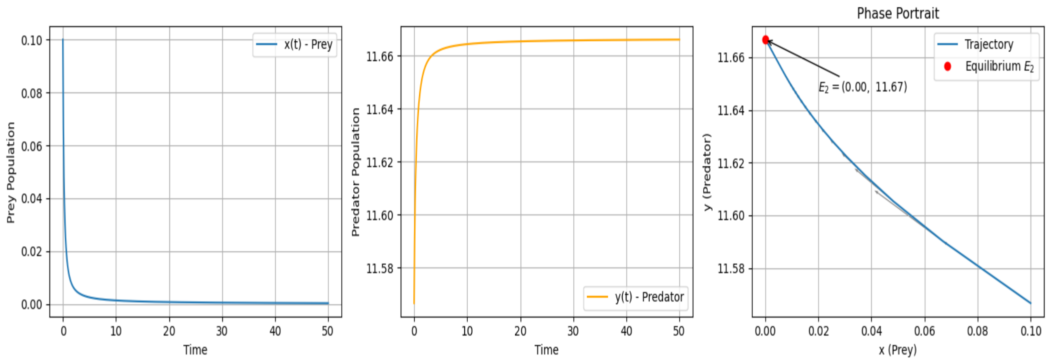

Figure 1: The figure shows the stability of the equilibrium point $E_2 = (0, \frac{r_c - r_b}{(r_c-1)\alpha})$ of the model for the initial conditions $x_0 = 0.1, y_0 = 11.57, r_b = 0.2, r_c = 1.6, K = 20, \alpha = 0.2, q = 0.8$.

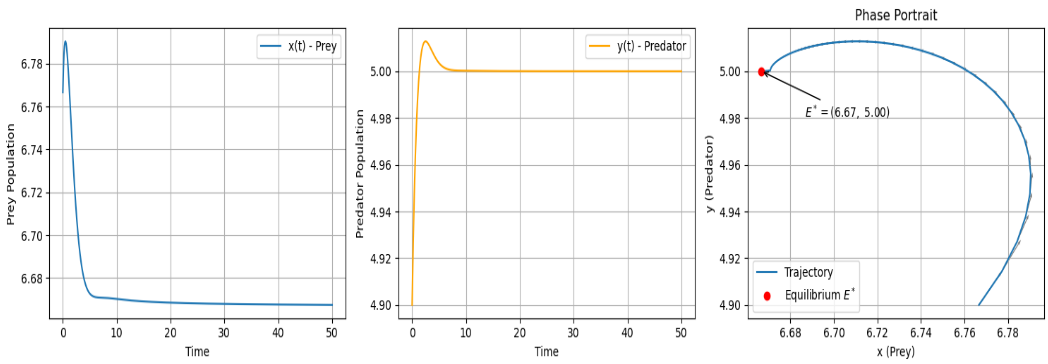

Figure 2: The figure shows the stability of the endemic equilibrium $E^* = (\frac{r_b-1}{r_b}K, 1/\alpha)$ of the model for the initial conditions initial conditions $x_0 = 6.77, y_0 = 4.9, r_b = 1.5, r_c = 1.8, K = 20, \alpha = 0.2, q = 0.9$.

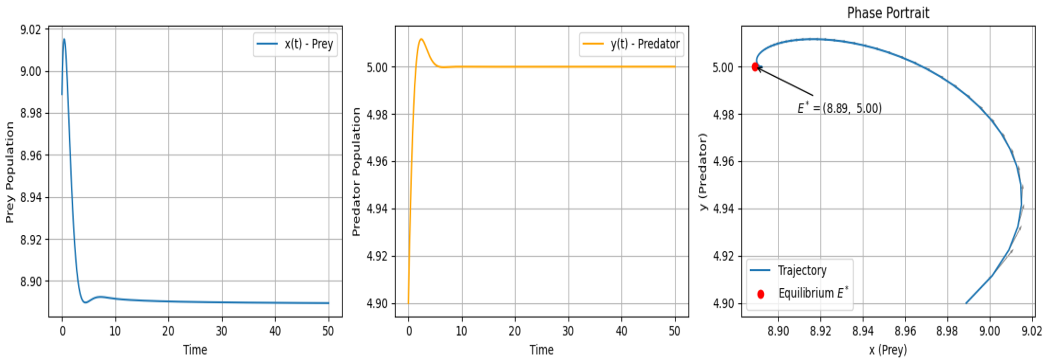

Figure 3: The figure shows the stability of the endemic equilibrium $E^* = (\frac{r_b-1}{r_b}K, 1/\alpha)$ of the model for the initial conditions $x_0 = 8.99, y_0 = 4.9, r_b = 1.8, r_c = 1.5, K = 20, \alpha = 0.2, q = 0.9$.

