Analytical and Numerical Study of a New Class of Fractional Delay Differential Equations with Caputo and Riemann–Liouville Derivatives

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Abstract

We study the analytical properties of the proposed model, focusing on the existence and uniqueness of solutions. By applying Picard's iterative method under relaxed Lipschitz conditions on the nonlinear functions, we establish results without requiring classical contraction assumptions. In addition, we develop a numerical scheme based on integral approximations and demonstrate its convergence under the same set of assumptions.

Our results contribute to the theoretical and numerical understanding of fractional systems with delay, offering a robust approach to modeling real-world phenomena with non-local, nonlinear, and memory-dependent dynamics.

Keywords: Fractional differential equations; Caputo derivative; Riemann-Liouville derivative; Delay differential equations; Picard iteration; Numerical approximation; Mixed derivatived.
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1 Introduction

Fractional differential equations (FDEs) offer a versatile mathematical framework for modeling complex dynamic behaviors across diverse scientific fields, including physics, biology, and engineering. By incorporating fractional derivatives, these equations extend traditional derivatives, providing a more accurate description of phenomena that are non-local and non-stationary.

A particularly intriguing and valuable area of research involves the integration of both Riemann-Liouville and Caputo fractional derivatives in differential equations that include a delay term. This approach has several scientific and practical advantages, such as modeling non-local temporal dynamics [11], offering flexibility in representing real-world phenomena [13, 14], and enabling thorough mathematical analysis and practical applications in technology and engineering [12].

In this study, we focus on a specific class of FDEs that combine Caputo and Riemann-Liouville fractional derivatives along with a delay term introduced by Hallaci et al[1]. The

equation under study is:

$$\begin{aligned} &\forall t \in [0, T], \ {}^{RL}D^{\alpha}\ {}^{C}D^{\beta}\varphi(t) - g(t, \varphi(t-r)) = f(t, \varphi(t-r)), \\ &\forall t \in [-r, 0], \varphi(t) = \phi(t), \\ &\lim_{t \longrightarrow 0} t^{1-\alpha}\, {}^{C}D^{\beta}\varphi(t) = 0, \ \varphi'(0) = \varphi_0. \end{aligned} \tag{1.1}$$

This equation illustrates the intricate connection between fractional calculus, delay dynamics, and nonlinearity, making it a key area of interest in both theoretical and applied mathematics. The incorporation of Caputo and Riemann-Liouville derivatives provides a robust framework for describing systems with memory and non-local effects [10], while the delay term captures the time-dependent interactions within the system. In this paper, we investigate the analytical properties and numerical solutions of this fractional delay differential equation. We explore the existence and uniqueness of solutions through Picard's iteration method, subject to certain Lipschitz conditions. For the numerical approach, we apply integration methods to approximate the solutions. Our study aims to enhance the understanding of fractional calculus and its applications within dynamical systems theory, advancing the modeling and analytical techniques for complex phenomena. In our method, we use Picard iteration to establish the existence and uniqueness of solutions under relaxed Lipschitz conditions on f and g avoiding the need for contraction assumptions as required in previous studies [1, 2, 11, 12]. These conditions are sufficient to ensure the convergence of the numerical method, thus extending the range of applicable scenarios and improving the robustness of our model in addressing real-world complexities.

2 Problem position

In this section, we introduce the fractional differential problem to be studied in this paper and his integral equivalent form. Let $f, g : [0, T] \times \mathrm{R} \to \mathrm{R}$ be two continuous functions with $f(t, 0) = g(t, 0) = 0$, where $T > 0$. Our goal is to solve numerically the following fractional differential equation (FDE) with a delay $r > 0$

$$\begin{aligned} &\forall t \in [0, T], \ {}^{RL}D^{\alpha}\ {}^{C}D^{\beta}\varphi(t) - g(t, \varphi(t-r)) = f(t, \varphi(t-r)), \\ &\forall t \in [-r, 0], \varphi(t) = \phi(t), \\ &\lim_{t \longrightarrow 0} t^{1-\alpha}\, {}^{C}D^{\beta}\varphi(t) = 0, \ \varphi'(0) = \varphi_0. \end{aligned} \tag{2.1}$$

