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# A STUDY ON THREE SPECIES SYN-ECOSYSTEM CONSISTING OF HOST, COMMENSAL AND MUTUALISM

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**ABSTRACT:** The present paper deals with an investigation on three species ($S_1$, $S_2$, $S_3$) synecology consisting of host, commensal and mutualism. In this system the first species is host of the second species and the third species is a commensal of the second species. Further the first and the third species are mutuals. Here all three species are having limited resources quantized by the respective carrying capacities. The mathematical model equations constitute a set of three first order non-linear simultaneous coupled differential equations in the strengths $N_1$, $N_2$, $N_3$ of $S_1$, $S_2$, $S_3$ respectively. In all, eight equilibrium points of the model are identified. The system would be stable, if all the characteristic roots are negative, in case they are real and have negative real parts, in case they are complex. Trajectories of the perturbations over the equilibrium points are illustrated. Criteria for the global stability of normal steady state is derived by constructing suitable a Liapunov's function. Further the numerical solutions for the growth rate equations are computed using Runga Kutta fourth order method.

**Keywords:** Characteristic root, Equilibrium State, Mutualism, Liapunov's function, Unstable.

**2010 Mathematics Subject Classification:** 92D25, 92D40

## 1. Introduction

Ecology is the study of the interactions between organisms and their environment. The organisms include animals and plants, the environment includes the surroundings of animals. The study of living things (plants and animals) in connection to their environments and habits is known as ecology. This discipline of knowledge is a branch of evolutionary biology purported to explain how or to what extent the living beings are regulated in nature. Allied to

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the problem of population regulation is the problem of species distribution- prey-predator, competition and so on. The subject of ecology can be broadly sub-divided as auto-ecology (the study of single species populations) and synecology (the study of two or more communities). Synecological studies lead to the concept of the eco-system. This concept is a direct outcome of the intensive work of several life scientists/biologists and botanists of many generations. An eco-system may be considered as a unit that includes animals, plants and the physical environment in which these live. This area of knowledge seeks to explain how many different kinds of plants and animals can live together in the same place for many generations. Animals and plants share the same habitat. Sometimes they can only share for so long before some locally go extinct, but there are other circumstances when many different kinds persist in a habitat indefinitely. As such, ecology may also be referred as the study of distribution and abundance of species under habitat availing the same resources. The Ecological interactions can be broadly classified as Ammensalism, Competition, Commensalism, Neutralism, Mutualism, Predation and so on. Significant researches in the area of theoretical ecology have been discussed by Gillman [3] and by Kot [4]. Several ecologists and mathematicians contributed to the growth of this area of knowledge. Mathematical ecology can be broadly divided into two main sub-divisions, Autecology and Synecology, which are described in the treatises of Anna Sher [1], Arumugam [2] and Sharma [28].

Mathematical Modeling plays a key role in providing insight into the mutual relationships (positive, negative) between the interacting species. Several authors Ma [6], Moghadas [7], Murray [8] and Sze-Bi Hsu [30] were introduced the general concepts of Modeling in Biological Science. Srinivas [29] studied the competitive ecosystem of two species and three species with limited and unlimited resources. Later, Narayan [9] studied prey-predator ecological models with partial cover for the prey and alternate food for the predator. Further, Kumar [5] studied some mathematical models of ecological commensalism. The present author Prasad [10-27] investigated continuous and discrete models on two, three and four species syn-ecosystems.

## Notation

$N_i(t)$ : The population strength of $S_i$ at time $t$, $i = 1,2,3$

$t$ : Time instant

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$a_i$ : Natural growth rate of $S_i$, $i = 1, 2, 3$

$a_{ii}$ : Self inhibition coefficients of $S_i$, $i = 1, 2, 3$

$a_{12}$ : Interaction coefficients of $S_1$ due to $S_2$

$a_{13}$ : Interaction coefficients of $S_1$ due to $S_3$

$a_{23}$ : Interaction coefficients of $S_2$ due to $S_3$

$k_i = \frac{a_i}{a_{ii}}$ : Carrying capacity of $S_i$, $i = 1, 2, 3$

Further the variables $N_1, N_2, N_3$ are non-negative and the model parameters $a_1, a_2, a_3, a_{11}, a_{22}$, $a_{12}, a_{13}, a_{23}$ are assumed to be non-negative constants.

**Basic Equations :**

The model equations for syn ecosystem is given by the following system of first order non-linear ordinary differential equations.

$$\frac{dN_1}{dt} = a_1 N_1 - a_{11} N_1^2 + a_{13} N_1 N_3. \tag{1}$$

$$\frac{dN_2}{dt} = a_2 N_2 - a_{22} N_2^2 + a_{21} N_1 N_2 \,. \tag{2}$$

$$\frac{dN_3}{dt} = a_3 N_3 - a_{33} N_3^2 + a_{32} N_2 N_3 + a_{31} N_1 N_3. \tag{3}$$

## 2. Equilibrium States:

The system under investigation has eight equilibrium states given by

$$\frac{dN_i}{dt} = 0 \,. \text{ i= 1,2,3.} \tag{4}$$

(i)Fully washed out state.

$$E_1 : \bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = 0.$$

(ii)States in which only one of the three species is survives while the other two are not.

$$E_2 : \bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = k_3 \,.$$

$$E_3 : \bar{N}_1 = 0, \bar{N}_2 = k_2, \bar{N}_3 = 0.$$

$$E_4 : \bar{N}_1 = k_1, \bar{N}_2 = 0, \bar{N}_3 = 0.$$

(iii)States in which only two of the three species are survives while the other one is not

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$$E_5 : \bar{N}_1 = k_1 , \bar{N}_2 = k_2 + \frac{a_{21}k_1}{a_{22}} , \bar{N}_3 = 0.$$

$$E_6 : \bar{N}_1 = \frac{a_1 a_{33} + a_3 a_{13}}{a_{11}a_{33} - a_{13}a_{31}} , \bar{N}_2 = 0 , \bar{N}_3 = \frac{a_1 a_{31} + a_3 a_{11}}{a_{11}a_{33} - a_{13}a_{31}} .$$

$$E_7 : \bar{N}_1 = 0 , \bar{N}_2 = k_2 , \bar{N}_3 = k_3 + \frac{a_{32}k_2}{a_{33}} .$$

(iv)The co-existent state (or) normal steady state.

$$E_8 : \bar{N}_1 = \frac{a_{13}a_{22}a_3 + a_{13}a_{32}a_2 + a_{33}a_{22}a_1}{a_{11}a_{22}\, a_{33} - a_{13}a_{21}a_{32} - a_{31}a_{13}a_{22}}; \ \bar{N}_2 = \frac{a_{21}a_{13}a_3 + (a_{11}a_{33} - a_{13}a_{31})a_2 + a_{21}a_{33}a_1}{a_{11}a_{22}a_{33} - a_{13}a_{21}a_{32} - a_{31}a_{13}a_{22}};$$

$$\bar{N}_3 = \frac{a_{11}a_{22}a_3 + a_{11}a_{32}a_2 + (a_{21}a_{32} + a_{31}a_{22})a_1}{a_{11}a_{22}a_{33} - a_{13}a_{21}a_{32} - a_{31}a_{13}a_{22}} .$$

## 3. Stability of the Equilibrium States:

Let $N = (N_1 , N_2 , N_3) = \bar{N} + U$ .

