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Approximation of solution for Mckean-Vlasov SDEs under G-Brownian motion

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ABSTRACT

This paper investigates the McKean-Vlasov stochastic differential equation (MVSDEs), called also mean-field stochastic differential equations stochastic differential equations (MFSDEs), driven by G-Brownian motion. Note that the coefficients depend not only on the state variable, but also in its marginal distribution, and the solutions of such equation are known in the literature as nonlinear diffusions. Under the assumption of Lipschitz continuous coefficients, we establish the existence and uniqueness of solutions using the method of successive approximations, also known as Picard iteration scheme. The approach leverages the contractive properties of the iteration scheme to rigorously demonstrate convergence to a unique solution. Furthermore, we analyze the stability of the solution with respect to initial condition and coefficients. By introducing small perturbations, we derive quantitative bounds that highlight the continuous dependence of the solution on these parameters. Our results contribute to the theoretical understanding of McKean-Vlasov SDEs in the context of G-expectation theory. These equations are important due to their significant role in a wide range of fields.

Keywords: McKean-Vlasov stochastic differential equation, G-Brownian motion, G-expectation, Picard successive approximations, Stability.

1. INTRODUCTION

McKean-Vlasov stochastic differential equations (MVSDEs) were first introduced in statistical physics by Kac [10], they appeared as the stochastic counterpart of the Vlasov equation in plasma dynamics [20]. With a thorough survey by Sznitman [19], McKean [13] continued this work and developed the probabilistic underpinning of a theory for these equations. These equations appear as the asymptotic behavior of systems comprising many weakly interacting particles in the limit that the number of particles goes to infinity. This well-known phenomenon is called the propagation of chaos in the literature. The key property of these equations is that the coefficients depend on the state variable X_t and its marginal distribution P_{X_t}. These equations have proven valuable in diverse fields, including simulating crowd behavior, modeling opinion dynamics, optimizing oil exploration, analyzing network congestion, and studying market dynamics in mathematical finance. The existence and uniqueness of solutions for MVSDEs have been extensively investigated, with foundational results found in works such as [3, 9]. Further research has explored scenarios with less regular coefficients, as seen in [4, 5, 6, 15].

In recent years, MVSDEs have become central to the theory of mean-field games (MFG). This theory, developed independently in 2006 by Lions and Lasry [11] as well as Huang, Malhamé, and Caines [8], seeks to approximate Nash equilibria for differential games involving a large number of players (See [2]). Since its introduction, MFG theory and the broader field of mean-field control have seen rapid development, This growth is driven by their relevance to game theory, financial mathematics, communication networks, and resource management. For the latest advancements and a comprehensive bibliography on this subject, readers can refer to [3].

The development of advanced financial tools, including risk management and uncertainty quantification, has given rise to the creation of G-expectation theory by Peng [16, 17]. Under this theory, Peng introduced fundamental concepts like G-normal distribution and G-Brownian motion, along with a corresponding Itô stochastic calculus. Building on this foundation, Peng [17] and Gao [7] established the existence and uniqueness of solutions for stochastic differential equations driven by G-Brownian motion (G-SDEs) under Lipschitz conditions through the contraction mapping principle. Progressing further, Lin [12] extended this to pathwise uniqueness for non-Lipschitz G-SDEs with bounded coefficients, while Bai and Lin [1] extended the theory to G-SDEs governed by integral-Lipschitz coefficients. This progress naturally led to the study of mean-field G-SDEs, which account for interactions between systems. Sun [18] initially demonstrated the existence and uniqueness of solutions for mean-field backward stochastic differential equations driven by G-Brownian motion with Lipschitz conditions. More recently, the authors in [14] utilized the Banach fixed-point theorem to prove the existence and uniqueness of solutions for mean-field SDEs driven by G-Brownian motion.

This paper aims to investigate the fundamental properties of McKean-Vlasov stochastic differential equations driven by G-Brownian motion (G-MVSDEs), focusing on their existence, uniqueness, and stability properties. To establish the existence of solutions, we employ the successive approximation method, also known as Picard’s iterative method, which serves as a robust framework for constructing solutions step by step. We further investigate the stability of G-MVSDEs, we examine how small perturbations in the initial conditions or coefficients influence the trajectories. All results are derived under the assumption of Lipschitz continuity with respect to both the state variable and the distribution.