where $0 < \alpha \leq 1$, $1 < \beta \leq 2$, ϕ is a given continuous function and φ is the unknown to be found in $C^0[0, T]$, ${}^{RL}D^{\alpha}$ the Riemann-Liouville fractional derivative is given, for $\alpha \in]n-1, n[$, $n \in \mathrm{N}$, by

$$\forall t \geq 0, \ {}^{RL}D^{\alpha}\psi(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1}\, \psi(s)\, ds,$$

and ${}^{C}D^{\alpha}$ is the Caputo fractional derivative, which is given for $\alpha \in]n-1, n[$, $n \in \mathrm{N}$ by

$$\forall t \geq 0, \quad {}^{C}D^{\alpha}\psi(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1}\, \psi^{(n)}(s)\, ds,$$

where, Γ is the Euler gamma function defined for α by

$$\Gamma(\alpha) = \int_0^{\infty} e^{-\eta}\eta^{\alpha-1}d\eta.$$

let $k \in \mathbb{R}$ satisfies that $|k| \leq \frac{\lambda-1}{2}$ clerly $\lambda + k > 0$ Hallaci et al. [1] showed that (FDE) 2.1 is equivalent to the following Volterra type integral equation (VIE),

$$\begin{cases} \varphi(t) = \phi(0)e^{-kt} + \frac{1-e^{-kt}}{k}\varphi'(0) + k\int_0^t e^{-k(t-s)}\varphi(s)ds \\ \quad + \frac{1}{\Gamma(\alpha+\beta-1)}\int_0^t\int_0^t e^{-k(t-s)}(s-u)^{\alpha+\beta-2}\, dsf(u,\varphi(u-r))du \\ \quad + \frac{1}{\Gamma(\beta-1)}\int_0^t\int_0^u e^{-k(t-s)}(s-u)^{\beta-2}\, dsg(u,\varphi(u-r))du, \ \forall\, t \in [0,T] \\ \varphi(t) = \phi(t), \ \forall\, t \in [-r,0] \end{cases} \tag{2.2}$$

In the next section, we are going to deal directly with the integral version (VIE) of our problem. Firstly, we show that (FIE) has a unique solution $\varphi \in C_\lambda[0,T]$ the class of all continous functions defined on $[0,T]$ with the norm,

$$\forall \psi \in C_\lambda[0,T], |\psi|_\lambda = \max_{0\leq t\leq T} |e^{-\lambda t}\psi(t)|.$$

for all positive real number $\lambda > 1$, Also $C_r = C([-r,0])$ is endowed with norm

$$\forall \psi \in C_r, |\psi|_r = \max_{0\leq t\leq T} |\psi(t)|.$$

Secondly, we use numerical integration to build an approximation of φ.

3 Analytical approach

Our goal in this section is to build acceptable hypotheses in order to obtain a unique solution u of the following (VIE).

$$\begin{cases} \varphi(t) = \phi(0)e^{-kt} + \frac{1-e^{-kt}}{k}\varphi'(0) + k\int_0^t e^{-k(t-s)}\varphi(s)ds \\ \quad + \frac{1}{\Gamma(\alpha+\beta-1)}\int_0^t\int_0^t e^{-k(t-s)}(s-u)^{\alpha+\beta-2}\, dsf(u,\varphi(u-r))du \\ \quad + \frac{1}{\Gamma(\beta-1)}\int_0^t\int_0^u e^{-k(t-s)}(s-u)^{\beta-2}\, dsg(u,\varphi(u-r))du, \ \forall\, t \in [0,T] \\ \varphi(t) = \phi(t), \ \forall\, t \in [-r,0] \end{cases} \tag{3.1}$$

Changing $u - r$ by μ, we get

$$\begin{cases} \text{For } u = 0 \quad \text{so } \mu = -r \\ \text{For } u = t \quad \text{so } \mu = t - r \end{cases}$$

$$\begin{cases} \varphi(t) = \phi(0)e^{-kt} + \frac{1-e^{-kt}}{k}\varphi'(0) + k\int_0^t e^{-k(t-s)}\varphi(s)ds \\ \quad + \frac{1}{\Gamma(\alpha+\beta-1)}\int_{t-r}^t\int_{\mu+r}^t e^{-k(t-s)}(s-\mu-r)^{\alpha+\beta-2}\, dsf(\mu+r,\varphi(\mu))d\mu \\ \quad + \frac{1}{\Gamma(\beta-1)}\int_{-r}^t\int_{\mu+r}^t e^{-k(t-s)}(s-\mu-r)^{\beta-2}\, dsg(\mu+r,\varphi(\mu))d\mu, \ \forall\, t \in [0,T] \\ \varphi(t) = \phi(t), \ \forall\, t \in [-r,0] \end{cases} \tag{3.2}$$