Where $U = (u_1 , u_2 , u_3)^T$ is a small perturbation over the equilibrium state

$\bar{N} = (\bar{N}_1 , \bar{N}_2 , \bar{N}_3 )$ .

The basic equations are quasi-linearized to obtain the equations for the perturbed state as,

$$\text{i.e} \quad \frac{dU}{dt} = AU \tag{5}$$

where

$$A = \begin{bmatrix} a_1 - 2a_{11}\bar{N}_1 + a_{13}\bar{N}_3 & 0 & a_{13}\bar{N}_1 \\ a_{21}\bar{N}_2 & a_2 - 2a_{22}\bar{N}_2 + a_{21}\bar{N}_1 & 0 \\ a_{31}\bar{N}_3 & a_{32}\bar{N}_3 & a_3 - 2a_{33}\bar{N}_3 + a_{32}\bar{N}_2 + a_{31}\bar{N}_1 \end{bmatrix} \tag{6}$$

The characteristic equation for the system is $det\,[A - \lambda I] = 0.$ (7)

The equilibrium state is stable , if all the roots of the equation are negative in case they are real or have negative real parts, in case they are complex .

## 4.1. Stability of fully washed out state:

In this case, we have

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$$A = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \tag{8}$$

The characteristic equation is $(\lambda - a_1)(\lambda - a_2)(\lambda - a_3) = 0$ .

The characteristic roots of above equation are $a_1 , a_2 , a_3$ .Since all the three roots are positive . Hence, the fully washed out state is unstable and the solution of the above equation are

$$u_1 = u_{10}\, e^{a_1 t} \; ; \; u_2 = u_{20}\, e^{a_2 t} \; ; \; u_3 = u_{30}\, e^{a_3 t} . \tag{9}$$

Where $u_{10} , u_{20} , u_{30}$ are the initial values of $u_1 , u_2 , u_3$ respectively.

**Trajectories of perturbations:**

The trajectories in $u_1 - u_2$ and $u_2 - u_3$ planes are $\left[\frac{u_1}{u_{10}}\right]^{\frac{1}{a_1}} = \left[\frac{u_2}{u_{20}}\right]^{\frac{1}{a_2}} = \left[\frac{u_3}{u_{30}}\right]^{\frac{1}{a_3}}$ . (10)

## 4.2. Equilibrium state $E_2$ : $\bar{N}_1 = 0 , \bar{N}_2 = 0 , \bar{N}_3 = k_3$ .

In this case, we have

$$A = \begin{bmatrix} a_1 + a_{13}k_3 & 0 & 0 \\ 0 & a_2 & 0 \\ a_{31}k_3 & a_{32}k_3 & -a_3 \end{bmatrix} \tag{11}$$

The characteristic roots are $a_1 + a_{13}k_3$ , $a_2$ and $-a_3$ . Since one of the three roots is negative , hence the state is **unstable** and the solutions are,.

$u_1 = u_{10} e^{(a_1 + a_{13}k_3)t}$ , $u_2 = u_{20}\, e^{a_2 t}$ and

$$u_3 = (u_{30} - \phi_1 - \phi_2)e^{-a_3 t} + \phi_1 e^{a_2 t} + \phi_2 e^{(a_1 + a_{13}k_3)t}. \tag{12}$$

Where $\phi_1 = \frac{a_{32}k_3 u_{20}}{a_2 + a_3} > 0$ and $\phi_2 = \frac{a_{31}k_3 u_{10}}{a_1 + a_{13}k_3 + a_3} > 0.$

**Trajectories of perturbations:**

The trajectories in $u_1 - u_3$ and $u_2 - u_3$ planes are

$$(u_{30} - \phi_1 - \phi_2)\left[\frac{u_1}{u_{10}}\right]^{\frac{-a_3}{a_1 + a_{13}k_3}} + \phi_1\left[\frac{u_1}{u_{10}}\right]^{\frac{a_2}{a_1 + a_{13}k_3}} + \phi_2\left[\frac{u_1}{u_{10}}\right]$$

$$= (u_{30} - \phi_1 - \phi_2)\left[\frac{u_2}{u_{20}}\right]^{\frac{-a_3}{a_2}} + \phi_1\left[\frac{u_2}{u_{20}}\right] + \phi_2\left[\frac{u_2}{u_{20}}\right]^{\frac{a_1 + a_{13}k_3}{a_2}} . \tag{13}$$

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## 4.3. Equilibrium state $E_3$ : $\bar{N}_1 = 0$ , $\bar{N}_2 = k_2$ , $\bar{N}_3 = 0$.

In this case, we have

$$A = \begin{bmatrix} a_1 & 0 & 0 \\ a_{21}k_2 & -a_2 & 0 \\ 0 & 0 & a_3 + a_{32}k_2 \end{bmatrix} \tag{14}$$

The characteristic roots are $a_1$ , $-a_2$ and $a_3 + a_{32}k_2$ . Since one of the three roots is negative , hence the state is **unstable** and the solutions are,

$u_1 = u_{10}\, e^{a_1 t}$ , $u_2 = (u_{20} - \phi_3)e^{-a_2 t} + \phi_3 e^{a_1 t}$

And $u_3 = u_{30}e^{(a_3+a_{32}k_2)t}$ (15)

Where $\phi_3 = \frac{a_{21}k_2u_{10}}{a_1+a_2} > 0$

**Trajectories of perturbations:**

The trajectories in $u_1 - u_2$ and $u_2 - u_3$ planes are

$$(u_{20} - \phi_3)\left[\frac{u_1}{u_{10}}\right]^{\frac{-a_2}{a_1}} + \phi_3\left[\frac{u_1}{u_{10}}\right] = (u_{20} - \phi_3)\left[\frac{u_3}{u_{30}}\right]^{\frac{-a_2}{a_3+a_{32}k_2}} + \phi_3\left[\frac{u_3}{u_{30}}\right]^{\frac{a_1}{a_3+a_{32}k_2}}. \tag{16}$$

## 4.4. Equilibrium state $E_4$ : $\bar{N}_1 = k_1$ , $\bar{N}_2 = 0$ , $\bar{N}_3 = 0$.

In this case, we have

$$A = \begin{bmatrix} -a_1 & 0 & a_{13}k_1 \\ 0 & a_2 + a_{21}k_1 & 0 \\ 0 & 0 & a_3 + a_{31}k_1 \end{bmatrix} \tag{17}$$