This study is organized to guide the reader through its core elements. Section 2 introduces the fundamental definitions and properties of the G-framework. Section 3 establishes the necessary assumptions and framework for our analysis. Section 4 proves the existence and uniqueness of the G-MVSDE solution via successive approximation. Finally, Section 5 examines the solution’s stability with respect to initial conditions and coefficients, under Lipschitz conditions.

2. THE G-FRAMEWORK

This section presents the foundational concepts and key findings of G-stochastic calculus framework.

A nonlinear expectation $\hat{E}$ is a functional mapping $\mathcal{H} \to \mathbb{R}$ that satisfies the following properties, if $X, Y \in \mathcal{H}$, $c \in \mathbb{R}$, and $\lambda \geq 0$:

Monotonicity : if $X \geq Y$ then $\hat{E}[X] \geq \hat{E}[Y]$

Constant preserving : $\hat{E}[c] = c$

Subadditivity : $\hat{E}[X + Y] \leq \hat{E}[X] + \hat{E}[Y]$

Positive homogeneity : $\hat{E}[\lambda X] = \lambda \hat{E}[X]$

Where $\mathcal{H}$ denote a lattice consisting of real-valued functions. The space $\mathcal{H}$ serves as a collection of random variables, with the additional assumption that every function in $\mathcal{H}$ is bounded.

Define $\Omega := \{\omega \in C([0, T]; \mathbb{R}^n) : \omega_0 = 0\}$ as the space the space of all $\mathbb{R}-$valued continuous functions with $\omega_0 = 0$, equipped with the distance :

$$d(\omega^1, \omega^2) = \sum_{k=1}^{\infty} 2^{-k} \left(\max_{t \in 0,k]} |\omega_t^1 - \omega_t^2| \wedge 1 \right).$$

Denote by $(\Omega, \mathcal{F})$ the canonical space equipped with its natural filtration $(\mathcal{F}_t)_{t \geq 0}$ such that for each $t \geq 0$ and $s > t$, we have $\mathcal{F}_s \subset \mathcal{F}_t$. The canonical process associated is denoted by $\omega = (\omega_t)_{t \geq 0}$.

For each fixed $T \in 0; \infty)$, the corresponding space of random variables is defined as follows:

$$L_{ip}(\mathcal{F}_T) = \left\{ \varphi(B_{t_1}, B_{t_2}, \ldots, B_{t_n}) : t_1, \ldots, t_n \in 0; T], \varphi \in lip(\mathbb{R}^n), n \in \mathbb{N} \right\},$$

where $lip(\mathbb{R}^n)$ is the set of all real-valued functions defined on $\mathbb{R}^n$ that are both bounded and Lipschitz continuous. Here, $\mathbb{R}$ is identified as Ω, and $lip(\mathbb{R})$ is referred to as $\mathcal{H}$.

Remark 1: $L_{ip}(\mathcal{F}_t) \subseteq L_{ip}(\mathcal{F}_T)$ *for* $t \leq T$, *and* $L_{ip}(\mathcal{F}) = \cup_{m=1}^{\infty} L_{ip}(\mathcal{F}_m)$.

We define the canonical space and specify $B_t(\omega) = \omega_t$ for $t \in 0, \infty)$ and $\omega \in \Omega$.

The canonical process $(B_t)_{t \geq 0}$ is called $G-$Brownian motion under nonlinear expectation $\hat{E}$ defined on $L_{ip}(\mathcal{F})$, provided that for each $T > 0$, $n = 1,2, \ldots.$, and for every function $\varphi \in lip(\mathbb{R}^n)$, where $0 \leq t_1 < \ldots < t_n \leq T$, we have :

$$\hat{E}\left[\varphi(B_{t_1}, B_{t_2} - B_{t_1}, B_{t_3} - B_{t_2}, \ldots, B_{t_n} - B_{t_{n-1}})\right] = \varphi_n, \text{ where} \varphi \in \mathbb{R}.$$

Definition 2: *The sublinear expectation* $\hat{E} : L_{ip}(\Omega) \to \mathbb{R}$ *defined above is called a G-expectation and the corresponding canonical process* B *is called a* $G-$*Brownian motion.*