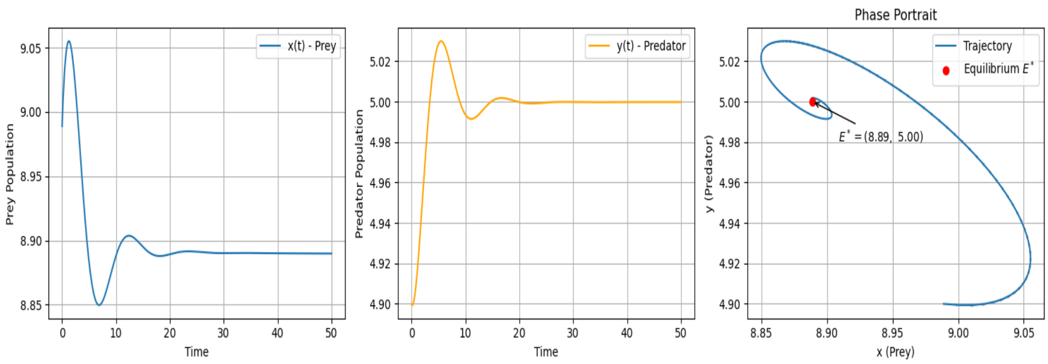

Figure 4: The figure shows the stability of the endemic equilibrium $E^* = (\frac{r_b-1}{r_b}K, 1/\alpha)$ of the model for the initial conditions $x_0 = 8.99, y_0 = 4.9, r_b = 1.8, r_c = 0.5, K = 20, \alpha = 0.2, q = 0.9$.

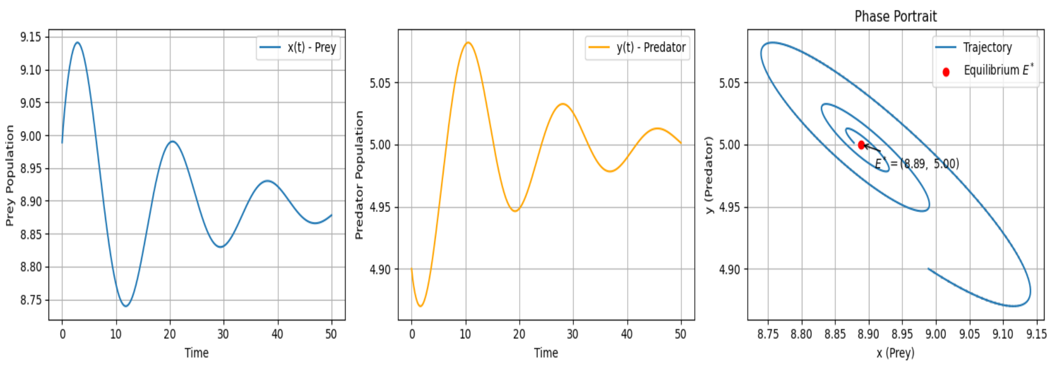

Figure 5: The figure shows the stability of the endemic equilibrium $E^* = (\frac{r_b-1}{r_b}K, 1/\alpha)$ of the model for the initial conditions $x_0 = 8.99, y_0 = 4.9, r_b = 1.8, r_c = 0.2, K = 20, \alpha = 0.2, q = 0.9$.

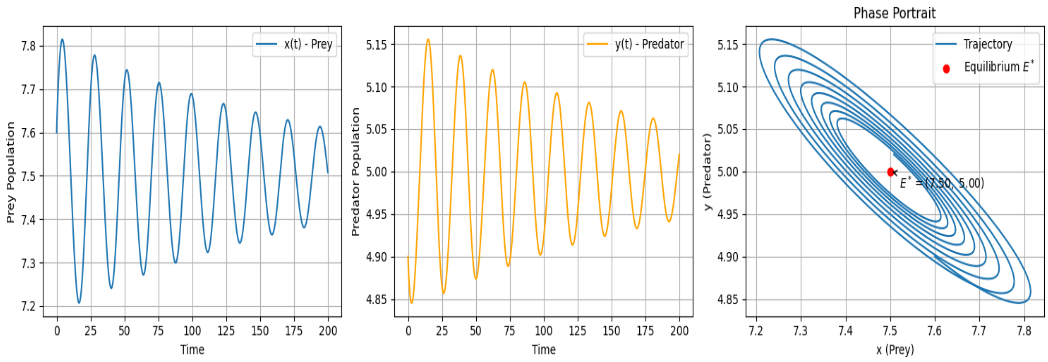

Figure 6: The figure shows the stability of the endemic equilibrium $E^* = (\frac{r_b-1}{r_b}K, 1/\alpha)$ of the model for the initial conditions $x_0 = 7.6, y_0 = 4.9, r_b = 1.6, r_c = 0.2, K = 20, \alpha = O.2, q = 0.8$.