The characteristic roots are $-a_1$ , $a_2 + a_{21}k_1$ and $a_3 + a_{31}k_1$ . Since one of the three roots is negative, hence the state is **unstable** and the solutions are,

$u_1 = (u_{10} - \phi_4)e^{-a_1 t} + \phi_4 e^{(a_3+a_{31}k_1)t}$ ,

$u_2 = u_{20}e^{(a_2+a_{21}k_1)t}$ and $u_3 = u_{30}e^{(a_3+a_{31}k_1)t}$ . (18)

where $\phi_4 = \frac{a_{13}k_1u_{30}}{a_1+a_3+a_{31}k_1} > 0$

**Trajectories of perturbations:**

The trajectories in $u_1 - u_2$ and $u_2 - u_3$ planes are

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$$u_1 = (u_{10} - \phi_4)\left[\frac{u_2}{u_{20}}\right]^{\frac{-a_1}{a_2+a_{21}k_1}} + \phi_4\left[\frac{u_2}{u_{20}}\right]^{\frac{a_3+a_{31}k_1}{a_2+a_{21}k_1}} \text{ and}$$

$$\left[\frac{u_2}{u_{20}}\right]^{\frac{1}{a_2+a_{21}k_1}} = \left[\frac{u_3}{u_{30}}\right]^{\frac{1}{a_3+a_{31}k_1}}. \quad (19)$$

## 4.5. Equilibrium state $E_5$ : $\bar{N}_1 = k_1, \bar{N}_2 = k_2 + \frac{a_{21}k_1}{a_{22}}, \bar{N}_3 = 0$.

In this case, we have

$$A = \begin{bmatrix} -a_1 & 0 & a_{13}k_1 \\ a_{21}\left(k_2 + \frac{a_{21}k_1}{a_{22}}\right) & -(a_2 + a_{21}k_1) & 0 \\ 0 & 0 & \phi_5 \end{bmatrix} \quad (20)$$

The characteristic roots are $-a_1$ , $-(a_2 + a_{21}k_1)$ and $\phi_5$ . Since one of the three roots is positive , hence the state is **unstable** and the solution are,

$$u_1 = (u_{10} - \phi_6)e^{-a_1 t} + \phi_6 e^{\phi_5 t}$$

$$u_2 = (u_{20} - \phi_8 - \phi_9)e^{-(a_2+a_{21}k_1)t} + \phi_8 e^{-a_1 t} + \phi_9 e^{\phi_5 t}$$

$$\text{and } u_3 = u_{30}e^{\phi_5 t}. \quad (21)$$

Where $\phi_5 = \frac{a_{11}a_{22}a_3 + a_{11}a_{32}a_2 + a_{21}a_{32}a_1 + a_{31}a_{22}a_1}{a_{11}a_{22}} > 0$ , $\phi_6 = \frac{a_{13}k_1u_{30}}{\phi_5 + a_1} > 0$.

$\phi_7 = a_{21}\left(k_2 + \frac{a_{21}k_1}{a_{22}}\right) > 0$, $\phi_8 = \frac{\phi_7(u_{10} - \phi_6)}{a_2 + a_{21}k_1 - a_1} > 0$ and $\phi_9 = \frac{\phi_7\phi_6}{a_2 + a_{21}k_1 + \phi_5} > 0$

**Trajectories of perturbations:**

The trajectories in $u_1 - u_3$ and $u_2 - u_3$ planes are

$$u_1 = (u_{10} - \phi_6)\left[\frac{u_3}{u_{30}}\right]^{\frac{-a_1}{\phi_5}} + \phi_6\left[\frac{u_3}{u_{30}}\right] \text{ and}$$

$$u_2 = (u_{20} - \phi_8 - \phi_9)\left[\frac{u_3}{u_{30}}\right]^{\frac{-(a_2+a_{21}k_1)}{\phi_5}} + \phi_8\left[\frac{u_3}{u_{30}}\right]^{\frac{-a_1}{\phi_5}} + \phi_9\left[\frac{u_3}{u_{30}}\right] \quad (22)$$

## 4.6. Equilibrium state $E_6$ : $\bar{N}_1 = \frac{a_1a_{33} + a_3a_{13}}{a_{11}a_{33} - a_{13}a_{31}}, \bar{N}_2 = 0, \bar{N}_3 = \frac{a_1a_{31} + a_3a_{11}}{a_{11}a_{33} - a_{13}a_{31}}$

In this case, we have

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$$A = \begin{bmatrix} -a_{11}\overline{N}_1 & 0 & a_{13}\overline{N}_1 \\ 0 & a_2 + a_{21}\overline{N}_1 & 0 \\ a_{31}\overline{N}_3 & a_{32}\overline{N}_3 & -a_{33}\overline{N}_3 \end{bmatrix} \tag{23}$$

The characteristic roots are $-(a_2 + a_{21}\overline{N}_1)$

and $\frac{-(a_{11}\overline{N}_1+a_{33}\overline{N}_3)\pm\sqrt{(a_{11}\overline{N}_1-a_{33}\overline{N}_3)^2+4a_{13}a_{31}\overline{N}_1\overline{N}_3}}{2}$

If $a_{11}\overline{N}_1 + a_{33}\overline{N}_3 > 0$ then the state is **stable**, Otherwise **unstable**, and the solution curves are given by

$$u_1 = e^{-\alpha t}(c_1\cosh t\beta + c_2\sinh t\beta) + R_1 e^{\phi_{10}t}$$

$$u_2 = u_{20}e^{\phi_{10}t} \text{ and } u_3 = e^{-\alpha t}(c_1\cosh t\beta + c_2\sinh t\beta) + R_2 e^{\phi_{10}t} \tag{24}$$

Where $\alpha = \frac{a_{11}\overline{N}_1+a_{33}\overline{N}_3}{2}$ , $\beta = \frac{\sqrt{(a_{11}\overline{N}_1-a_{33}\overline{N}_3)^2+4a_{13}a_{31}\overline{N}_1\overline{N}_3}}{2}$

$$R_1 = \frac{a_{32}a_{13}\overline{N}_1\overline{N}_3u_{20}}{(\phi_{10})^2+(a_{11}\overline{N}_1+a_{33}\overline{N}_3)\phi_{10}+(a_{11}a_{33}-a_{13}a_{31})\overline{N}_1\overline{N}_3}$$

$$\phi_{10} = \frac{a_{21}a_{33}a_1+(a_{11}a_{33}-a_{13}a_{31})a_2+a_{21}a_{13}a_3}{a_{11}a_{33}-a_{13}a_{31}} \; ; a_{13}a_{31} \neq a_{11}a_{33}$$

$$R_2 = \frac{a_{32}u_{20}\phi_{10}\overline{N}_3+a_{11}\overline{N}_1}{(\phi_{10})^2+(a_{11}\overline{N}_1+a_{33}\overline{N}_3)\phi_{10}+(a_{11}a_{33}-a_{13}a_{31})\overline{N}_1\overline{N}_3}$$

**Trajectories of perturbations:**

The trajectories in $u_1 - u_2$ and $u_2 - u_3$ planes are

$$u_1 = \left[\frac{u_2}{u_{20}}\right]^{\frac{-\alpha}{\phi_{10}}}(c_1\cosh t\beta + c_2\sinh t\beta) + R_1\left[\frac{u_2}{u_{20}}\right] \text{ and}$$

$$u_3 = \left[\frac{u_2}{u_{20}}\right]^{\frac{-\alpha}{\phi_{10}}}(c_1\cosh t\beta + c_2\sinh t\beta) + R_2\left[\frac{u_2}{u_{20}}\right]. \tag{25}$$

## 4.7. Equilibrium state $E_7$ : $\overline{N}_1 = 0, \overline{N}_2 = k_2, \overline{N}_3 = k_3 + \frac{a_{32}k_2}{a_{33}}$ .