In addition, for $p \geq 1$ we define $L_G^p(\Omega)$ as a completion of $L_{ip}(\Omega)$ under the norme :

$$||X||_{L_G^p(\mathcal{F}_T)} = \left(\hat{E}[|X|^p]\right)^{1/p}.$$

Definition 3: *In the sublinear expectation space* $(\Omega, \mathcal{H}, \hat{E})$*, the canonical process* $(B_t)_{t\geq 0}$ *is defined as a* G *−Brownian motion if it satisfies the following properties, for* $t, s \geq 0$:

1. $B_0 = 0$.
2. The increment $B_{t+s} - B_t \overset{d}{=} \sqrt{s}X$, where X is G −normal distributed.
3. The increment $B_{t+s} - B_t$ is independent from $(B_{t_1}, B_{t_2}, \dots, B_{t_n})$ for each $n \in N$, and $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n \leq t$.

We now define the space $M_G^{0,p}(0,T)$, which consists of $\mathbb{R}^d$ −valued elementary processes that are $\mathcal{F}$ −progressively measurable.

We focus on a specific class of simple processes characterized by :

$$\eta(t) = \sum_{i=0}^{n-1} \xi_{t_i}(\omega)\mathbf{I}_{[t_i,t_{i+1}]}(t),$$

where ξ_i belong to $L_G^p(\Omega_{t_i})$ for $i = 0,1,\dots,n-1$ and $M_G^{p,0}(0,T)$ is the collection containing the above type of processes.

The space $M_G^p(0,T)$ is defined as the completion of $M_G^{0,p}(0,T)$ with respect to the norm :

$$||\eta||^p_{M_G^p(0,T)} = \int_0^T \hat{E}[|\eta_v|^p]ds$$

and for $1 \leq p \leq q$ we have : $M_G^p(0,T) \supset M_G^q(0,T)$.

Definition 4: *For any* $\eta_t \in M_G^{0,2}(0,T)$*, the G-Itô integral can be described by the following properties:*

$$I(\eta) = \int_0^T \eta_v dB_v = \sum_{i=0}^{N-1} \xi_i\left(B_{t_{i+1}} - B_{t_i}\right).$$

The mapping $I: M_G^{0,2}(0,T) \to L_G^2(\mathcal{F}_T)$ is both linear and continuous, which could be extended to a continuous extension $I: M_G^2(0,T) \to L_G^2(\mathcal{F}_T)$. It holds that :

1. $\hat{E}\left[\int_0^T \eta_s dB_s\right] = 0$.
2. $\hat{E}\left[\left(\int_0^T \eta_s dB_s\right)^2\right] \leq \int_0^T \hat{E}[\eta_t^2]dt$.

A continuous and non-decreasing process denoted as $\{\langle B\rangle_t, t \geq 0\}$, with the initial condition $\langle B\rangle_0 = 0$, called the quadratic variation process of G −Brownian motion B.

This quadratic variation process can be formilated within $L_G^2(\mathcal{F}_T)$ as a continuous, $d \times d$ −symmetric-matrix-valued process, defined by :

$$\langle B \rangle_t = B_t \otimes B_t - 2\int_0^t B_s \otimes dB_s.$$

Proposition 5: *Consider η and ζ as elements of $M_G^2(0,T)$, with $0 \leq s \leq r \leq t \leq T$. Then, within the space $L_G^1(\mathcal{F}_T)$, the following properties hold :*

1. $\int_s^t \eta_v dB_v = \int_s^r \eta_v dB_v + \int_r^t \eta_v dB_v$.
2. $\int_s^t (\alpha\eta_v + \zeta_v) dB_v = \alpha \int_s^t \eta_v dB_v + \int_s^t \zeta_v dB_v$, if α is bounded and in $L_G^1(\mathcal{F}_s)$.
3. $\hat{E}\left[X + \int_r^T \eta_v dB | \mathcal{F}_s\right] = \hat{E}[X]$ where $X \in L_G^1(\mathcal{F})$.