But, for all $t \in [-r, 0]$, $\varphi(t) = \phi(t)$, then $\forall\, t \in [0, T]$

$$\begin{aligned}\varphi(t) &= h(t) + k\int_0^t e^{-k(t-s)}\varphi(s)ds \\ &+ \frac{1}{\Gamma(\alpha+\beta-1)}\int_{t-r}^{t}\int_{\mu+r}^{t} e^{-k(t-s)}(s-\mu-r)^{\alpha+\beta-2}\, dsf(\mu+r, \varphi(\mu))d\mu \\ &+ \frac{1}{\Gamma(\beta-1)}\int_0^t\int_{\mu+r}^{t} e^{-k(t-s)}(s-\mu-r)^{\beta-2}\, dsg(\mu+r, \varphi(\mu))d\mu,\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}h(t) &= \frac{1}{\Gamma(\alpha+\beta-1)}\int_{-r}^{0}\int_{\mu+r}^{t} e^{-k(t-s)}(s-\mu-r)^{\alpha+\beta-2}\, dsf(\mu+r, \phi(\mu))d\mu \\ &+ \frac{1}{\Gamma(\beta-1)}\int_{-r}^{0}\int_{\mu+r}^{t} e^{-k(t-s)}(s-\mu-r)^{\beta-2}\, dsg(\mu+r, \phi(\mu))d\mu \\ &+ \phi(0)e^{-kt} + \frac{1-e^{-kt}}{k}\varphi'(0)\end{aligned} \tag{3.4}$$

Theorem 3.1

Assume that f and g verify the following Lipshitz proprieties

$$(H_1): \exists L_1 > 0,\ \forall t \in [0, T],\ \forall x, \bar{x} \in \mathrm{R}\ |f(t, x) - f(t, \bar{x})| \le L_1|x - \bar{x}|,$$
$$(H_2): \exists L_2 > 0,\ \forall t \in [0, T],\ \forall x, \bar{x} \in \mathrm{R}\ |g(t, x) - g(t, \bar{x})| \le L_2|x - \bar{x}|.$$

Then, equation (3.3) admits a unique solution $\varphi \in C_\lambda[0, T]$.

Proof. We introduce a Picard sequence $\{u_n\}_{n\in\mathrm{N}}$ as $\forall t \in [0, T],\ \forall n \ge 0$

$$\begin{aligned}&\varphi_0 = h(t), \\ &\varphi_{n+1}(t) = h(t) + k\int_0^t e^{-k(t-s)}\varphi_n(s)ds \\ &+ \frac{1}{\Gamma(\alpha+\beta-1)}\int_0^{t-r}\int_{\mu+r}^{t} e^{-k(t-s)}(s-\mu-r)^{\alpha+\beta-2}\, dsf(\mu+r, \varphi_n(\mu))d\mu \\ &+ \frac{1}{\Gamma(\beta-1)}\int_0^{t-r}\int_{\mu+r}^{t} e^{-k(t-s)}(s-\mu-r)^{\beta-2}\, dsg(\mu+r, \varphi_n(\mu))d\mu,\end{aligned} \tag{3.5}$$

For ease of manipulation it is convenient to introduce

$$\psi_n(t) = \varphi_n(t) - \varphi_{n-1}(t),\ n = 1, 2, \ldots,$$

With

$$\psi_0(t) = h(t)$$

To facilitate the calculations, we assume the following notations:

$$K_1(t,\mu) = \int_{\mu+r}^{t} e^{-k(t-s)}(s-\mu-r)^{\alpha+\beta-2}\,ds$$
$$K_2(t,\mu) = \int_{\mu+r}^{t} e^{-k(t-s)}(s-\mu-r)^{\beta-2}\,ds$$

