Prey predator model with mixed predator behavior and a safe region for prey

BENSID Yazid[1,3], HELAL Nacera[3,3], LAKMECHE Abdelkader[3] and HELAL Mohamed[3]

[1] ESAA-Tlemcen, Algeria, yazid.bensid@essa-tlemcen.dz,
[2] Higher School of Computer Sciences, Sidi Bel Abbes, 22000, Algeria, n.helal@esi-sba.dz,
[3] Biomathematics Laboratory, Univ. Sidi Bel Abbes, 22000, Algeria, lakmeche@yahoo.fr,
[3] Biomathematics Laboratory, Univ. Sidi Bel Abbes, 22000, Algeria, mhelal_abbes@yahoo.fr.

Received: 25/02/2025 **Revised:** 27/03/2025 **Accepted:** 28/04/2025

Abstract
In this paper, we investigate a prey predator model where the prey can move freely between two patches at different rates. We assume that the prey is subject to an Allee effect and that the predator has a mixed predating behavior modeled by a constant $0 < \lambda < 1$. The steady states of the model are calculated and their local stability is investigated.

Introduction

Conservation of endangered species is one of the most important topics in biomathematics [9]. Endangered species are often subject to Allee effect [13]. Named after the American ecologist Warder Clyde Allee [1], this phenomenon describes the fact that under a critical size a, population declines to the point of extinction. This phenomenon is due to different factors such as the difficulty to find mates for reproduction, and failing to self-defense against other species or predators. When the endangered specie is a prey for one or more predators it could vanish if the population size drops below the critical size a.

When trying to model prey predator interaction, it is important to take into account the predator behavior. There are two main families of predators: the specialist predators which feed only on one prey specie [7] [10] and generalist predators which have more choices for prey [14] [12].

in the literature, there are two main strategies when dealing with the conservation of endangered prey specie. The first one is to remove a certain number of predator individuals. This strategy requires the existence of a super-predator or more simply a chasing effort. A more ecologically friendly solution is to set a refuge region which access is forbidden to predators [8] [11] [3].

The case when a single population lives and moves across different regions have been extensively studied [6] [5] [4].

Predator behavior is critical in this situation, in fact in the case of specialist predators this strategy could potentially lead to the extinction of predators which depend only on one prey.

In [2], authors studied a mathematical model involving a specialist predator and a prey subject to the Allee effect. The prey can move freely between two regions A and B with region B being forbidden to predators.
Our main objective is to extend this study to different types of predator behaviors. This paper is organized as follows:
In the first section, we present our mathematical model and define the parameters involved.
In the second section, we present our main results which are the investigation of steady states and the study of their local stability.
Numerical simulations are performed in section 3 in order to illustrate our theoretical results.

Presentation of the model

We assume two regions A and B.
Prey individuals can move freely between regions A and B with different rates m_{AB} and m_{BA} when access to region B is only allowed to prey and thus predator individuals are constrained in region A.
Let us denote $A(t)$ the total number of prey individuals in the region A which is subject to Allee effect with a critical size a. This population is supposed to grow logistically with a a carrying capacity K and a growth rate r. The prey suffers from predator's attacks which is expressed by the term bAP.
$B(t)$ represents the prey population in the free-predator region B. Its growth is modeled through a logistic equation with carrying capacity H and growth rate r.
Finally, $P(t)$ is the total population of predators. e represents the chasing effort.
Following, we take $0 < \lambda < 1$ as a parameter to describe the predator's behavior. When $\lambda = 1$, the predator becomes specialist which is the case already studied in [2]. In this case, in the absence of prey $A(t)$, predator population decrease with a fixed rate n since it cannot feed on other preys.
When $\lambda = 0$, the predator becomes generalist. In the absence of prey $A(t)$, its growth follows a logistic equation with carrying capacity L and growth rate q.
Our main objective is to study the model when $0 < \lambda < 1$, that is when predator's behavior is not purely generalist nor purely specialist.
The model is then expressed by a system of three ordinary differential equations

$$\begin{cases} \dfrac{dA}{dt} &= rA\left(\dfrac{A}{K}-\dfrac{a}{K}\right)\left(1-\dfrac{A}{K}\right)-bAP-m_{BA}A+m_{AB}B, \\ \dfrac{dP}{dt} &= ebAP-\lambda nP+(1-\lambda)q\left(1-\dfrac{P}{L}\right)P, \\ \dfrac{dB}{dt} &= rB\left(1-\dfrac{B}{H}\right)+m_{BA}A-m_{AB}B. \end{cases} \quad \text{[sysb1]}$$

After a suitable change of variables, system ([sysb1]) becomes,

$$\begin{cases}\dfrac{dA}{dt} &= A(A-a)(1-A)-AP-m_{BA}A+m_{AB}B,\\ \dfrac{dP}{dt} &= \left[eA-\lambda n+(1-\lambda)q\left(1-\dfrac{P}{l}\right)\right]P,\\ \dfrac{dB}{dt} &= B(1-kB)+m_{BA}A-m_{AB}B.\end{cases}$$

Main results

Existence of steady states

The system ([sysb1]) admits the following steady states

1. The trivial steady state $E_0=(0,0,0)$ which exists always.
2. If $\lambda<\hat{\lambda}$, then there exists free prey steady state $E_2(0,P_2^*,0)$, where $\hat{\lambda}=\frac{q}{n+q}$ and $P_2^*=l-\frac{\lambda nl}{(1-\lambda)q}$.
3. If one of the following conditions is satisfied, then there exists free predator steady state $E_s(A_s^*,0,B_s^*)$,
    a. If $0<k\leq k_0$, then system[sysb1] has a unique equilibrium.
    b. If $k_0<k\leq k_1$, then system[sysb1] has at least one equilibrium.
    c. If $k>k_1$, then system[sysb1] has three equilibria.

    where $k_0=m_{AB}\frac{(1-m_{AB})}{f(A_+)}$ and $k_1=k_0+\frac{m_{AB}^2m_{BA}A_+}{f^2(A_+)}$.
4. Moreover, if one of the following conditions is satisfied, then there exists coexistence steady state $E_c(A_c^*,P_c^*,B_c^*)$,
    a. If $0<k\leq k_0$, then system[sysb1] has a unique equilibrium.
    b. If $k_0<k\leq k_1$, then system[sysb1] has at least one equilibrium.
    c. If $k>k_1$, then system[sysb1] has three equilibria.

    where $k_0=m_{AB}\frac{(1-m_{AB})}{f(A_+)}$ and $k_1=k_0+\frac{m_{AB}^2m_{BA}A_+}{f^2(A_+)}$.