Lemma 6: *Let $\eta \in M_G^2(0,T)$, then we have :*

$$\hat{E}\left[\left(\int_0^T \eta_t dB_t\right)^2\right] = \hat{E}\left[\int_0^T \eta_s^2 d\langle B \rangle_s\right]$$
$$\leq \overline{\sigma}^2 \hat{E}\left[\int_0^T \eta_s^2 dt\right].$$

The following Burkholder Davis Gundy type inequalities are presented here, simplified to the one-dimensional case.

Lemma 7: *If $p \geq 1$ and $\eta \in M_G^p(0,T)$, there exists a positive constant $\overline{\sigma}$ such that $\frac{d\langle B \rangle_t}{dt} \leq \overline{\sigma}$ quasi-surely, then :*

$$\hat{E}\left[\sup_{0\leq t\leq T} \left|\int_0^t \eta_r d\langle B \rangle_r\right|^p\right] \leq \overline{\sigma}^{2p} T^{p-1} \int_0^T \hat{E}[|\eta_r|^p] dr,$$

Lemma 8: *For each $p \geq 2$ and $\eta \in M_G^p(0,T)$, we have :*

$$\hat{E}\left[\sup_{0\leq t\leq T} \left|\int_0^t \eta_r dB_r\right|^p\right] \leq C_p \hat{E}\left[\left|\int_s^t |\eta_r|^p dr\right|^{\frac{p}{2}}\right],$$

where C_p is a positive constant depending only on p and T.

3. FORMULATION OF THE PROBLEM

3.1 Distances Between Measures: Wasserstein Distances

In the study of the McKean-Vlasov SDE, which characterize the mean-field limit of interacting particle systems, quantifying the distance between probability measures is fundamental. This quantification is vital for evaluating the convergence of the empirical measure of a finite system to its limiting distributions and for establishing stability properties of the associated nonlinear process. Among the various metrics available, the Wasserstein metric stands out due to its compatibility with the topological structure of probability spaces and its geometric sensitivity.

Formally, define $\mathcal{P}_p(\mathbb{R}^d)$ as the space of probability measures on $\mathbb{R}^d$ with finite p −th moment, i.e., $\mu \in \mathcal{P}_p(\mathbb{R}^d)$ if $\int_{\mathbb{R}^d} |x|^p d\mu(x) < \infty$, where $p \geq 1$.For two measures $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^d)$, the p −Wasserstein distance is defined as:

$$W_p(\mu,\nu) = \inf_{\pi \in \Pi(\mu,\nu)} [\int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^p d\pi(x,y)]^{1/p}$$

Where $\Pi(\mu,\nu)$ represents is the set of all couplings (or transport plans) between μ and ν, i.e., probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals μ and ν.

When $\mu = \mathbb{P}_X$ and $\nu = \mathbb{P}_Y$ represent the probability distributions of $\mathbb{R}^d$-valued random variable X and Y with finite p −th moments, the following holds:

$$W_p(\mu,\nu) \leq \hat{\mathbb{E}} \quad |X-Y|^p]^{1/p} \tag{1}$$

3.2 Hypotheses

Consider $(\Omega, \mathcal{F}, P)$ as a probability space equipped with a filtration satisfying the usual conditions. Let (B_t) be a d-dimensional $(\mathcal{F}, P)$ −Brownian motion.

In this paper, we explore a McKean-Vlasov stochastic differential equation driven by a G-Brownian motion (G-MVSDE) given by the following form:

$$\begin{cases} dX_t = b(t, X_t, \mathbb{P}_{X_t})dt + g(t, X_t, \mathbb{P}_{X_t})d\langle B\rangle_t + \sigma(t, X_t, \mathbb{P}_{X_t})dB_t \\ X_0 = x, \end{cases} \tag{2}$$

whose coefficients b, g and σ are functions of both the state process and its distribution. We impose the following conditions on the functions b, g and σ where :

$$b : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathbb{R}^d$$
$$\sigma, g : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^d,$$