We obtain, $\forall t \in [0,T],\ \forall n \geq 1$, we have

$$\begin{aligned}
|e^{-\lambda t}\psi_n(t)| &= |e^{-\lambda t}(\varphi_{n+1}(t)-\varphi_n(t))| \\
&\leq ke^{-\lambda t}\int_0^t e^{-k(t-s)}|\varphi_n(s)-\varphi_{n-1}(s)|ds \\
&+ \frac{e^{-\lambda t}}{\Gamma(\alpha+\beta-1)}\int_0^{t-r} K_1(t,\mu)\,|f(\mu+r,\varphi_n(\mu))-f(\mu+r,\varphi_{n-1}(\mu))|d\mu \\
&+ \frac{e^{-\lambda t}}{\Gamma(\beta-1)}\int_0^{t-r} K_2(t,\mu)\,|g(\mu+r,\varphi_n(\mu))-g(\mu+r,\varphi_{n-1}(\mu))|d\mu
\end{aligned}$$

Applying the lipschitz condition H_1 and H_2 gives

$$\begin{aligned}
|e^{-\lambda t}\psi_n(t)| &\leq k\int_0^t e^{-(k+\lambda)(t-\mu)}|e^{-\lambda\mu}(\varphi_n(\mu)-\varphi_{n-1}(\mu))|d\mu \\
&+ \frac{L_f}{\Gamma(\alpha+\beta-1)}\int_0^{t-r} e^{-\lambda\mu}K_1(t,\mu)\,|\varphi_n(\mu)-\varphi_{n-1}(\mu)|d\mu \\
&+ \frac{L_g}{\Gamma(\beta-1)}\int_0^{t-r} e^{-\lambda\mu}K_2(t,\mu)\,|\varphi_n(\mu)-\varphi_{n-1}(\mu)|d\mu
\end{aligned}$$

With $0 \leq t \leq T$ and $0 \leq \mu \leq t-r$

$$\begin{aligned}
K_1(t,\mu) &= \int_{\mu+r}^{t} e^{-k(t-s)}(s-\mu-r)^{\alpha+\beta-2}\,ds \leq \int_{\mu+r}^{t}(s-\mu-r)^{\alpha+\beta-2}\,ds \\
&\leq \frac{(t-\mu-r)^{\alpha+\beta-1}}{\alpha+\beta-1} \leq \frac{(T-r)^{\alpha+\beta-1}}{\alpha+\beta-1} \\
K_2(t,\mu) &= \int_{\mu+r}^{t} e^{-k(t-s)}(s-\mu-r)^{\beta-2}\,ds \leq \int_{\mu+r}^{t}(s-\mu-r)^{\beta-2}\,ds \\
&\leq \frac{(T-r)^{\beta-1}}{\beta-1}
\end{aligned}$$

So that

$$\begin{aligned}
|e^{-\lambda t}\psi_n(t)| &\leq k\int_0^t |e^{-\lambda\mu}(\varphi_n(\mu)-\varphi_{n-1}(\mu))|d\mu \\
&+ \frac{L_f(T-r)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\int_0^{t-r} e^{-\lambda\mu}|\varphi_n(\mu)-\varphi_{n-1}(\mu)|d\mu \\
&+ \frac{L_g(T-r)^{\beta-1}}{\Gamma(\beta)}\int_0^{t-r} e^{-\lambda\mu}|\varphi_n(\mu)-\varphi_{n-1}(\mu)|d\mu \\
&\leq k\int_0^t |e^{-\lambda\mu}(\varphi_n(\mu)-\varphi_{n-1}(\mu))|d\mu \\
&+ \frac{L_f(T-r)^{\alpha+\beta-}}{\Gamma(\alpha+\beta)}\int_0^t e^{-\lambda\mu}|\varphi_n(\mu)-\varphi_{n-1}(\mu)|d\mu \\
&+ \frac{L_g(T-r)^{\beta}}{\Gamma(\beta)}\int_0^t e^{-\lambda\mu}|\varphi_n(\mu)-\varphi_{n-1}(\mu)|d\mu \\
&\leq k+\frac{L_f(T-r)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}+\frac{L_g(T-r)^{\beta-1}}{\Gamma(\beta)}\int_0^t e^{-\lambda\mu}|\psi_{n-1}(\mu)|d\mu
\end{aligned}$$

By repeating the last inequality n-times we have $\forall\, t \in [0,T]$,

$$||\psi_n(t)||_\lambda \leq k+\frac{L_f(T-r)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}+\frac{L_g(T-r)^{\beta-1}}{\Gamma(\beta)}\frac{T^n}{n!}H \tag{3.6}$$