Local stability

Next, we establish the stability of steady states in the following cases.

Case1: When $\lambda<\hat{\lambda}$, we have the following results:

Theorem 1.

1. *The trivial equilibrium $E_0(0,0,0)$ is unstable.*
2. *Prey free equilibrium $E_2(0,P_2^*,0)$ is unstable.*
3. *Predator free equilibrium $E_1(A_1^*,0,B_1^*)$ is unstable.*
4. *Suppose $1+a-2A_c^*<0$.*
    a. *If $0<k\leq k_0$, then the unique coexistence equilibrium $E_c(A_c^*,P_c^*,B_c^*)$ is locally asymptotically stable.*
    b. *If $k_0<k\leq k_1$, then there is at least one coexistence equilibrium $E_c(A_c^*,P_c^*,B_c^*)$ locally asymptotically stable.*
    c. *If $k>k_1$, the system [sysb1] admits three coexistence equilibria:*

$E_c^- < E_c^0 < E_c^+$ *furthermore:*

i. E_c^- *is locally asymptotically stable.*

ii. E_c^+ *is locally asymptotically stable.*

iii. *Stability of* E_c^0 *remains unknown.*

Case2: When $\lambda > \hat{\lambda}$, then we have the following results:

Theorem 2.

1. *The trivial equilibrium* $E_0(0,0,0)$ *is unstable.*
2.
   a. *If* $e > \frac{\varepsilon(n+q)}{A_1^*}$, *then any predator free equilibrium* $E_1(A_1^*,0,B_1^*)$ *is unstable.*

   b. *If* $e < \frac{\varepsilon(n+q)}{A_1^*}$, *and* $1 + a - 2A_1^* < 0$, *then:*

   i. *If* $0 < k \leq k_0$,*then any predator free equilibrium* $E_1(A_1^*,0,B_1^*)$ *is locally asymptotically stable.*

   ii. *If* $k_0 < k \leq k_1$, *then system [sysb1] has at least one predator free equilibrium* $E_1(A_1^*,0,B_1^*)$ *locally asymptotically stable.*

   iii. *If* $k > k_1$, *the system [sysb1] admits three predator free equilibria:* $E_1^- < E_1^0 < E_1^+$

   1. E_1^- *locally asymptotically stable.*
   2. E_1^+ *locally asymptotically stable.*
   3. *Stability of* E_1^0 *remains unknown.*
3. *Suppose* $1 + a - 2A_c^* < 0$.

   a. *If* $0 < k \leq k_0$, *then the unique coexistence equilibrium* $E_c(A_c^*,P_c^*,B_c^*)$ *is locally asymptotically stable.*

   b. *If* $k_0 < k \leq k_1$, *then there is at least one coexistence equilibrium* $E_c(A_c^*,P_c^*,B_c^*)$ *locally asymptotically stable.*

   c. *If* $k > k_1$, *the system [sysb1] admits three coexistence equilibria:* $E_c^- < E_c^0 < E_c^+$ *furthermore:*

   i. E_c^- *is locally asymptotically stable.*

   ii. E_c^+ *is locally asymptotically stable.*

   iii. *Stability of* E_c^0 *remains unknown.*

Proof. The Jacobian of the system at the equilibrium point is given by:

$$J = \begin{pmatrix} J_{11} & -A^* & m_{AB} \\ eP^* & J_{22} & 0 \\ m_{BA} & 0 & J_{33} \end{pmatrix}$$

with

$$J_{11} = (A^*-a)(1-A^*) + A^*[(1-A^*)-(A^*-a)]-P^*-m_{BA}$$

$$J_{22} = eA^*-\lambda n + (1-\lambda)q-2\frac{(1-\lambda)q}{l}P^*$$

$$J_{33} = -2kB^* + (1-m_{AB}).$$

If $A^* \neq 0$, then we obtain

$$m_{BA} = (A^* - a)(1 - A^*) - P^* + m_{AB}\frac{B^*}{A^*}.$$

That is

$$J_{11} = A^*(1 + a - 2A^*) - m_{AB}\frac{B^*}{A^*}$$
$$J_{22} = eA^* - \lambda n + (1-\lambda)q - 2\frac{(1-\lambda)q}{l}P^*.$$

If $P^* \neq 0$, then from second equation of ([sysb2]), we obtain

$$eA^* - \lambda n + (1-\lambda)q = (1-\lambda)q\frac{P^*}{l}.$$

Thus

$$J_{22} = -\frac{(1-\lambda)q}{l}P^* = -eA^* + \lambda n - (1-\lambda)q.$$

If $B^* \neq 0$, then from third equation of ([sysb2]), we obtain

$$kB^{*2} + (m_{AB} - 1)B^* - m_{BA}A^* = 0.$$

That is,

$$1 - m_{AB} = kB^* - m_{BA}\frac{A^*}{B^*}.$$

Then

$$J_{33} = -2kB^* + \left(kB^* - m_{BA}\frac{A^*}{B^*}\right) = -kB^* - m_{BA}\frac{A^*}{B^*} < 0.$$