In this case, we have

$$A = \begin{bmatrix} \left(a_1 + \frac{a_{13}\tau}{a_{33}}\right) & 0 & 0 \\ a_{21}k_2 & -a_2 & 0 \\ a_{31}\left(k_3 + \frac{a_{32}k_2}{a_{33}}\right) & a_{32}\left(k_3 + \frac{a_{32}k_2}{a_{33}}\right) & -\tau \end{bmatrix} \tag{26}$$

Where $\tau = a_3 + a_{32}k_2 > 0$ .

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The characteristic roots are $a_1 + \frac{a_{13}\tau}{a_{33}}$ , $-a_2$ and $-\tau$ . Since one of the three roots is positive , hence the state is **unstable** and the solution curves.

$$u_1 = u_{10}e^{\phi_{11}t} \ , \ u_2 = (u_{20} - \phi_{12})e^{-a_2t} + \phi_{12}e^{\phi_{11}t}$$

And $u_3 = (u_{30} - R_3 - R_4)e^{-\tau t} + R_3e^{-a_2t} + R_4e^{\phi_{11}t}$ (27)

Where $\phi_{11} = \frac{a_{22}a_{33}a_1 + a_{13}a_{32}a_2 + a_{13}a_{22}a_3}{a_{22}a_{33}} > 0$; $\phi_{12} = \frac{a_{21}k_2u_{10}}{\phi_{11}+a_2} > 0$,$R_3 = \frac{a_{32}(u_{20}-\phi_{12})\tau}{a_{33}(\tau-a_2)} > 0$ ; $\tau \neq a_2$ . And $R_4 = \frac{a_{32}\phi_{12}\tau + a_{31}u_{10}\tau}{a_{33}(\tau+\phi_{11})} > 0$ .

**Trajectories of perturbations:**

The trajectories in $u_1 - u_2$ and $u_1 - u_3$ planes are

$$u_2 = (u_{20} - \phi_{12})\left[\frac{u_1}{u_{10}}\right]^{\frac{-a_2}{\phi_{11}}} + \phi_{12}\left[\frac{u_1}{u_{10}}\right] \quad \text{and}$$

$$u_3 = (u_{30} - R_3 - R_4)\left[\frac{u_1}{u_{10}}\right]^{\frac{-\tau}{\phi_{11}}} + R_3\left[\frac{u_1}{u_{10}}\right]^{\frac{-a_2}{\phi_{11}}} + R_4\left[\frac{u_1}{u_{10}}\right] . \qquad (28)$$

## 4.8. Equilibrium state $E_8$ :

In this case, we have

$$A = \begin{bmatrix} -a_{11}\overline{N}_1 & 0 & a_{13}\overline{N}_1 \\ a_{21}\overline{N}_2 & -a_{22}\overline{N}_2 & 0 \\ a_{31}\overline{N}_3 & a_{32}\overline{N}_3 & -a_{33}\overline{N}_3 \end{bmatrix} \qquad (29)$$

The characteristic equation of above matrix is $\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0$ . (30)

Where $b_1 = a_{11}\overline{N}_1 + a_{22}\overline{N}_2 + a_{33}\overline{N}_3$

$$b_2 = a_{11}a_{22}\overline{N}_1\overline{N}_2 + a_{22}a_{33}\overline{N}_2\overline{N}_3 + (a_{11}a_{33} - a_{13}a_{31})\overline{N}_1\overline{N}_3$$

$$b_3 = (a_{11}a_{22}a_{33} - a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22})\overline{N}_1\overline{N}_2\overline{N}_3. \qquad (31)$$

According to Routh-Hurwitz's criteria, the necessary and sufficient conditions for local stability of co-existent points are $b_1 > 0$ , $b_3 > 0$ and $b_3(b_1b_2 - b_3) > 0$ . (32)

It is evident that $b_1 > 0$ and $a_{11}a_{22}a_{33}\overline{N}_1\overline{N}_2\overline{N}_3 > (a_{13}a_{21}a_{32} + a_{13}a_{31}a_{22})\overline{N}_1\overline{N}_2\overline{N}_3$. Thus the stability of co-existent state is determined by the sign of $b_1b_2 - b_3$ . By direct calculations we obtain

$$b_1b_2 - b_3 = a_{11}^2\overline{N}_1^2\omega_1 + a_{22}^2\overline{N}_2^2\omega_2 + a_{33}^2\overline{N}_3^2\omega_3$$

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$$+(2a_{11}a_{22}a_{33} + a_{13}a_{21}a_{32})\bar{N}_1\bar{N}_2\bar{N}_3 - a_{13}a_{31}\bar{N}_1\bar{N}_3\omega_2.$$

Here $\omega_1 = a_{22}\bar{N}_2 + a_{33}\bar{N}_3$ , $\omega_2 = a_{11}\bar{N}_1 + a_{33}\bar{N}_3$ and $\omega_3 = a_{22}\bar{N}_2 + a_{11}\bar{N}_1$.

Hence the co-existent state is locally asymptotically stable.