Where

$$H = k\int_0^t |h(0)|ds+\frac{L_f(T-r)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\int_0^t |f(u+r,h(0)|du+\frac{L_g(T-r)^{\beta-1}}{\Gamma(\beta)}\int_0^t |g(u+r,h(0)|du.$$

proving that $\sum_{n=1}^{\infty}\psi_n(t)$ is uniformly convergent in $C_\lambda[0,T]$

i.e

$$\forall t\in[0,T]: \sum_{n=0}^{\infty}\psi_n(t) = \sum_{n=1}^{\infty}(\varphi_n(t)-\varphi_{n-1}(t))+h(t) = \lim_{n\longrightarrow+\infty}\varphi_n(t)=\varphi(t). \tag{3.7}$$

To prove that $\varphi(t)$ defined by (3.7) satisfies of (3.3), set

$$\varphi(t)=\varphi_n(t)+\delta_n(t) \tag{3.8}$$

From (3.5)

$$\begin{aligned}
\varphi(t)-\delta_{n+1}(t) &= h(t)+k\int_0^t e^{-k(t-s)}(\varphi(s)-\delta_n(s))\,ds \\
&+ \frac{1}{\Gamma(\alpha+\beta-1)}\int_0^{t-r}\int_{\mu+r}^t e^{-k(t-s)}(s-\mu-r)^{\alpha+\beta-2}\,dsf(\mu+r,(\varphi(\mu)-\delta_n(\mu)))d\mu \\
&+ \frac{1}{\Gamma(\beta-1)}\int_0^{t-r}\int_{\mu+r}^t e^{-k(t-s)}(s-\mu-r)^{\beta-2}\,dsg(\mu+r,(\varphi(\mu)-\delta_n(\mu)))d\mu,
\end{aligned}$$

Applying Theorem 4.1 page 52 from [4], we obtain

$$\lim_{n\to+\infty} |\delta_n(t)| = 0$$

To show uniqueness, we assume the existence of another continuous solution of 3.3 denoted$\tilde{\varphi}$.
Then, $\forall [0, T]$

$$|\varphi(t) - \tilde{\varphi}(t)| \leq \left(k + \frac{L_f (T-r)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + \frac{L_g (T-r)^{\beta-1}}{\Gamma(\beta)}\right) \int_0^t |\varphi(\mu) - \tilde{\varphi}(\mu)| d\mu.$$

We use Lemma 3 from [3] to obtain $\forall t \in [0, T]$, $\varphi(t) - \tilde{\varphi}(t) = 0$ proving the uniqueness. ■

4 Numerical approach

From the previous section, we understand that if (H_1) and (H_2) are verified, the system

$$\varphi(t) = \phi(t), \quad t \in [-r, 0], \tag{4.1}$$

$$\begin{aligned}\varphi(t) = \phi(0)e^{-kt} + \frac{1-e^{-kt}}{\int k_{t-r}} \int_t \varphi_0 + k \int_0^t e^{-k(t-s)} \varphi(s) ds \\ + \frac{1}{\Gamma(\alpha+\beta-1)} \int_{t-r} \oint_t \int_{u+r} e^{-k(t-s)} (s-u-r)^{\alpha+\beta-2} ds f(u+r, \varphi(u)) du \\ + \frac{1}{\Gamma(\beta-1)} \int_0 \int_{u+r} e^{-k(t-s)} (s-u-r)^{\beta-2} ds g(u+r, \varphi(u)) du\end{aligned} \tag{4.2}$$

admits a unique solution $\varphi \in C_\lambda[-r, T]$.

The objective of this section is to build a numerical approximation of the exact solution φ. For that, we use Nyström method because of its efficiency to deal with this type of Volterra equation as show in many previous works [2, 3, 6, 7, 5, 8]. Let us introduce a subdivision of the interval $[-r, T]$ in the following sense.
For an integer $N \geq 2$, we define $h = \frac{r}{N}$ and $M \in \mathbb{N}$, such that $-r + (M-1)h < T \leq -r + Mh$ let now $\{t_i\}_{i=1}^M$ be a subdivision of $[-r, T]$ such that

$$\begin{aligned} t_i &= -r + ih, \quad 0 \leq i \leq M-1, \\ t_M &= T. \end{aligned}$$