Stability of the trivial equilibrium

At trivial equilibrium E_0, the characteristic equation is given by

$$(J_{22} - X)(X^2 - Tr(M_1)X + ((m_{AB} - 1) - m_{BA}) = 0,$$

where $M_1 = \begin{pmatrix} J_{11} & m_{AB} \\ m_{BA} & J_{33} \end{pmatrix} = \begin{pmatrix} -a - m_{BA} & m_{AB} \\ m_{BA} & 1 - m_{AB} \end{pmatrix}$

J_{22} is an eigenvalue of J_0.

$$J_{22} = -\lambda n + (1-\lambda)q$$

If $\lambda < \hat{\lambda}$, then $J_{22} < 0$

Thus, E_0 is unstable.

suppose $\lambda > \hat{\lambda}$, then $J_{22} < 0$

E_0 is locally asymptotically stable if $tr(M_1) < 0$ and $\det(M_1) > 0$.

$$\begin{aligned}\det(M_1) &= (-a - m_{BA})(1 - m_{AB}) - m_{BA}m_{AB} \\ &= -a + am_{AB} - m_{BA} \\ &= a(m_{AB} - 1) - m_{BA}\end{aligned}$$

Since, $m_{AB} < 1$, then $\det(M_1) < 0$

Therefore, E_0 is unstable.

Stability of the prey free equilibrium $E_2(0,P_2^*,0)$

The Jacobian is given by:

$$J_2 = \begin{pmatrix} J_{11} & 0 & m_{AB} \\ eP_2^* & J_{22} & 0 \\ m_{BA} & 0 & J_{33} \end{pmatrix}$$

$$\begin{aligned} P_{J_2}(X) &= \begin{vmatrix} J_{11}-X & 0 & m_{AB} \\ eP_2^* & J_{22}-X & 0 \\ m_{BA} & 0 & J_{33}-X \end{vmatrix} \\ &= (J_{22}-X)\begin{vmatrix} J_{11}-X & m_{AB} \\ m_{BA} & J_{33}-X \end{vmatrix} \\ &= (J_{22}-X)\left(X^2-Tr(M_2)X+\det(M_2)\right) \end{aligned}$$

with $M_2 = \begin{pmatrix} J_{11} & m_{AB} \\ m_{BA} & J_{33} \end{pmatrix}$

$$\begin{aligned} J_{22} &= -\lambda n + (1-\lambda)q - 2\frac{(1-\lambda)q}{l}\frac{-\lambda(n+q)+q}{(1-\lambda)q}l \\ &= -\lambda n + (1-\lambda)q + 2[\lambda(n+q)-q] \\ &= -\lambda n + q - \lambda q + 2\lambda n + 2\lambda q - 2q \\ &= \lambda(n+q)-q \end{aligned}$$

Recall that P_2^* exists if and only if $\lambda < \hat{\lambda}$

Thus $J_{22} = \lambda(n+q)-q < 0$

$$\begin{aligned} \det(M) &= J_{11}J_{33}-m_{AB}m_{BA} \\ &= (-a-P_2^*)(1-m_{AB})-m_{BA} \end{aligned}$$

Since $m_{AB} < 1$, then $det(M) < 0$

We conclude that E_2 is unstable.

Stability of the predator free equilibrium $E_1(A_1^*,0,B_1^*)$

The Jacobian is given by:

$$J_1 = \begin{pmatrix} J_{11} & -A_1^* & m_{AB} \\ 0 & J_{22} & 0 \\ m_{BA} & 0 & J_{33} \end{pmatrix}$$

$$\begin{aligned} P_{J_1}(X) &= \begin{vmatrix} J_{11}-X & -A_1^* & m_{AB} \\ 0 & J_{22}-X & 0 \\ m_{BA} & 0 & J_{33}-X \end{vmatrix} \\ &= (J_{22}-X)\left[X^2-Tr(M_2)X+\det(M_2)\right] \end{aligned}$$

With:

$$M_3 = \begin{pmatrix} J_{11} & m_{AB} \\ m_{BA} & J_{33} \end{pmatrix}$$

J_1 has an eigenvalue $J_{22} = eA^* - \lambda n + (1-\lambda)q$

If $\lambda < \hat{\lambda}$, then $\lambda = \hat{\lambda} - \varepsilon$ with $\varepsilon = \hat{\lambda} - \lambda$

$$\begin{aligned} J_{22} &= eA_1^* - \lambda n + (1-\lambda)q \\ &= eA_1^* + q - \lambda(n+q) \\ &= eA_1^* + q - (\hat{\lambda}-\varepsilon)(n+q) \\ &= eA_1^* + q - \left(\frac{q}{n+q}-\varepsilon\right)(n+q) \\ &= eA_1^* + \varepsilon(n+q) > 0 \end{aligned}$$

Thus, E_1 is unstable.

If $\lambda > \hat{\lambda}$, then $\lambda = \hat{\lambda} + \varepsilon$ with $\varepsilon = \lambda - \hat{\lambda}$

$$\begin{aligned} J_{22} &= eA_1^* - \lambda n + (1-\lambda)q \\ &= eA_1^* + q - \lambda(n+q) \\ &= eA_1^* + q - (\hat{\lambda} + \varepsilon)(n+q) \\ &= eA_1^* + q - \left(\frac{q}{n+q} + \varepsilon\right)(n+q) \\ &= eA_1^* - \varepsilon(n+q) \end{aligned}$$

If $e > \frac{\varepsilon(n+q)}{A_1^*}$, then E_1 is unstable.