When

$$[a_{11}^2\bar{N}_1^2\omega_1 + a_{22}^2\bar{N}_2^2\omega_2 + a_{33}^2\bar{N}_3^2\omega_3 + (2a_{11}a_{22}a_{33} + a_{13}a_{21}a_{32})\bar{N}_1\bar{N}_2\bar{N}_3] > a_{13}a_{31}\bar{N}_1\bar{N}_3\omega_2. \quad (32)$$

The solution of the perturbation equations is:

$$u_1 = A_1e^{s_1t} + B_1e^{s_2t} + C_1e^{s_3t};\ u_2 = A_2e^{s_1t} + B_2e^{s_2t} + C_2e^{s_3t};$$

$$u_3 = A_3e^{s_1t} + B_3e^{s_2t} + C_3e^{s_3t}. \quad (33)$$

Here $A_1 = \frac{u_{10}(s_1^2+s_1T_1+U_1)+u_{20}V_1+u_{30}(a_{13}a_{22}\bar{N}_1\bar{N}_2+s_1a_{13}\bar{N}_1)}{(s_1-s_2)(s_1-s_3)}$

$$B_1 = \frac{u_{10}(s_2^2+s_2T_1+U_1)+u_{20}V_1+u_{30}(a_{13}a_{22}\bar{N}_1\bar{N}_2+s_2a_{13}\bar{N}_1)}{(s_2-s_1)(s_2-s_3)}$$

$$C_1 = \frac{u_{10}(s_3^2+s_3T_1+U_1)+u_{20}V_1+u_{30}(a_{13}a_{22}\bar{N}_1\bar{N}_2+s_3a_{13}\bar{N}_1)}{(s_3-s_1)(s_3-s_2)}.$$

$$A_2 = \frac{u_{10}(a_{21}\bar{N}_2s_1+a_{21}a_{33}\bar{N}_1\bar{N}_3)+u_{20}(s_1^2+s_1T_2+U_2)+u_{30}V_2}{(s_1-s_2)(s_1-s_3)}$$

$$B_2 = \frac{u_{10}(a_{21}\bar{N}_2s_2+a_{21}a_{33}\bar{N}_1\bar{N}_3)+u_{20}(s_2^2+s_2T_2+U_2)+u_{30}V_2}{(s_2-s_1)(s_2-s_3)}$$

$$C_2 = \frac{u_{10}(a_{21}\bar{N}_2s_3+a_{21}a_{33}\bar{N}_1\bar{N}_3)+u_{20}(s_3^2+s_3T_2+U_2)+u_{30}V_2}{(s_3-s_1)(s_3-s_2)}.$$

$$A_3 = \frac{u_{10}(a_{31}\bar{N}_3s_1+W_3)+u_{20}(a_{32}\bar{N}_3s_1+V_3)+u_{30}(s_1^2+s_1T_3+U_3)}{(s_1-s_2)(s_1-s_3)}$$

$$B_3 = \frac{u_{10}(a_{31}\bar{N}_3s_2+W_3)+u_{20}(a_{32}\bar{N}_3s_2+V_3)+u_{30}(s_2^2+s_2T_3+U_3)}{(s_2-s_1)(s_2-s_3)}$$

$$C_3 = \frac{u_{10}(a_{31}\bar{N}_3s_3+W_3)+u_{20}(a_{32}\bar{N}_3s_3+V_3)+u_{30}(s_3^2+s_3T_3+U_3)}{(s_3-s_1)(s_3-s_2)}.$$

Where $T_1 = a_{22}\bar{N}_2 + a_{33}\bar{N}_3$ , $U_1 = a_{22}a_{33}\bar{N}_2\bar{N}_3$ , $V_1 = a_{13}a_{32}\bar{N}_1\bar{N}_3$.

$T_2 = a_{11}\bar{N}_1 + a_{33}\bar{N}_3$ , $U_2 = (a_{11}a_{33} - a_{13}a_{31})\bar{N}_1\bar{N}_3$ , $V_2 = a_{13}a_{31}\bar{N}_1\bar{N}_2$ .

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$W_3 = a_{21}a_{32}\overline{N}_1\overline{N}_3 + a_{22}a_{31}\overline{N}_2\overline{N}_3$ , $T_3 = a_{11}\overline{N}_1 + a_{22}\overline{N}_2$ , $U_3 = a_{11}a_{22}\overline{N}_1\overline{N}_2$ and $V_3 = a_{32}a_{11}\overline{N}_1\overline{N}_3$.

## 5. Liapunov's function for global stability :

In section 4, we discussed the local stability of all eight equilibrium states. From which only two states $E_6 = (\overline{N}_1, \overline{N}_2, \overline{N}_3)$ and $E_8 = (\overline{N}_1, \overline{N}_2, \overline{N}_3)$ are stable and rest of them are unstable. We now examine the global stability of dynamical system (1), (2) and (3) at these two states by suitable Liapunov's function.

**Theorem 1 :**The equilibrium state is $E_6$ : $(\overline{N}_1, \overline{N}_2, \overline{N}_3)$ globally asymptotically stable

**Proof** : Let us consider the following Liapunovs function .

$$L(N_1, N_2, N_3) = N_1 - \overline{N}_1 - \overline{N}_1 \ln\left(\frac{N_1}{\overline{N}_1}\right) + l_2\left[N_3 - \overline{N}_3 - \overline{N}_3 ln\left(\frac{N_3}{\overline{N}_3}\right)\right]. \qquad (34)$$

Where $l_2$ is a suitable constant to be determined as in the subsequent steps. Now, the time derivative of L , along with solutions of (1) and (3) can be written as

$$\frac{dL}{dt} = \left(\frac{N_1 - \overline{N}_1}{N_1}\right)\frac{dN_1}{dt} + l_2\left(\frac{N_3 - \overline{N}_3}{N_3}\right)\frac{dN_3}{dt} \;. \qquad (35)$$

$$\frac{dL}{dt} = -\left[\sqrt{a_{11}}(N_1 - \overline{N}_1) + \sqrt{l_2 a_{33}}(N_3 - \overline{N}_3)\right]^2 + \left[2\sqrt{a_{11} l_2 a_{33}} + (a_{13} + l_2 a_{31})\right]$$

$$(N_1 - \overline{N}_1)(N_3 - \overline{N}_3). \qquad (36)$$

The positive constant $l_2$ as so chosen that , the coefficient of $(N_1 - \overline{N}_1)(N_3 - \overline{N}_3)$ in (36) vanish.

$$\frac{dL}{dt} < 0, \text{ when } 2\sqrt{a_{11} l_2 a_{33}} = -(a_{13} + l_2 a_{31}) \qquad (37)$$

Hence $E_6$ is **globally asymptotically stable.**

**Theorem 2 :** The equilibrium state is $E_8$ : $(\overline{N}_1, \overline{N}_2, \overline{N}_3)$ globally asymptotically stable