With this subdivision, we use trapezoidal weights $\{\xi_i\}_{i=0}^M$given by:

$$\begin{aligned} \xi_0 &= \frac{h}{2}, \\ \xi_i &= h, \quad 1 \leq i \leq M-1, \\ \xi_M &= \frac{T + r - (M-1)h}{2} \end{aligned}$$

as a numerical integration method. This is the most adequate method with our equation because it verifies (see [4]):

$$\forall \varphi \in C_\lambda[-r, T], \quad \lim_{h \longrightarrow 0} \sum_{i=0}^{M} \xi_i \varphi(t_i) - \int_{-r}^{T} \varphi(t)dt = 0. \tag{4.3}$$

the following discrte system

$$\varphi_i = \phi(t_i) \quad 0 \leq i \leq N \tag{4.4}$$

$$\begin{aligned} \varphi_i &= h(t_i) + k \sum_{j=0}^{i-1} \xi_i e^{t_i - t_j} \varphi_j \\ &+ \frac{1}{\Gamma(\alpha + \beta - 1)} \sum_{j=0}^{i-N} \xi_j \sum_{k=j+N}^{k=i} \xi_k e^{-k(t_i - t_k)} (t_k - t_j - r)^{\alpha+\beta-2} f(t_j + r, \varphi_j) \\ &+ \frac{1}{\Gamma(\beta - 1)} \sum_{j=0}^{i-N} \xi_j \sum_{k=j+N}^{k=i} \xi_k e^{-k(t_i - t_k)} (t_k - t_j - r) g(t_j + r, \varphi_j) \qquad \text{for } N+1 \leq i \leq M \end{aligned} \tag{4.5}$$

where, $\varphi_i \approx \varphi(t_i)$ for $0 \leq i \leq M$.

Unlike system obtained with Nyström method in [3, 9] the delay term gives us an explicit numerical systems (4.4)-(4.5) which facilitates the convergence study and programming.

To study the convergence of our numerical method, we introduce the discrete error $\{\Delta_i\}_{i=0}^{M}$ as

$$\Delta_i = \varphi_i - \varphi(t_i), \quad 0 \leq i \leq M.$$

Now, we introduce the consistence errors for : $N+1 \leq i \leq 2N$,

$$\begin{aligned} \hat{\delta}_i^1 &= \int_0^{t_i} e^{-k(t_i - s)} \phi(s)ds - \sum_{j=0}^{i-1} \xi_i e^{t_i - t_j} \phi(t_j), \\ \hat{\delta}_i^2 &= \int_{t_i - r}^{0} \int_{u+r}^{t_i} e^{-k(t_i - s)} (s - u - r)^{\alpha+\beta-2} \, dsf(u + r, \phi(u))du \\ &- \sum_{j=0}^{i-N} \xi_j \sum_{k=j+N}^{k=i} \xi_k e^{-k(t_i - t_k)} (t_k - t_j - r)^{\alpha+\beta-2} f(t_j + r, \phi(t_j)) \ , \\ \hat{\delta}_i^3 &= \int_{t_i - r}^{0} \int_{u+r}^{t_i} e^{-k(t_i - s)} (s - u - r)^{\beta-2} \, dsg(u + r, \phi(u))du \\ &- \sum_{j=0}^{i-N} \xi_j \sum_{k=j+N}^{k=i} \xi_k e^{-k(t_i - t_k)} (t_k - t_j - r) g(t_j + r, \phi(t_j)) \ . \end{aligned}$$

But, for $2N+1 \leq i \leq M$, ($T >> r$ and $\hbar$ small enough to obtain $M \geq 2N+2$), the consistence errors are given by

$$\delta_i^1 = \int_0^{t_i} e^{-k(t_i-s)}\varphi(s)ds - \sum_{j=0}^{i-1} \xi_j e^{t_i-t_j}\varphi_j,$$

$$\delta_i^2 = \int_{t_i-r}^{0}\int_{u+r}^{t_i} e^{-k(t_i-s)}(s-u-r)^{\alpha+\beta-2}\,dsf(u+r,\varphi(u))du$$
$$- \sum_{j=N}^{i-N}\xi_j \sum_{k=j+N}^{k=i} \xi_k e^{-k(t_i-t_k)}(t_k-t_j-r)^{\alpha+\beta-2}f(t_j+r,\varphi(t_j))\Big),$$

$$\delta_i^3 = \int_{t_i-r}^{0}\int_{u+r}^{t_i} e^{-k(t_i-s)}(s-u-r)^{\beta-2}\,dsg(u+r,\varphi(u))du$$
$$- \sum_{j=N}^{i-N}\xi_j \sum_{k=j+N}^{k=i} \xi_k e^{-k(t_i-t_k)}(t_k-t_j-r)g(t_j+r,\varphi(t_j))\Big).$$