If $e < \frac{\varepsilon(n+q)}{A_1^*}$, we need to calculate $Tr(M_3)$ and $\det(M_3)$.

$$Tr(M_3) = J_{11} + J_{33} = A^*(1 + a - 2A^*) - m_{AB}\frac{B^*}{A^*} - kB^* - m_{BA}\frac{A^*}{B^*}$$

Suppose $1 + a - 2A^* < 0$, then $Tr(M_3) < 0$

$$\det(M_2) = [A_1^*(2A_1^* - 1 - a)]\left(kB_1^* + m_{BA}\frac{A_1^*}{B_1^*}\right) + km_{AB}\frac{B_1^{*2}}{A_1^*}$$

If $1 + a - 2A_1^* < 0$, then $\det(M_2) > 0$

Let us find a condition so that $1 + a - 2A_1^* < 0$

$$A_+ = \frac{(a+1) - \sqrt{\Delta_f}}{3} < \frac{a+1}{3} < \frac{a+1}{2}$$

$$f(A) = A[(A-a)(A-1) + m_{BA}]$$

$$\Rightarrow f'(A) = [(A-a)(A-1) + m_{BA}] + A[2A - (a+1)]$$

thus, $f\left(\frac{a+1}{2}\right) = \left(\frac{a+1}{2}\right) f'\left(\frac{a+1}{2}\right)$

We observe that $f'\left(\frac{a+1}{2}\right)$ and $f\left(\frac{a+1}{2}\right)$ have the same sign.

$$f'\left(\frac{a+1}{2}\right) = \left[\left(\frac{a+1}{2} - a\right)\left(\frac{a+1}{2} - 1\right) + m_{BA}\right]$$

$$\Rightarrow f'\left(\frac{a+1}{2}\right) = \left(\frac{1-a}{2}\right)\left(\frac{a-1}{2}\right) + m_{BA} = m_{BA} - \frac{1}{4}(1-a)^2$$

1. If $m_{BA} < \frac{1}{4}(1-a)^2$, then: $f'\left(\frac{a+1}{2}\right) < 0$ and $f\left(\frac{a+1}{2}\right) < 0$

   We deduce that $A_+ < \frac{a+1}{2} < A_-$

   In the case where the equilibrium is unique:

   $\frac{a+1}{2} < A^* \Rightarrow 1 + a - 2A_1^* < 0$ and the equilibrium is locally asymptotically stable.

   If there are three equilibria:

   $\frac{a+1}{2} < A_1^+ \Rightarrow 1 + a - 2A_1^+ < 0$ and the equilibrium is locally asymptotically stable.

   The stability of the two remaining equilibria is unknown.

Stability of the coexistence equilibrium $E_c(A_c^*, P_c^*, B_c^*)$

The Jacobian is given by:

$$J_c = \begin{pmatrix} J_{11} & -A_c^* & m_{AB} \\ eP_c^* & J_{22} & 0 \\ m_{BA} & 0 & J_{33} \end{pmatrix}$$

$$P_{J_c}(X) = \begin{vmatrix} J_{11}-X & -A_c^* & m_{AB} \\ eP_c^* & J_{22}-X & 0 \\ m_{BA} & 0 & J_{33}-X \end{vmatrix} = eP_c^*A_c^*(J_{33}-X) + (J_{22}-X)\begin{vmatrix} J_{11}-X & m_{AB} \\ m_{BA} & J_{33}-X \end{vmatrix}$$
$$= eP_c^*A_c^*(J_{33}-X) + (J_{22}-X)\left[X^2-(J_{11}+J_{33})X + (J_{11}J_{33}-m_{AB}m_{BA})\right]$$
$$= eP_c^*A_c^*(J_{33}-X) + (J_{22}-X)\left[X^2-Tr(M_4)X + \det(M_4)\right]$$

with:

$$M_4 = \begin{pmatrix} J_{11} & m_{AB} \\ m_{BA} & J_{33} \end{pmatrix}$$
$$P_{J_c}(X) = -X^3 + [J_{22} + Tr(M_4)]X^2 - [\det(M_4) + J_{22}Tr(M_4) + eP_c^*A_c^*]X + eP_c^*A_c^*J_{33} + J_{22}\det(M_4)$$
$$-P_{J_c}(X) = X^3 - [J_{22} + Tr(M_4)]X^2 + [\det(M_4) + J_{22}Tr(M_4) + eP_c^*A_c^*]X - eP_c^*A_c^*J_{33} - J_{22}\det(M_4)$$
$$= X^3 + a_2X^2 + a_1X + a_0$$

Lemma 1 (Rooth criteria). *The system is stable if and only if all the following terms are strictly positive:*

a_0, a_1, a_2, a_3 *and* c_1 *where* $c_1 = a_2a_1 - a_3a_0$

Suppose that $1 + a - 2A_c^* < 0$.

$$a_2 = -J_{22} - Tr(M_4) = Tr(J)$$
$$= -J_{11} - J_{22} - J_{33}$$
$$= -\left[A^*(1 + a - 2A^*) - m_{AB}\frac{B^*}{A^*}\right] + \left[\frac{(1-\lambda)q}{l}P^*\right] + kB^* + m_{BA}\frac{A^*}{B^*} > 0$$

$$a_1 = \det(M_4) + J_{22}Tr(M_4) + eP_c^*A_c^*$$
$$= J_{11}J_{33} - m_{AB}m_{BA} + J_{22}(J_{11} + J_{33}) + eP_c^*A_c^*$$
$$= \left[A_c^*(1 + a - 2A_c^*) - m_{AB}\frac{B_c^*}{A_c^*}\right]\left[-kB^* - m_{BA}\frac{A_c^*}{B_c^*}\right] - m_{AB}m_{BA}$$
$$- \left(\frac{(1-\lambda)q}{l}P^*\right)\left[A_c^*(1 + a - 2A_c^*) - m_{AB}\frac{B_c^*}{A_c^*} - kB^* - m_{BA}\frac{A_c^*}{B_c^*}\right] + eP_c^*A_c^*$$
$$= [A_c^*(1 + a - 2A_c^*)]\left[-kB^* - m_{BA}\frac{A_c^*}{B_c^*}\right] + km_{AB}\frac{B_c^{*2}}{A_c^*}$$
$$- \left(\frac{(1-\lambda)q}{l}P^*\right)\left[A_c^*(1 + a - 2A_c^*) - m_{AB}\frac{B_c^*}{A_c^*} - kB^* - m_{BA}\frac{A_c^*}{B_c^*}\right] + eP_c^*A_c^* > 0$$