**Proof** : Let us consider the following Liapunovs function .

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$$L(N_1, N_2, N_3) = N_1 - \bar{N}_1 - \bar{N}_1 \ln\left(\frac{N_1}{\bar{N}_1}\right) + l_1\left[N_2 - \bar{N}_2 - \bar{N}_2 ln\left(\frac{N_2}{\bar{N}_2}\right)\right]$$

$$+l_2\left[N_3 - \bar{N}_3 - \bar{N}_3\, ln\left(\frac{N_3}{\bar{N}_3}\right)\right]. \quad (38)$$

where $l_1$ and $l_2$ are suitable constants to be determined as in the subsequent steps. Now the time derivative of L , along with solutions of (1),(2) and (3) can be written as ,

$$\frac{dL}{dt} = \left(\frac{N_1-\bar{N}_1}{N_1}\right)\frac{dN_1}{dt} + l_1\left(\frac{N_2-\bar{N}_2}{N_2}\right)\frac{dN_2}{dt} + l_2\left(\frac{N_3-\bar{N}_3}{N_3}\right)\frac{dN_3}{dt} . \quad (39)$$

$$= (N_1 - \bar{N}_1)(a_1 - a_{11}N_1 + a_{13}N_3) + l_1(N_2 - \bar{N}_2)(a_2 - a_{22}N_2 + a_{21}N_1)$$

$$+ l_2(N_3 - \bar{N}_3)(a_3 - a_{33}N_3 + a_{32}N_2 + a_{31}N_1).$$

$$= -a_{11}(N_1 - \bar{N}_1)^2 + a_{13}(N_1 - \bar{N}_1)(N_3 - \bar{N}_3)$$

$$+ l_1[-a_{22}(N_2 - \bar{N}_2)^2 + a_{21}(N_2 - \bar{N}_2)(N_1 - \bar{N}_1)]$$

$$+l_2[-a_{33}(N_3 - \bar{N}_3)^2 + a_{32}(N_3 - \bar{N}_3)(N_2 - \bar{N}_2) + a_{31}(N_3 - \bar{N}_3)(N_1 - \bar{N}_1)] .$$

Hence

$$\frac{dL}{dt} = -\left[\sqrt{a_{11}}(N_1 - \bar{N}_1) + \sqrt{l_1 a_{22}}(N_2 - \bar{N}_2) + \sqrt{l_2 a_{33}}(N_3 - \bar{N}_3)\right]^2$$

$$+\left(2\sqrt{a_{11} l_1 a_{22}} + l_1 a_{21}\right)(N_1 - \bar{N}_1)(N_2 - \bar{N}_2)$$

$$+\left(2\sqrt{l_1 l_2 a_{22} a_{33}} + l_2 a_{32}\right)(N_2 - \bar{N}_2)(N_3 - \bar{N}_3)$$

$$+\left(2\sqrt{l_2 a_{33} a_{11}} + (a_{13} + l_2 a_{31})\right)(N_3 - \bar{N}_3)(N_1 - \bar{N}_1). \quad (40)$$

The positive constants $l_1$ and $l_2$ as so chosen that , the coefficient of

$(N_1 - \bar{N}_1)(N_2 - \bar{N}_2)$ , $(N_2 - \bar{N}_2)(N_3 - \bar{N}_3)$ and $(N_3 - \bar{N}_3)(N_1 - \bar{N}_1)$ in (40) vanish.

Then we have $l_1 = \frac{4a_{11}a_{22}}{a_{21}^2} > 0$ and $l_2 = \frac{16a_{11}a_{22}^2 a_{33}}{a_{21}^2 a_{32}^2} > 0$ . with this choice of the constants $l_1$ and $l_2$.

$$\frac{dL}{dt} = -\sqrt{a_{11}}\left[(N_1 - \bar{N}_1) + \frac{2a_{22}}{a_{21}}(N_2 - \bar{N}_2) + \frac{4a_{22}a_{33}}{a_{21}a_{32}}(N_3 - \bar{N}_3)\right]^2. \quad (41)$$

Which is negative definite. Hence, the $E_8$ state is **globally asymptotically stable.**

## 6. Numerical Approach

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The numerical solutions to growth rate equations (1), (2), and (3) were derived using the Runge-Kutta 4th order technique for specified values of the model's various parameters and initial conditions. Figures 1 through 6 present the results.

**Table**

| Fig. No. | $a_1$ | $a_2$ | $a_3$ | $a_{11}$ | $a_{22}$ | $a_{33}$ | $a_{13}$ | $a_{21}$ | $a_{31}$ | $a_{32}$ | $N_1$ | $N_2$ | $N_3$ | $t^*$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.414 | 1.43 | 0.89 | 1.8 | 2.2 | 1.07 | 0.6 | 1.33 | 0.13 | 0.43 | 3.924 | 0.223 | 5.616 | 0.72 |
| 2 | 0.3 | 2.0 | 2.13 | 1.83 | 2.8 | 2.63 | 1.97 | 1.4 | 1.6 | 1.6 | 2.64 | 1.232 | 2.64 | -- |
| 3 | 0.43 | 0.13 | 1.0 | 5.67 | 2.13 | 1.63 | 0.2 | 2.033 | 4.0 | 1.6 | 1.89 | 2.64 | 8.028 | -- |
| 4 | 0.73 | 0.53 | 2.37 | 3.53 | 1.33 | 8.23 | 2.33 | 0.27 | 0.89 | 7.73 | 5.0 | 5.0 | 5.0 | -- |
| 5 | 1.23 | 1.3 | 0.9 | 3.37 | 1.27 | 1.7 | 2.37 | 1.4 | 0.57 | 1.33 | 1.856 | 2.24 | 2.528 | -- |
| 6 | 1.0 | 0.3 | 0.27 | 0.57 | 1.0 | 1.93 | 0.01 | 1.07 | 2.2 | 0.27 | 2.8 | 1.632 | 3.824 | 0.48 |

By taking above values we have drawn some figures.