$(\mathbf{H_1})$ b, g and σ are measurable and continuous with respect to (x, μ), Moreover, there exists a constant $C > 0$ such that :

$$|b(t,x,\mu)| + |g(t,x,\mu)| + |\sigma(t,x,\mu)| \leq C(1+|x|)$$

$(\mathbf{H_2})$ b, g and σ satisfies a Lipschitz condition in (x, μ) uniformly in t, i.e., there exists $L > 0$ such that:

$$\begin{aligned}
&|b(t,x,\mu_1) - b(t,y,\mu_2)| \leq L(|x-y| + W_2(\mu_1,\mu_2)) \\
&|g(t,x,\mu_1) - g(t,y,\mu_2)| \leq L(|x-y| + W_2(\mu_1,\mu_2)) \\
&|\sigma(t,x,\mu_1) - \sigma(t,y,\mu_2)| \leq L(|x-y| + W_2(\mu_1,\mu_2)).
\end{aligned}$$

Theorem 9: *Under* $(\boldsymbol{H}_1)$ *and* $(\boldsymbol{H}_2)$, *for any specified initial value* $X_0 \in \mathbb{R}^d$, *the G-MVSDE (2) admits a unique strong solution.*

Proof: See [14].

4. CONSTRUCTION OF SOLUTION BY APPROXIMATION

Assume that the functions $b(t,x,\mu)$, $g(t,x,\mu)$ and $\sigma(t,x,\mu)$ satisfy the conditions $(\mathbf{H_1})$ and $(\mathbf{H_2})$.The objective is to establish the convergence of the Picard iteration scheme, which serves as an effective numerical approach for computing the unique solution of (2). We initialize the sequence by setting $(X_t^0) = x$ for all $t \in [0,T]$ and recursively define (X_t^{n+1}) as the solution to the following equation.

$$\begin{cases} dX_t^{n+1} = b(t, X_t^n, \mathbb{P}_{X_t^n})dt + g(t, X_t^n, \mathbb{P}_{X_t^n})d\langle B\rangle_t + \sigma(t, X_t^n, \mathbb{P}_{X_t^n})dB_t \\ X_0^{n+1} = x. \end{cases}$$

Theorem 10: *Assuming conditions* $(\boldsymbol{H_1})$ *and* $(\boldsymbol{H_2})$*hold, the sequence* (X^n) *approaches the unique solution of the equation (2) as* n *increases, i.e.*

$$\lim_{n\to+\infty} \hat{E}[\sup_{t\leq T}|X_t^n - X_t|^2] = 0$$

Proof: Let $n \geq 0$, using the inequality $|a+b+c+d|^2 \leq 4(|a|^2 + |b|^2 + |c|^2 + |d|^2)$, we get:

$$\begin{aligned}
&|X_t^{n+1} - X_t^n|^2 \leq 3\left(\int_0^t \left|b(s, X_s^n, P_{X_s^n}) - b(s, X_s^{n-1}, P_{X_s^{n-1}})\right|ds\right)^2 \\
&+3\left(\int_0^t \left|g(s, X_s^n, P_{X_s^n}) - g(s, X_s^{n-1}, P_{X_s^{n-1}})\right|d\langle B\rangle_s\right)^2 \\
&+3\left(\int_0^t \left|\sigma(s, X_s^n, P_{X_s^n}) - \sigma(s, X_s^{n-1}, P_{X_s^{n-1}})\right|dB_s\right)^2
\end{aligned}$$

by taking the sublineair expectation and application of the usual arguments such as Schwarz inequality and Burkholder-Davis-Gundy inequality for the martingale part, we get we have:

$$\hat{E}\left[\sup_{t\leq T}|X_t^{n+1} - X_t^n|^2\right] \leq 3T\hat{E}\left[\int_0^t \left|b(s, X_s^n, P_{X_s^n}) - b(s, X_s^{n-1}, P_{X_s^{n-1}})\right|^2 ds\right]$$

$$+3\overline{\sigma}^4 T\hat{E}\left[\int_0^t \left|g(s,X_s^n,P_{X_s^n}) - g(s,X_s^{n-1},P_{X_s^{n-1}})\right|^2 ds\right]$$
$$+3C_2\hat{E}\left[\int_0^t \left|\sigma(s,X_s^n,P_{X_s^n}) - \sigma(s,X_s^{n-1},P_{X_s^{n-1}})\right|^2 ds\right]$$

as b,σ and g being Lipschitz continuous with respect to (x,μ), and the property of the $2-$Wasserstein distance (1), we obtain,

$$\hat{E}\left[\sup_{t\leq T}|X_t^n - X_t|^2\right] \leq 3(T+\overline{\sigma}^4T + C_2)L^2\int_0^t \hat{E}[|X_s^n - X_s^{n-1}|^2] + W_2^2(P_{X_s^n},P_{X_s^{n-1}})ds$$
$$\leq 6(T+\overline{\sigma}^4T + C_2)L^2\int_0^t \hat{E}[|X_s^n - X_s^{n-1}|^2]ds$$
$$\leq 6(T+\overline{\sigma}^4T + C_2)L^2\int_0^t \hat{E}\left[\sup_{s\leq t}|X_s^n - X_s^{n-1}|^2\right]ds$$