Using (4.3), it is obvious to obtain that

$$\lim_{h\longrightarrow 0}\max_{N+1\le i\le 2N}|\hat{\delta}_i^1| = \lim_{h\longrightarrow 0}\max_{N+1\le i\le 2N}|\hat{\delta}_i^2| = \lim_{h\longrightarrow 0}\max_{N+1\le i\le 2N}|\hat{\delta}_i^3|$$
$$= \lim_{h\longrightarrow 0}\max_{2N+1\le i\le M}|\delta_i^1| = \lim_{h\longrightarrow 0}\max_{2N+1\le i\le M}|\delta_i^2|$$
$$= \lim_{h\longrightarrow 0}\max_{2N+1\le i\le M}|\delta_i^3| = 0.$$

Also, it is clear that

$$\Delta_i = 0, \quad 0 \le i \le N.$$

Theorem 4.1
Supposing that $(H_1), (H_2)$ *hold, then*

$$\lim_{h\longrightarrow 0}\max_{N\le i\le M}|\Delta_i| = 0.$$

Proof. For all $2N+1 \le i \le M$, we have:

$$
\begin{aligned}
|\Delta_i| &= |\varphi_i - \varphi(t_i)| \\
&= \Big| k \sum_{j=N}^{i-1} \xi_i e^{t_i - t_j} \varphi_j \\
&+ \frac{1}{\Gamma(\alpha+\beta-1)} \sum_{j=N}^{i-N} \xi_j \sum_{k=j+N}^{k=i} \xi_k e^{-k(t_i-t_k)} (t_k - t_j - r)^{\alpha+\beta-2} f(t_j + r, \varphi_j) \Big) \\
&+ \frac{1}{\Gamma(\beta-1)} \sum_{j=N}^{i-N} \xi_j \sum_{k=j+N}^{k=i} \xi_k e^{-k(t_i-t_k)} (t_k - t_j - r)^{\beta-2} g(t_j + r, \varphi_j) \Big) \\
&- k \int_0^{t_i} e^{-k(t_i-s)} \varphi(s) ds \\
&- \frac{1}{\Gamma(\alpha+\beta-1)} \int_0^{t_i - r} \int_{u+r}^{t_i} e^{-k(t_i-s)} (s - u - r)^{\alpha+\beta-2} \, ds f(u + r, \varphi(u)) du \\
&- \frac{1}{\Gamma(\beta-1)} \int_0^{t_i - r} \int_{u+r}^{t_i} e^{-k(t_i-s)} (s - u - r)^{\beta-2} \, ds g(u + r, \varphi(u)) du \Big| \\
&\leq kh \sum_{j=N}^{i-1} e^{t_i - t_j} (|\varphi_j - \varphi(t_j)|) \\
&+ \frac{h^2}{\Gamma(\alpha+\beta-1)} \sum_{j=N}^{i-N} \sum_{k=j+N}^{k=i} e^{-k(t_i-t_k)} (t_k - t_j - r)^{\alpha+\beta-2} |f(t_j + r, \varphi_j) - f(t_j + r, \varphi(t_j))| \Big) \\
&+ \frac{h^2}{\Gamma(\beta-1)} \sum_{j=N}^{i-N} \sum_{k=j+N}^{k=i} e^{-k(t_i-t_k)} (t_k - t_j - r)^{\beta-2} |g(t_j + r, \varphi_j) - g(t_j + r, \varphi(t_j))| \Big) \\
&+ \max_{2N+1 \leq i \leq M} |\delta_i^1| + \max_{2N+1 \leq i \leq M} |\delta_i^2| + \max_{2N+1 \leq i \leq M} |\delta_i^3| \\
&\leq \frac{\Xi}{\Gamma(\beta)} \sum_{j=N}^{i-N} |\Delta_j| + \max_{2N+1 \leq i \leq M} |\delta_i^1| + \max_{2N+1 \leq i \leq M} |\delta_i^2| + \max_{2N+1 \leq i \leq M} |\delta_i^3|,
\end{aligned}
$$