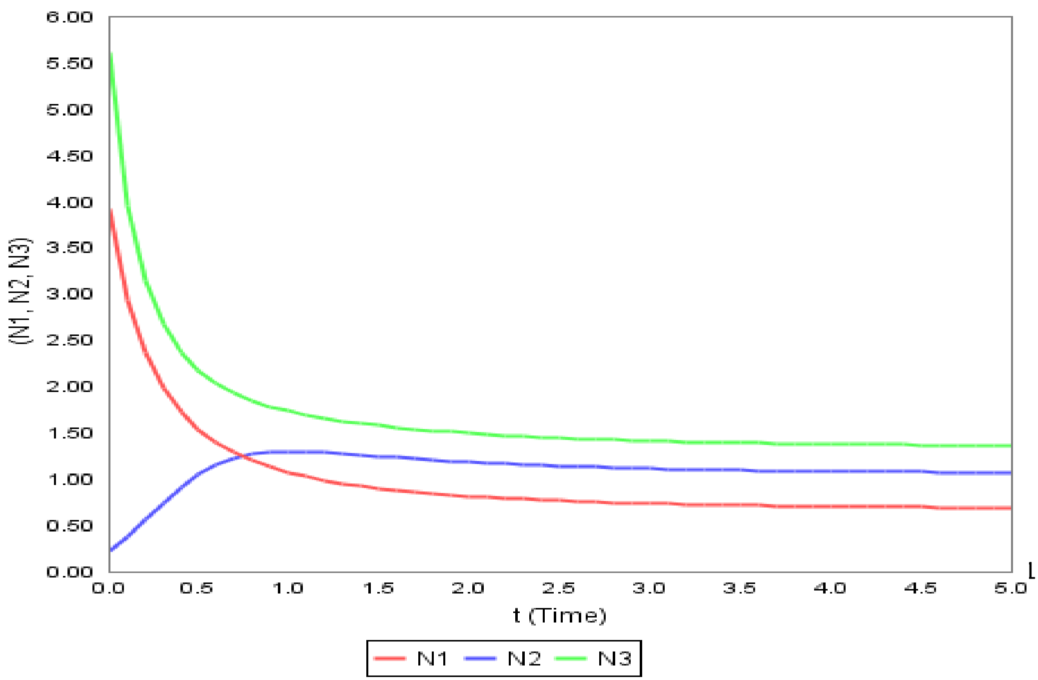

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**Figure 1**

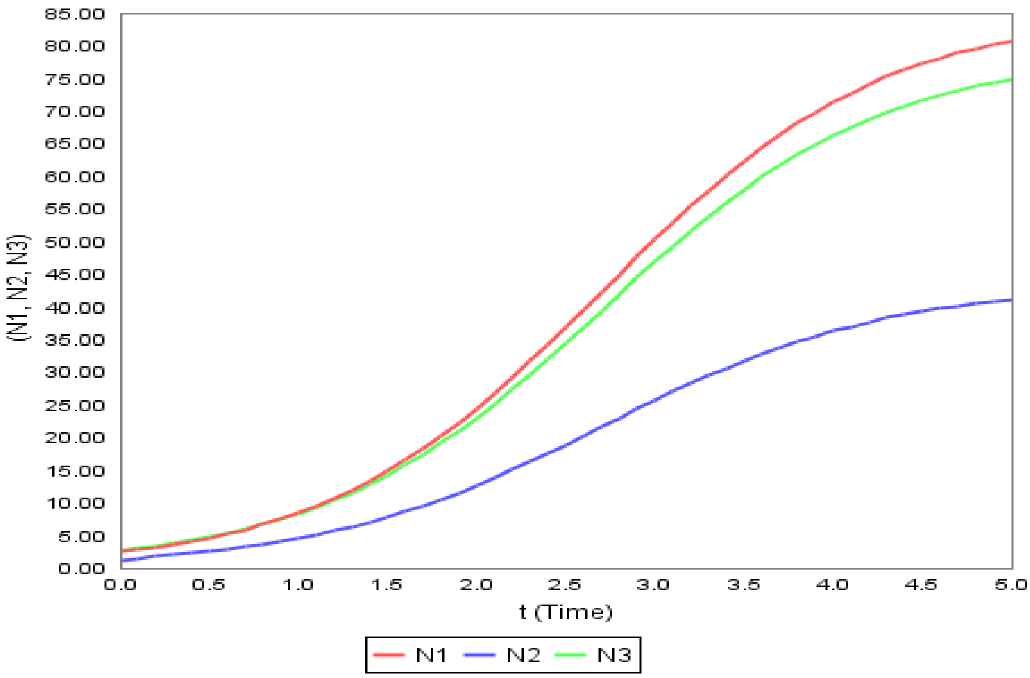


**Figure 2**

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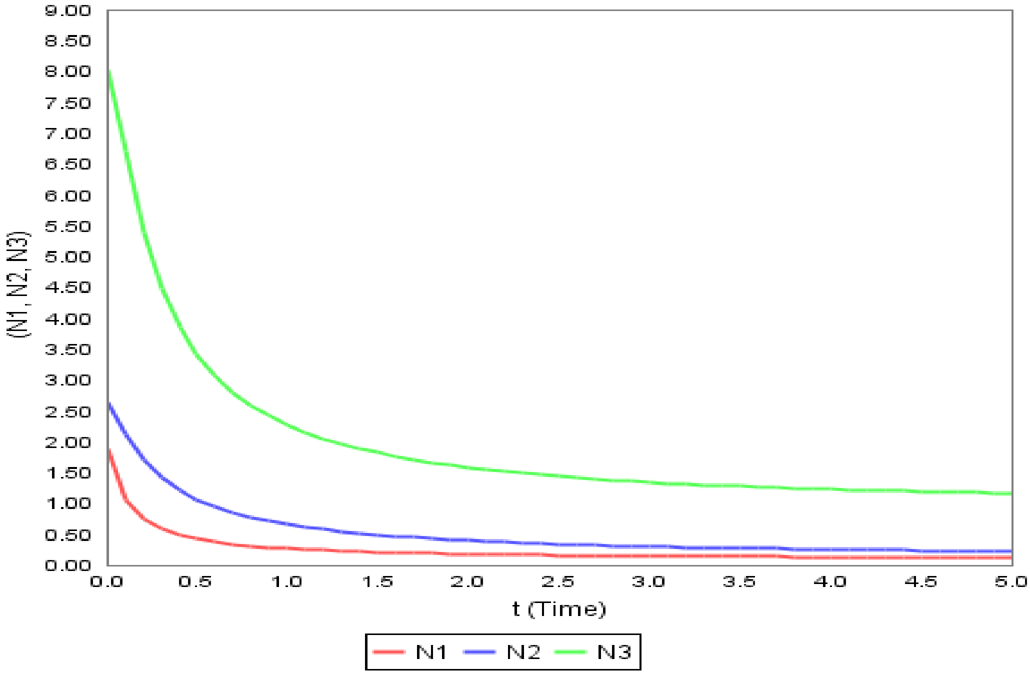