Then for all $n \geq 1$, and $t \leq T$

$$\hat{E}\left[\sup_{t\leq T}|X_t^1 - X_t^0|^2\right] \leq 3T\hat{E}\left[\int_0^t |b(s,x,\mu)|^2 ds\right] +$$
$$3\overline{\sigma}^4T\hat{E}\left[\int_0^t |g(s,x,\mu)|^2 ds\right] + 3C_2\hat{E}\left[\int_0^t |\sigma(s,x,\mu)|^2 ds\right]$$
$$\leq 3(T+\overline{\sigma}^4T + C_2)C(1+\hat{E}(|x|^2)T$$
$$\leq A_1T$$

The constant A_1 is determined solely by the parameters $C_2, M, T, \overline{\sigma}$ and $E[|x|^2]$. Thus, through an inductive argument on n, we deduce

$$\hat{E}[\sup_{t\leq T}|X_t^{n+1} - X_t^n|^2] \leq \frac{A_2^{n+1}T^{n+1}}{(n+1)!}.$$

This specifically indicates that (X_t^n) forms Cauchy sequence in $L^2(\Omega, \mathcal{C}([0,T],\mathbb{R}^d))$, a space known to be complete. Consequently, (X_t^n) converges to a limit process (X_t) which serves as the unique solution to (2).

5. VARIATION OF SOLUTIONS WITH RESPECT TO INITIAL CONDITION

In this part of work, we will examine the stability properties of G-MVSDEs under a small perturbations in the initial condition.

Let (X_t^n) represent the the unique solution of (3),

$$\begin{cases} dX_t^x = b(t,X_t^n,\mathbb{P}_{X_t^n})dt + g(t,X_t^n,\mathbb{P}_{X_t^n})d\langle B\rangle_t + \sigma(t,X_t^n,\mathbb{P}_{X_t^n})dB_t \\ X_0^n = x. \end{cases} \tag{3}$$

Theorem 11: *Assume that $b(t,x,\mu)$,$\sigma(t,x,\mu)$ and $g(t,x,\mu)$ satisfy H_1 and H_2, then*

$$\lim_{n\to+\infty}\hat{E}\left[\sup_{t\le T}|X_t^n-X_t|^2\right]=0$$

Proof: Assume that (x^n) be a sequence in $\mathbb{R}^d$ converging to x,

$$\begin{aligned}|X_t^n-X_t|^2 \le 4|x^n-x|^2 &+ 4\left|\int_0^t \left(b(s,X_s^n,P_{X_s^n})-b(s,X_s^n,P_{X_s^n})\right)ds\right|^2\\ &+4\left|\int_0^t \left(g(s,X_s^n,P_{X_s^n})-g(s,X_s^n,P_{X_s^n})\right)d\langle B\rangle_s\right|^2\\ &+4\left|\int_0^t \left(\sigma(s,X_s^n,P_{X_s^n})-\sigma(s,X_s^n,P_{X_s^n})\right)dB_s\right|^2\end{aligned}$$

by taking the sublineair expectation, we have

$$\begin{aligned}\hat{E}\left[\sup_{t\le T}|X_t^n-X_t|^2\right] \le 4\hat{E}[|x^n-x|^2] +&\\ &4\hat{E}\left[\sup_{s\le t}\int_0^t \left|b\left(s,X_s^n,P_{X_s^n}\right)\, b(s,X_s^n,P_{X_s^n})\right|ds\right]^2\\ &+4\hat{E}\left[\sup_{s\le t}\int_0^t \left|g(s,X_s^n,P_{X_s^n})-g(s,X_s^n,P_{X_s^n})\right|d\langle B\rangle_s\right]^2\\ &+4\hat{E}\left[\sup_{s\le t}\int_0^t \left|\sigma(s,X_s^n,P_{X_s^n})-\sigma(s,X_s^n,P_{X_s^n})\right|dB_s\right]^2\end{aligned}$$