$$a_0 = -eP_c^*A_c^*J_{33} - J_{22}\det(M_4)$$
$$= -eP_c^*A_c^*\left(-kB^* - m_{BA}\frac{A^*}{B^*}\right)$$
$$+ \left(\frac{(1-\lambda)q}{l}P^*\right)\left([A_c^*(1 + a - 2A_c^*) - m_{AB}\frac{B_c^*}{A_c^*}][-kB^* - m_{BA}\frac{A_c^*}{B_c^*}] - m_{AB}m_{BA}\right)$$
$$= -eP_c^*A_c^*\left(-kB^* - m_{BA}\frac{A^*}{B^*}\right) + \left(\frac{(1-\lambda)q}{l}P^*\right)\left([A_c^*(1 + a - 2A_c^*)][-kB^* - m_{BA}\frac{A_c^*}{B_c^*}] + km_{AB}\frac{B_c^{*2}}{A_c^*}\right)$$
$$\Rightarrow a_0 > 0$$

$$
\begin{aligned}
c_1 &= a_2 a_1 - a_3 a_0 \\
&= [-J_{22} - Tr(M_4)]\,[\det(M_4) + J_{22} Tr(M_4) + eP_c^* A_c^*] + eP_c^* A_c^* J_{33} + J_{22}\det(M_4) \\
&= -J_{22}[\det(M_4) + J_{22} Tr(M_4) + eP_c^* A_c^*] - Tr(M_4)\,[\det(M_4) + J_{22} Tr(M_4) + eP_c^* A_c^*] \\
&\quad + eP_c^* A_c^* J_{33} + J_{22}\det(M_4) \\
&= -J_{22}[J_{22} Tr(M_4) + eP_c^* A_c^*] - (J_{11} + J_{33})\,[\det(M_4) + J_{22} Tr(M_4) + eP_c^* A_c^*] + eP_c^* A_c^* J_{33} \\
&= -J_{22}[J_{22} Tr(M_4) + eP_c^* A_c^*] - J_{11}[\det(M_4) + J_{22} Tr(M_4) + eP_c^* A_c^*] \\
&\quad - J_{33}[\det(M_4) + J_{22} Tr(M_4)] > 0
\end{aligned}
$$

$$
\begin{aligned}
J_{22} Tr(M_4) &= J_{22}(J_{11} + J_{33}) \\
&= -\left(\frac{(1-\lambda)q}{l} P^*\right)\left[A_c^*(1 + a - 2A_c^*) - m_{AB}\frac{B_c^*}{A_c^*} - kB^* - m_{BA}\frac{A_c^*}{B_c^*}\right] > 0 \\
&= \left(\frac{(1-\lambda)q}{l} P^*\right)\left[A_c^*(2A_c^* - a - 1) + m_{AB}\frac{B_c^*}{A_c^*} + kB^* + m_{BA}\frac{A_c^*}{B_c^*}\right] > 0
\end{aligned}
$$

$$
\begin{aligned}
\det(M_4) &= J_{11} J_{33} - m_{AB} m_{BA} \\
&= \left[A_c^*(1 + a - 2A_c^*) - m_{AB}\frac{B_c^*}{A_c^*}\right]\left[-kB^* - m_{BA}\frac{A_c^*}{B_c^*}\right] - m_{AB} m_{BA} \\
&= [A_c^*(1 + a - 2A_c^*)]\left[-kB^* - m_{BA}\frac{A_c^*}{B_c^*}\right] + k m_{AB}\frac{B_c^{*2}}{A_c^*} > 0
\end{aligned}
$$

We conclude that the coexistence equilibrium $E_c(A_c^*, P_c^*, B_c^*)$ is locally asymptotically stable. □

Numerical simulations

Case $\lambda < \hat{\lambda}$

Choosing parameters as follow:

$a = 0.6;\ m_{BA} = 0.02;\ m_{AB} = 0.01;\ e = 0.01;\ n = 0.1;\ q = 0.8;\ \lambda = 0.0889;\ l = 0.5;\ k = 0.0701$

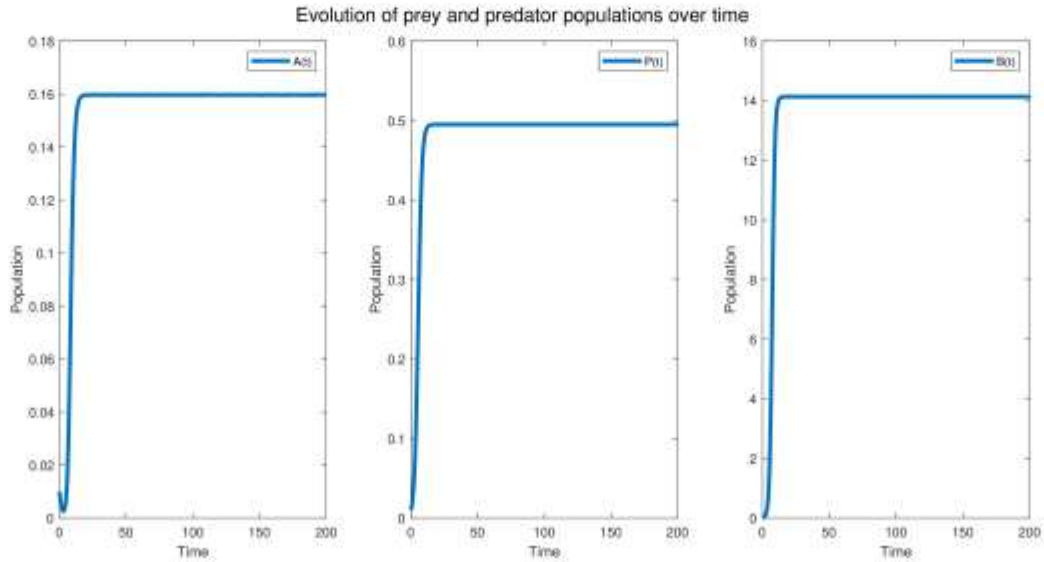
We obtain four steady states:

Trivial equilibrium $E_0(0,0,0)$

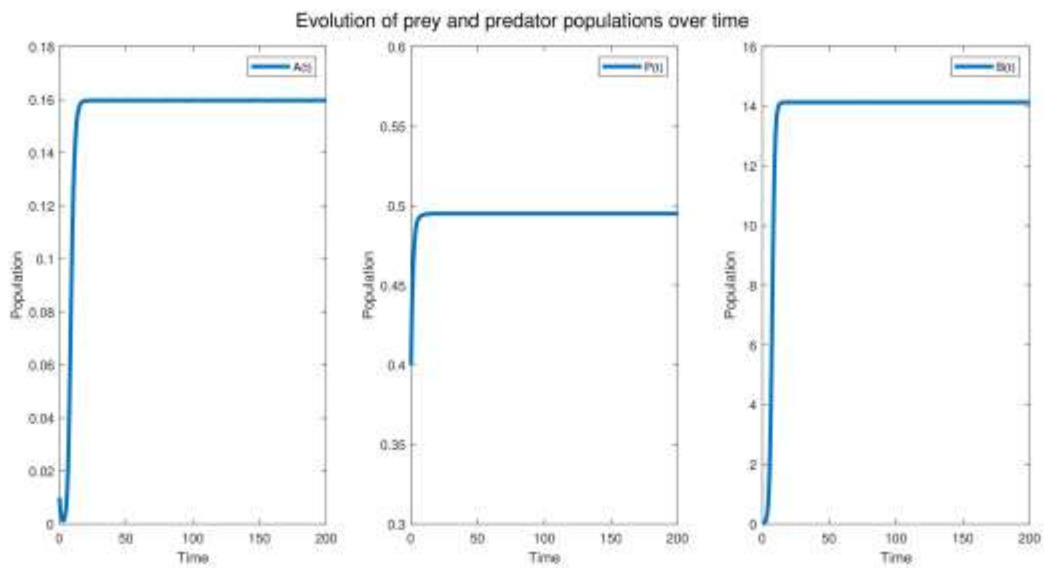
Prey-free equilibrium $E_1(0,\ 0.4939,\ 0)$

Predator-free equilibrium $E_s(1.1747,\ 0,\ 14.1520)$

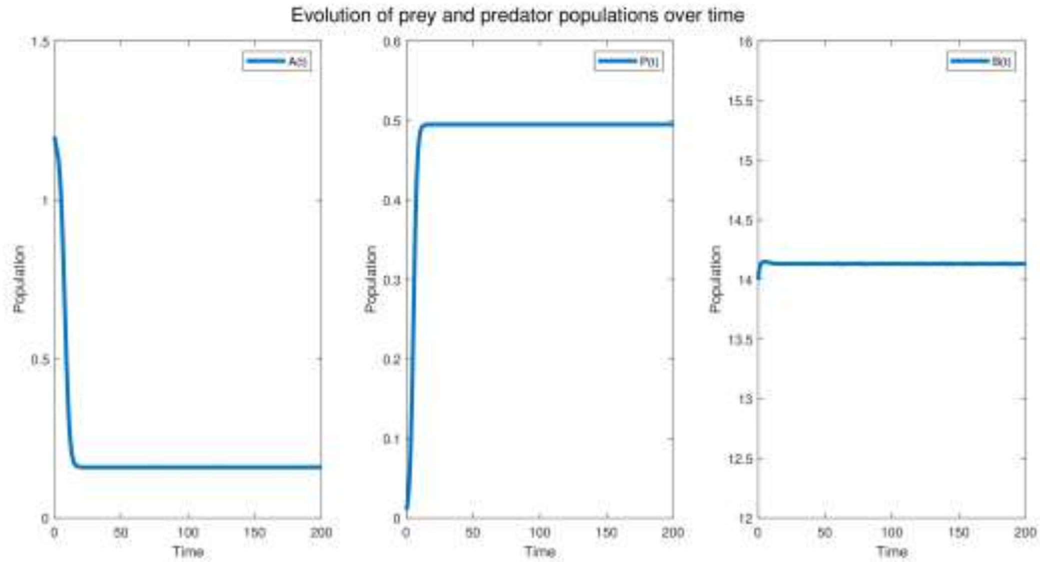
Coexistence equilibrium $E_c(0.1596,\ 0.4949,\ 14.1315)$



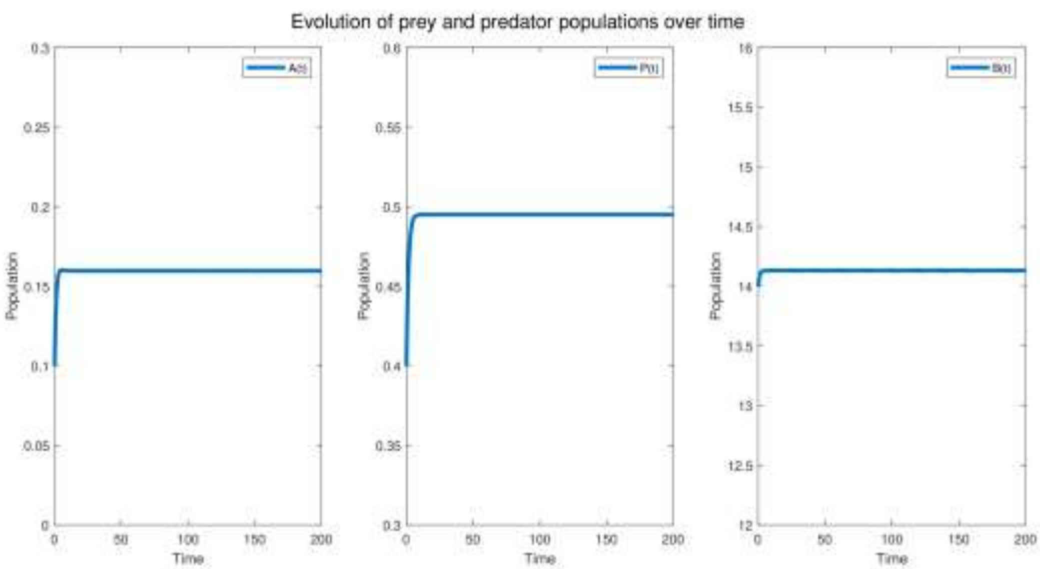
Initial conditions are taken close to the trivial equilibrium $E_0(0,0,0)$*. Solutions converge to the coexistence equilibrium* $E_c(0.1596,\ 0.4949,\ 14.1315)$.