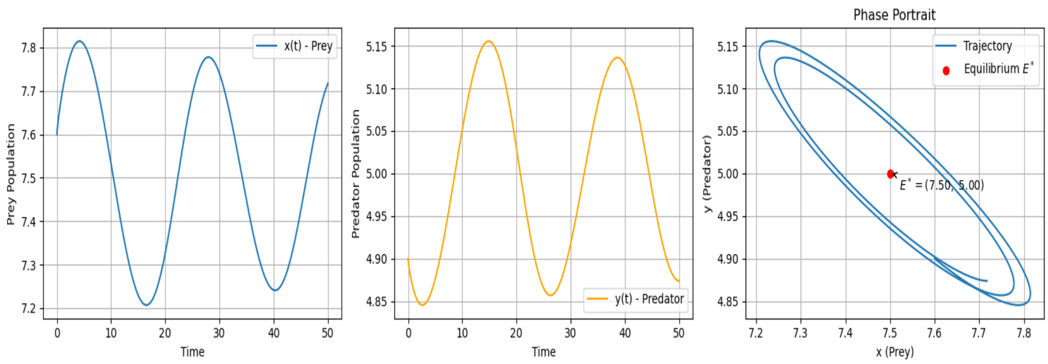

Figure 7: The figure shows the stability of the endemic equilibrium $E^* = (\frac{r_b-1}{r_b}K, 1/\alpha)$ of the model for the initial conditions $x_0 = 7.6, y_0 = 4.9, r_b = 1.6, r_c = 0.2, K = 20, \alpha = 0.2, q = 0.8$.

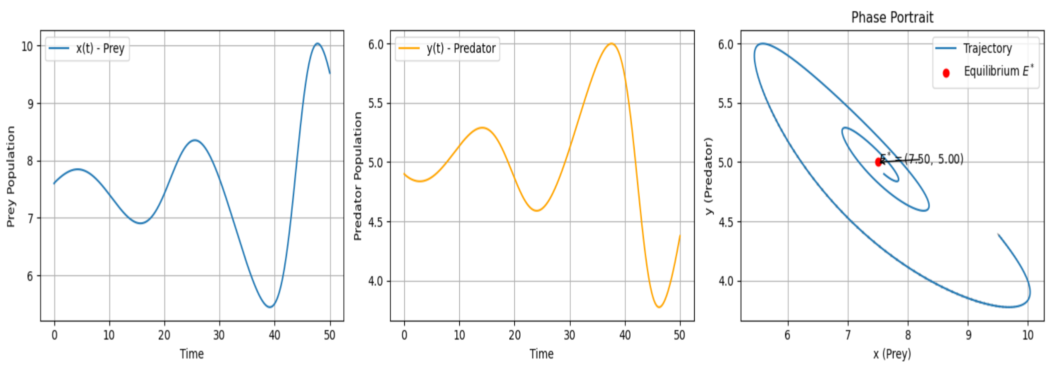

Figure 8: The figure shows the instability of the endemic equilibrium $E^* = (\frac{r_b-1}{r_b}K, 1/\alpha)$ of the model for initial conditions $x_0 = 7.6, y_0 = 4.9, r_b = 1.6, r_c = 0.2, K = 20, \alpha = 0.2, q = 0.88$.

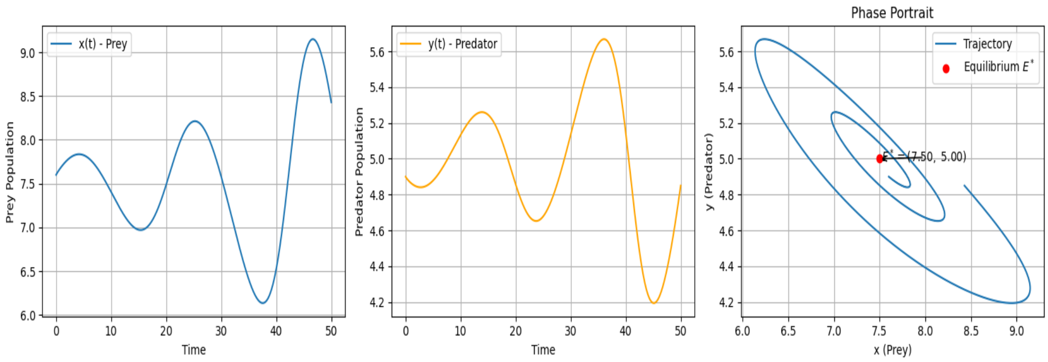

Figure 9: The figure shows the instability of the endemic equilibrium $E^* = (\frac{r_b-1}{r_b}K, 1/\alpha)$ of the model for initial conditions $x_0 = 7.6, y_0 = 4.9, r_b = 1.6, r_c = 0.2, K = 20, \alpha = 0.2, q = 0.88$.

7 Conclusion

In this work we have considered a fractional-order model of prey and predator populations, we have obtained conditions on the parameters of the model which give the local stability for each equilibrium. Furthermore, we have studied the global stability of the equilibrium points E_2 and E^* using the Lyapunov function, we have found sufficient conditions for their global stability. Finally, Numerical results are carried out to illustrate the feasibility of our main results.

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