we apply Schwarz and BDG type inequalities to obtain

$$\begin{aligned}\tilde{E}\left[\sup_{t\le T}|X_t^n-X_t|^2\right] \le 4\hat{E}[|x^n-x|^2] + 4T\hat{E}\left[\sup_{s\le t}\int_0^t \left|b(s,X_s^n,P_{X_s^n})-\right.\right.\\ \left.\left. b(s,X_s^n,P_{X_s^n})\right|^2 ds\right]\\ +4\overline{\sigma}^4 T\hat{E}\left[\sup_{s\le t}\int_0^t \left|g(s,X_s^n,P_{X_s^n})-g(s,X_s^n,P_{X_s^n})\right|^2 ds\right]\\ +4C_2\hat{E}\left[\sup_{s\le t}\int_0^t \left|\sigma(s,X_s^n,P_{X_s^n})-\sigma(s,X_s^n,P_{X_s^n})\right|^2 ds\right]\end{aligned}$$

The Lipschitz condition implies that

$$\hat{E}\left[\sup_{t\le T}|X_t^n-X_t|^2\right] \le 4\hat{E}[|x^n-x|^2] + 4(T+\overline{\sigma}^4 T+C_2)L^2\int_0^t \hat{E}\left[\sup_{s\le t}|X_s^n-X_s|^2\right] + W_2^2(P_{X_s^n},P_{X_s})ds$$

Since

$$W_2(P_{X_s^n},P_{X_s}) \le \hat{E}[|X_s^n-X_s|^2]^{1/2}$$

then, we have

$$\hat{E}\left[\sup_{t\leq T}|X_t^n - X_t|^2\right] \leq 4\hat{E}[|x^n - x|^2] + 8(T + \overline{\sigma}^4 T + C_2)L^2 \int_0^t \hat{E}\left[\sup_{s\leq t}|X_s^n - X_s|^2\right] ds$$

In conclusion, we apply Gronwall's lemma to establish that

$$\hat{E}\left[\sup_{t\leq T}|X_t^n - X_t|^2\right] \leq 4\hat{E}[|x^n - x|^2] e^{8(T+\overline{\sigma}^4 T + C_2)L^2 t}$$

As we mentioned at the beginning of the proof that x_n converging to x, we get $\hat{E}[\sup_{t\leq T}|X_t^n - X_t|^2] = 0$.

6. VARIATION OF SOLUTIONS WITH RESPECT TO THE COEFFICIENTS

Our main objectif in this section is to demonstrate the stability of the G-MVSDE under small perturbations in the coefficients b, g and σ. To achieve this, we define sequences of functions $(b_n), (g_n)$ and (σ_n), and examine the behavior of the associated equation :

$$\begin{cases} dX_t^n = b_n(t, X_t^n, \mathbb{P}_{X_t^n})dt + g_n(t, X_t^x, \mathbb{P}_{X_t^x})d\langle B\rangle_t + \sigma_n(t, X_t^n, \mathbb{P}_{X_t^n}) \\ X_0^n = x \end{cases} \tag{4}$$

The theorem below establishes that the solution exhibits continuous dependence on the coefficients.

Theorem 12: *Suppose that the functions $b(t,x,\mu)$, $b_n(t,x,\mu)$, $g(t,x,\mu)$, $g_n(t,x,\mu)$, $\sigma(t,x,\mu)$ and $\sigma_n(t,x,\mu)$ satisfy $(\boldsymbol{H_1})$ and $(\boldsymbol{H_2})$. Additionally, assume that for every $T > 0$, and every compact set K , there exists a positive constant C such that the following conditions are satisfied.*

1) $(|b_n(t,x,\mu)| + |g_n(t,x,\mu)| + |\sigma_n(t,x,\mu)|) \leq C(1 + |x|)$,
2) $\lim_{n\to\infty}\sup_{t\leq T}\sup_{x\in K}\sup_{\mu\in\mathcal{P}_2(\mathbb{R}^d)}||b_s|| + ||g_s|| + ||\sigma_s|| = 0$

such that

$$\begin{aligned} b_s &= b_n(t,x,\mu) - b(t,x,\mu) \\ g_s &= g_n(t,x,\mu) - g(t,x,\mu) \\ \sigma_s &= \sigma_n(t,x,\mu) - \sigma(t,x,\mu) \end{aligned}$$

then

$$\lim_{n\to\infty}\hat{E}\left[\sup_{t\leq T}|X_t^n - X_t|^2\right] = 0,$$

were (X_t^n) and (X_t) represent the solutions to equations (4) and (2) respectively.