Initial conditions are taken close to the prey free equilibrium $E_1(0,\ 0.4939,\ 0)$*. Solutions converge to the coexistence equilibrium* $E_c(0.1596,\ 0.4949,\ 14.1315)$.



Initial conditions are taken close to the predator free equilibrium $E_s(1.1747,\ 0,\ 14.1520)$*.*
Solutions converge to the coexistence equilibrium $E_c(0.1596,\ 0.4949,\ 14.1315)$*.*

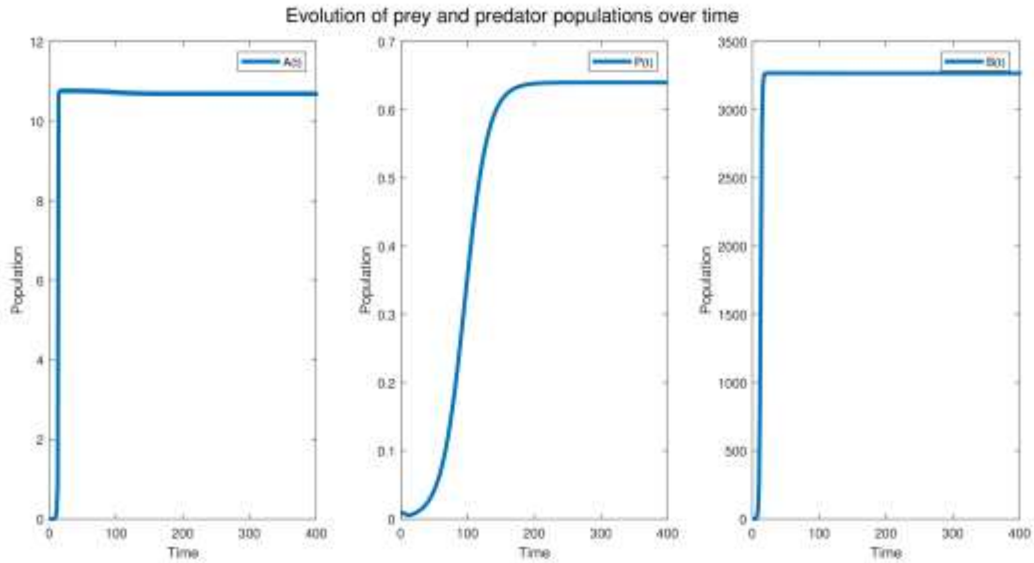


Initial conditions are taken close to the coexistence equilibrium $E_c(0.1596,\ 0.4949,\ 14.1315)$*.*
Solutions converge to the coexistence equilibrium.

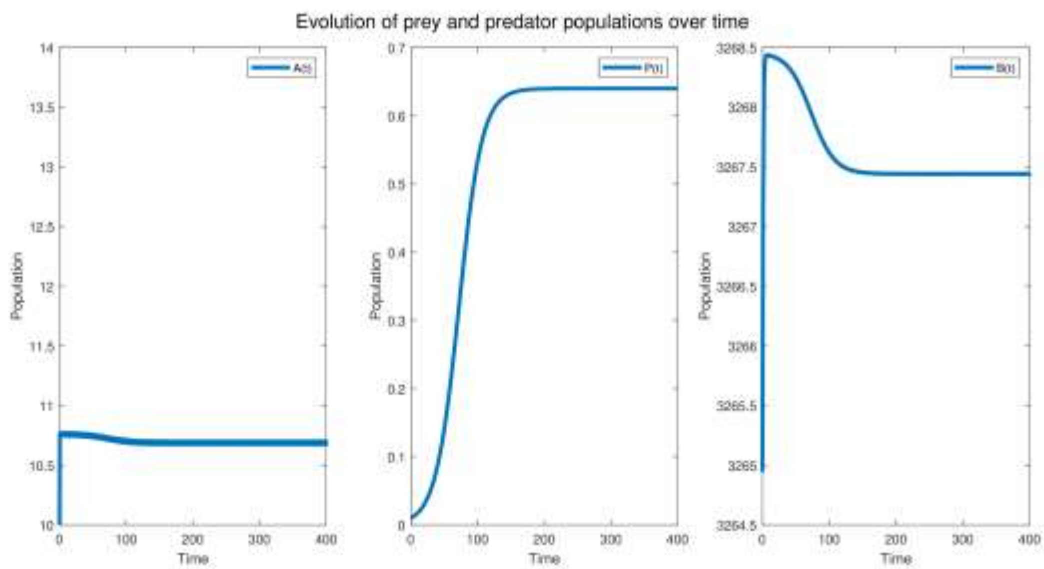
Case $\lambda > \hat{\lambda}$

Choosing parameters as follow:
a = 12 m_{BA} = 15.1250 m_{AB} = 0.01 e = 0.01 n = 0.1 q = 0.8 \lambda = 0.9444 l= 0.5 k=0.000318