Where

$$\Xi = (kh + h^2)(T - r)^{\alpha+\beta} \max\{L_1, L_2\}$$

Which means, for all $2N \leq i \leq M$

$$
\begin{aligned}
|\Delta_i| &\leq \frac{\Xi}{\Gamma(\beta) - \Xi} \sum_{j=N}^{i-N-1} |\Delta_j| \\
&+ \frac{\Gamma(\beta)}{\Gamma(\beta) - \Xi} \max_{2N+1 \leq i \leq M} |\delta_i^1| + |\delta_i^2| + |\delta_i^3| \Big) .
\end{aligned}
$$

Applying Theorem 7.1 page 101 from [4], we obtain for all $2N + 1 \leq i \leq M$,

$$|\Delta_i| \leq A_{N,M} \cdot \frac{\Gamma(\beta)}{\Gamma(\beta) - \Xi} \max_{2N+1 \leq i \leq M} |\delta_i^1| + \max_{2N+1 \leq i \leq M} |\delta_i^2| + \max_{2N+1 \leq i \leq M} |\delta_i^3| \Big) ,$$

where

$$A_{N,M} = \left(1 + \frac{\Xi}{\Gamma(\beta) - \Xi}\right)^{M-N}.$$

But,

$$\frac{1}{A_{N,M}} = \left(1 - \frac{r(k+h)\,(T-\tau)^{\alpha+\beta-1} \max\{L_1, L_2\}}{N\,\Gamma(\beta)}\right)^{M-N}.$$

Then,

$$A_{N,M} \sim \exp\left(\frac{rk + (T-\tau)^{\alpha+\beta-1} \max\{L_1, L_2\}}{\Gamma(\beta)}\right).$$

Now, for $N+1 \leq i \leq 2N$, we use the same previous steps to obtain,

$$|\Delta_i| \leq \frac{\Xi}{\Gamma(\beta)} \sum_{j=i-N}^{N} |\varepsilon_j| + \max_{N+1\leq i\leq 2N} |\delta_i^1| + \max_{N+1\leq i\leq 2N} |\delta_i^2 + \max_{N+1\leq i\leq 2N} |\delta_i^3|.$$

We use $|\Delta_j| = 0$, $i - N \leq j \leq N$, to achieve the proof. ■

5 Numerical example

We take the same analytical example studied in [1], i.e

$$\alpha = 1/2,\ \beta = 3/2,\ T = 4,\ f(t, \varphi(t-2)) = \frac{1}{5} e^{-4t} \varphi^2(t-2),$$
$$g(t, \varphi(t-2)) = -\frac{1}{4} e^{-4t} \sin(\varphi(t-2)),\ \lambda = 3,\ k = 2,\ \phi(t) = t\sin(10t).$$

If we take $\phi(t) = t\sin(10t)$, $r = 2$, the numerical results are presented in figure 1.

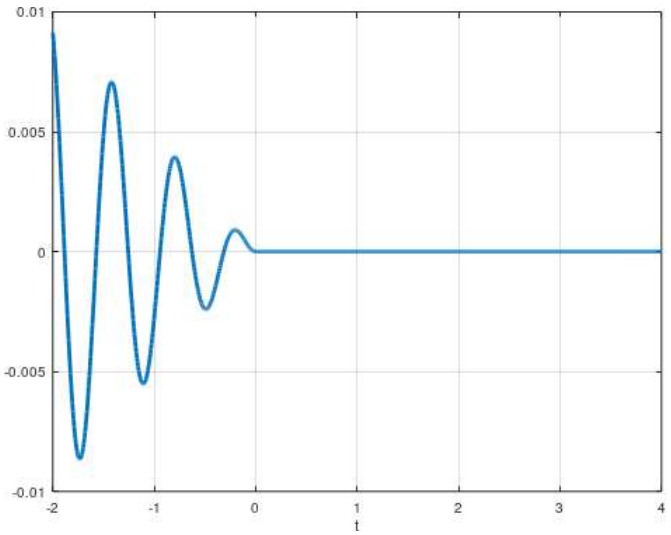

Figure 1: $\phi(t) = t\sin(10t)$, $r = 2$

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