Proof: For each $n \in \mathbb{N}$, let (X_t^n) be a solution of 4, then

$$\begin{aligned}|X_t^n - X_t|^2 \leq\ & 4\left|\int_0^t \left(b_n(s, X_s^n, P_{X_s^n}) - b_n(s, X_s, P_{X_s})\right)ds\right|^2 \\ & +4\left|\int_0^t \left(b_n(s, X_s, P_{X_s}) - b(s, X_s, P_{X_s})\right)ds\right|^2 \\ & +4\left|\int_0^t \left(g_n(s, X_s^n, P_{X_s^n}) - g_n(s, X_s, P_{X_s})\right)d\langle B\rangle_s\right|^2 \\ & +4\left|\int_0^t \left(g_n(s, X_s, P_{X_s}) - g(s, X_s, P_{X_s})\right)d\langle B\rangle_s\right|^2 \\ & +4\left|\int_0^t \left(\sigma_n(s, X_s^n, P_{X_s^n}) - \sigma_n(s, X_s, P_{X_s})\right)dB_s\right|^2 \\ & +4\left|\int_0^t \left(\sigma_n(s, X_s, P_{X_s}) - \sigma(s, X_s, P_{X_s})\right)dB_s\right|^2\end{aligned}$$

Using the Lipschitz condition and the BDG inequality, we can show that

$$\begin{aligned}\hat{E}\left[\sup_{t\leq T}|X_t^n - X_t|^2\right] \leq\ & 4(T + \overline{\sigma}^4 T + C_2)L^2 \int_0^t \hat{E}\left[\sup_{s\leq t}|X_s^n - X_s|^2\right] + W_2^2(P_{X_s^n}, P_{X_s})ds \\ & +4(T + \overline{\sigma}^4 T + C_2)\hat{E}\left[\left|\int_0^t \left(b_n(s, X_s, P_{X_s}) - b(s, X_s, P_{X_s})\right)ds\right|^2\right] \\ & +4(T + \overline{\sigma}^4 T + C_2)\hat{E}\left[\left|\int_0^t \left(g_n(s, X_s, P_{X_s}) - g(s, X_s, P_{X_s})\right)ds\right|^2\right] \\ & +4(T + \overline{\sigma}^4 T + C_2)\hat{E}\left[\left|\int_0^t \left(\sigma_n(s, X_s, P_{X_s}) - \sigma(s, X_s, P_{X_s})\right)ds\right|^2\right] \\ & \leq 8(T + \overline{\sigma}^4 T + C_2)L^2 \int_0^t \hat{E}\left[\sup_{s\leq t}|X_s^n - X_s|^2\right] ds + K_n\end{aligned}$$

with

$$\begin{aligned}K_n = 4(T + \overline{\sigma}^4 T + C_2)\hat{E}\Bigg[&\left|\int_0^t \left(b_n(s, X_s, P_{X_s}) - b(s, X_s, P_{X_s})\right)ds\right|^2 \\ & + \left|\int_0^t \left(g_n(s, X_s, P_{X_s}) - g(s, X_s, P_{X_s})\right)ds\right|^2 \\ & + \left|\int_0^t \left(\sigma_n(s, X_s, P_{X_s}) - \sigma(s, X_s, P_{X_s})\right)ds\right|^2\Bigg]\end{aligned}$$

By applying the Gronwall inequality, we can get

$$\hat{E}\left[\sup_{t\leq T}|X_t^n - X_t|^2\right] \leq K_n e^{8(T+\overline{\sigma}^4 T + C_2)L^2 t}.$$

as $n \to +\infty$ we have from $ii)$ $K_n \to 0$ which achieves the proof.

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