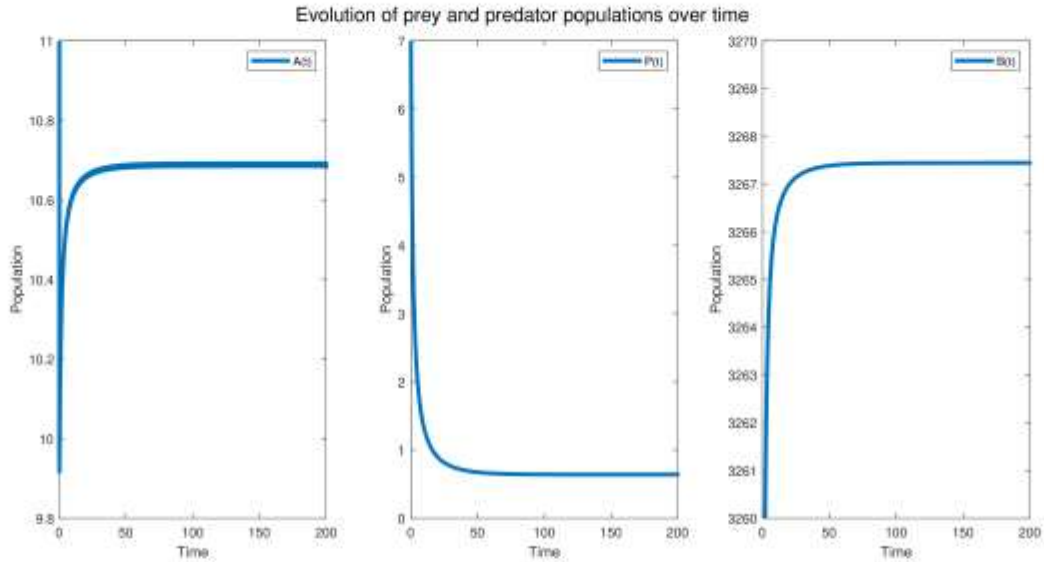
We obtain three steady states:
Trivial equilibrium $E_0(0,0,0)$
Predator-free equilibrium $E_s(10.7617,\ 0,\ 3268.4601)$
Coexistence equilibrium $E_c(10.6883,\ 0.6399,\ 3267.4412)$



Initial conditions are taken close to the trivial equilibrium $E_0(0,0,0)$*. Solutions converge to the coexistence equilibrium* $E_c(10.6883,\ 0.6399,\ 3267.4412)$



Initial conditions are taken close to the predator free equilibrium $E_s(10.7617,\ 0,\ 3268.4601)$*. Solutions converge to the coexistence equilibrium* $E_c(10.6883,\ 0.6399,\ 3267.4412)$*.*



Initial conditions are taken close to the coexistence equilibrium $E_c(10.6883,\ 0.6399,\ 3267.4412)$. Solutions converge to the coexistence equilibrium.

Conclusion

In study we have tried to capture the complexity of predator's behavior by setting a parameter $0 < \lambda < 1$ as in [2] .

Note that this model has not been studied previously.

Our main finding in this study is the existence of a critical value $\hat{\lambda}$. It is remarkable to see that if $\lambda > \hat{\lambda}$, the model behaves essentially as the specialist predator model studied in [2] .

On the other hand, when $\lambda < \hat{\lambda}$, the model behaves as a generalist predator.

We have demonstrated that in both cases, dynamics of the system are very different.

Therefore, we believe that predator's behavior should be taken into account when applying prey's conservation strategies.

In fact, in the case of a specialist-like predator (i.e.$\lambda > \hat{\lambda}$), setting a safe region for the prey can lead to the extinction of the predator because it has no alternative source of food.

In the other case (i.e. $\lambda < \hat{\lambda}$), this scenario can be avoided since predator has a generalist-like behavior and can feed on other species.

Acknowledgements

This work was partially supported by the DGRSDT (MESRS, Algeria), through PRFU research project C00L03UN220120220001.

.

[1] Warder Clyde Allee. A study in general sociology. , 1931.
[2] Malay Banerjee, Bob W. Kooi, and Ezio Venturino. A safe harbor can protect an endangered species from its predators. , 69:413–436, 2020.
[3] Joydev Ghosh, Banshidhar Sahoo, and Swarup Poria. Prey-predator dynamics with prey refuge providing additional food to predator. , 96:110–119, 2017.
[4] Ilkka Hanski. Metapopulation dynamics: does it help to have more of the same? , 4:113–114, 1989.
[5] Ilkka Hanski. A practical model of metapopulation dynamics. , 63:151–162, 1994.

[6] Ilkka Hanski. Metapopulation dynamics. , 396:41–49, 1998.
[7] Sundell Janne and Hannu Ylönen. Specialist predator in a multi-species prey community: boreal voles and weasels. , 3:51–63, 2008.
[8] Zhihui Ma, Wenlong Li, Yu Zhao, Wenting Wang, Hui Zhang, and Zizhen Li. Effects of prey refuges on a predator–prey model with a class of functional responses: The role of refuges. , 218:73–79, 2009.
[9] Richard S. Miller and Daniel B. Botkin. Endangered species: models and predictions: simulation models of endangered populations may indicate the outcomes of various management alternatives. , 62:172–181, 1974.
[10] François Mougeot, Xavier Lambin, Ruth Rodríguez-Pastor, Juan Romairone, and Juan Jose Luque-Larena. Numerical response of a mammalian specialist predator to multiple prey dynamics in mediterranean farmlands. , 100:p.e02776, 2019.
[11] Sahabuddin Sarwardi, Prashanta Kumar Mandal, and Santanu Ray. Analysis of a competitive prey–predator system with a prey refuge. , 110:133–148, 2012.
[12] William E. Snyder and Edward W. Evans. Ecological effects of invasive arthropod generalist predators. , 37:95–122, 2006.
[13] Gui-Quan Sun. Mathematical modeling of population dynamics with allee effect. , 85:1–12, 2016.
[14] W. O. C. Symondson, K. D. Sunderland, and M. H. Greenstone. Can generalist predators be effective biocontrol agents? , 47:561–594, 2002.