**Figure 3**

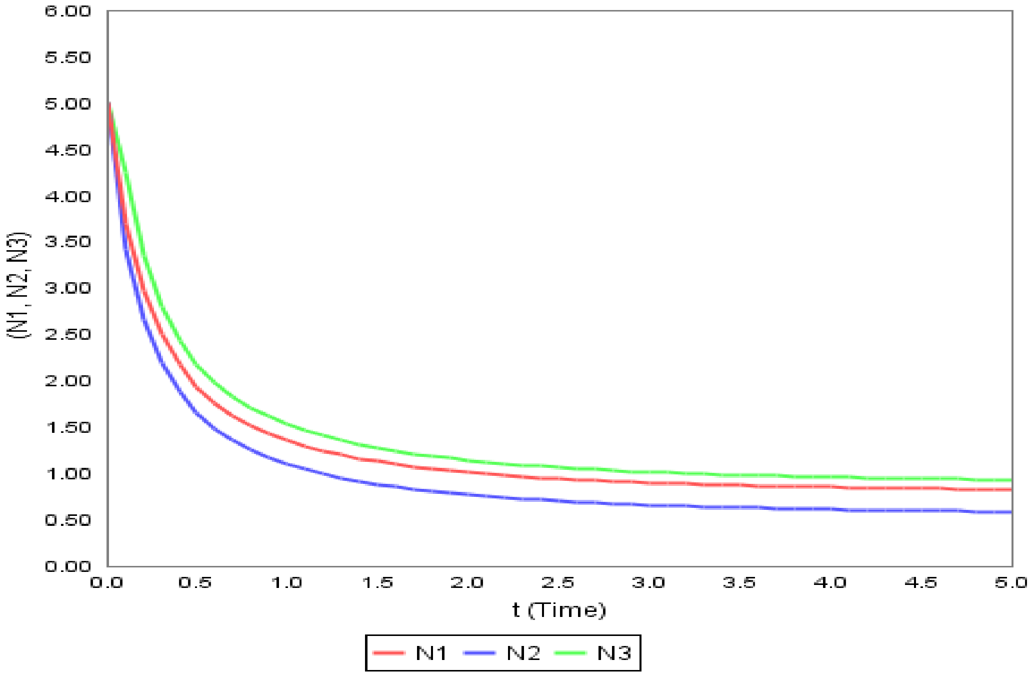


**Figure 4**

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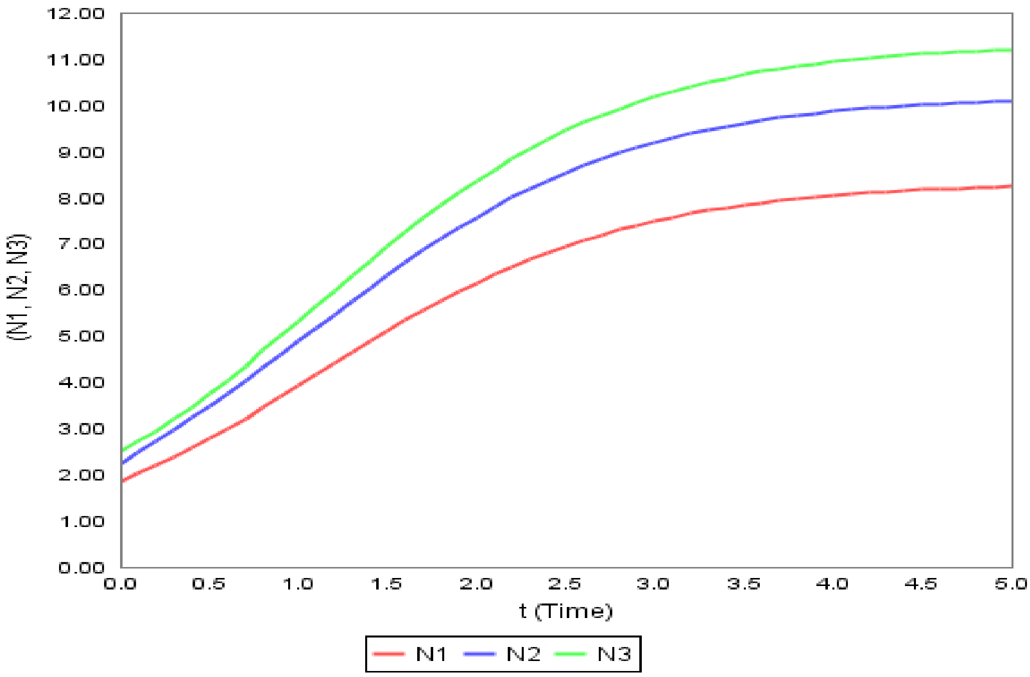


**Figure 5**

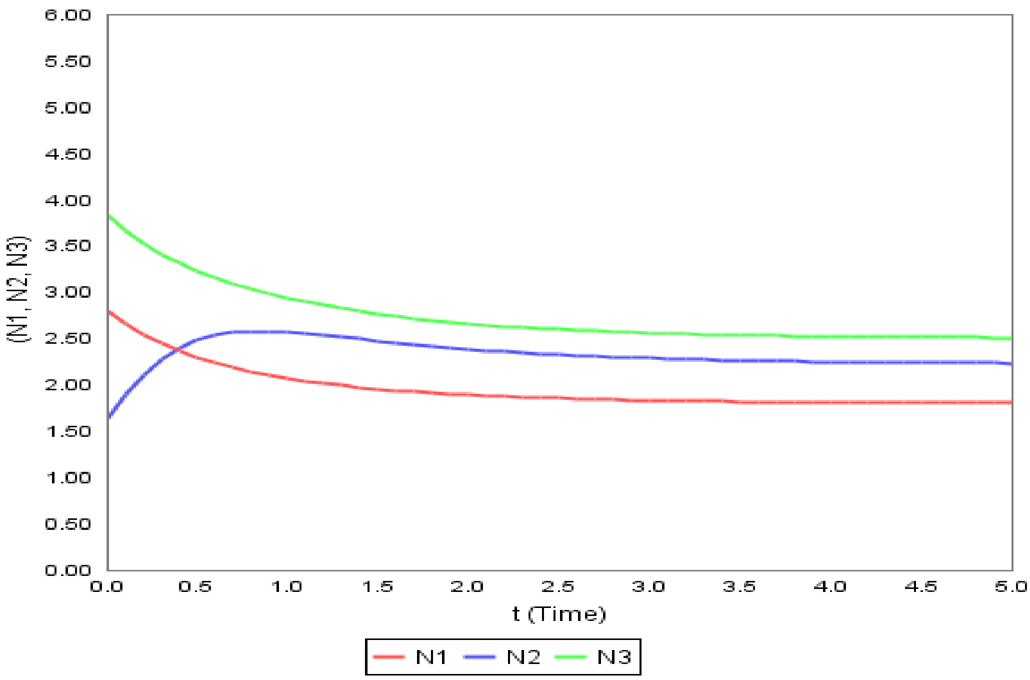


**Figure 6**

## Discussions:

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**Case (i):** In this case natural death rate of $S_3$ is less than of $S_1$. Initially the first species dominates over the second species till the time instant $t^*$= 0.72 and thereafter the dominance is reversed.

**Case (ii)** : In this case the self inhibition coefficient of the second species and the natural growth rate of the first species are identical and third species has least natural growth rate.

**Case (iii)** : In this case the self inhibition coefficients of $S_1$ , $S_2$ , $S_3$ are in decreasing order . Hera third species has the least natural death rate . we notice that the third species has a steep rise initially and then suffers a fall.

**Case (iv)** : In this case the self inhibition coefficients of $S_1$ , $S_2$ , $S_3$ are in decreasing order and there initial condition is same .Further it is evident that all the three species asymptotically converges to the equilibrium point.

**Case ( v)** : In this case the self inhibition coefficient of the second species and the natural growth rate of the third species are almost equal and first has the least natural growth rate.

**Case ( vi)** : In this case natural death rate of $S_3$ is less than of $S_1$. Initially the first species dominates over the second species till the time instant $t^*$= 0.48 and thereafter the dominance is reversed.

## 7. Conclusion

In this paper, we discussed the stability analysis of three species synecology consisting of host, commensal and mutualism. The model equations constitute a set of three first order non-linear coupled differential equations. All possible equilibrium states of the model are identified and the local stability is discussed. It is observed that, in all eight equilibrium states, only the equilibrium points $E_6$ and $E_8$ are locally stable. Further, the global stability of the system is established with the aid of suitably constructed Liapunov's function and the growth rates of the species are numerically estimated using Runge-Kutta fourth order method.

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