On the Solution of Fractional Differential Equations

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Abstract

This work provides a link between the fractional derivatives and ordinary derivatives by using a weighted linear combination model. We show that every fractional differ- ential equation has a dual form in the non-fractional differential equations. Solution of the non-fractional differential equation in the dual form leads to the solution of the corresponding fractional differential equation. This method reduces complex fractional problems to ordinary differential equations which opens up several ap- plications in engineering, physics and other domains. At the end, we solve several fractional differential equations found in the literature.

Keywords: Ordinary Derivative; Fractional Derivative; Fractional Differential Equations; Solution of the Fractional Differential Equations.

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1 Introduction

The history of fractional derivative is quite old. Basic idea was initiated with the question of L. Hospital in 1695 when he asked what does the fractional derivative $g^{(0.5)}(x)$ mean? Since then, many researchers attempted to answer the question.

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Some wellknown fractional order derivatives can be searched in literature with the names of Riemann- Liouville Forms, Caputo Form, Fabrizio derivative, and many more. We refer the reader to [1] - [8] for these derivatives, their corresponding equa- tions and solutions.

It is found, see Diethelm et al. [5], that fractional derivative defined through non-zero kernal should be avoided as it leads to incorrect results. The objective and purpose of this work is to link fractional derivative with non-fractional and to transform fractional differential equations to non-fractional, as several types of solu- tion techniques exist for non-fractional differential equations. Link shows fractional derivative possesses all the properties of the ordinary derivative and even general- izes some of them. Moreover, all fractional differential equations can be converted to non-fractional ones and vice-versa.

2 Link of Fractional Derivative with the Ordinary Derivative

In this section we link fractional derivative with ordinary derivative and then discuss the related geometry. We show that fractional derivatives satisfy almost all the rules and properties of the ordinary derivatives and every fractional differential equation can be expressed as an ordinary differential equation. In last, we solve several fractional differential equations given in the literature.

Throughout this work, we denote m and n as non-negative integers such that $m < n$. Let $g : \mathbb{R} \to \mathbb{R}$ be an n order differentiable function, the derivative of $g(x)$ with respect to x is in fact the limit $\lim_{\epsilon\to 0} \frac{g(x+\epsilon)-g(x)}{\epsilon} = \frac{d}{dx} g(x)$. In another notation, we write $\frac{d}{dx} g(x) = g^{(1)}(x)$. Repeating this limit process one gets $\frac{d}{dx} g^{(1)}(x) = \lim_{\epsilon\to 0} \frac{g^{(1)}(x+\epsilon)-g^{(1)}(x)}{\epsilon} = g^{(2)}(x)$. Continuing this one gets the nth order derivative $g^{(n)}(x)$. If n is replaced by a fractional number $\varsigma \in [m, n]$; $m, n \in \{0, 1, 2, ...\}$, then $g^{(\varsigma)}(x)$ is known as ς order fractional derivative of $g(x)$.

We link the fractional order derivative $g^{(\varsigma)}(x)$ with $g(x)$ as

$$g^{(\varsigma)}(x) = \alpha_{m,n}(\varsigma, x)g^{(n)}(x) + (1 - \alpha_{m,n}(\varsigma, x))g^{(m)}(x), \quad m \leq \varsigma \leq n, \tag{2.1}$$

with convention that $g^{(0)}(x) = g(x)$, where the function $\alpha_{m,n} : [m, n] \times \mathbb{R} \to [0, 1]$ satisfies the limits

$$\alpha_{m,n}(\varsigma, x) \to 0 \text{ as } \varsigma \to m,$$

while

$$\alpha_{m,n}(\varsigma, x) \to 1 \text{ as } \varsigma \to n.$$

Moreover, we assume that $\alpha_{0,1}(\varsigma, x) = 1$ if $g(x)$ is constant.

For $n = 1$ and $m = 0$ the expression (2.1) becomes

$$g^{(\varsigma)}(x) = (1 - \alpha_{0,1}(\varsigma, x))g(x) + \alpha_{0,1}(\varsigma, x)g^{(1)}(x), \quad (2.2)$$

where $\varsigma \in [0, 1]$.

In limit form, we can express (2.2) as

$$g^{(\varsigma)}(x) = \lim_{h\to 0} \frac{\alpha_{0,1}(\varsigma, x)g(x + h) - (\alpha_{0,1}(\varsigma, x) - (1 - \alpha_{0,1}(\varsigma, x))h)g(x)}{h}. \quad (2.3)$$

From (2.1) and Figure 1, one finds if $\varsigma \to m$ then $g^{(\varsigma)}(x) \to g^{(m)}(x)$, and if $\varsigma \to n$ then $g^{(\varsigma)}(x) \to g^{(n)}(x)$.

In the next, we study an example of the function $\alpha_{m,n}(\varsigma, x)$.

Theorem 2.1. *Let $g(x) = e^{ax}$, for any real number a, then*

$$\alpha_{m,n}(\varsigma, x) = \frac{a^{\varsigma} - a^m}{a^n - a^m}.$$

Proof. If $g(x) = e^{ax}$, for any real number a, then $g^{(n)}(x) = a^n e^{ax}$ for any positive integer n and therefore, $g^{(\varsigma)}(x) = a^{\varsigma} e^{ax}$, for any positive fractional number ς. Let $\varsigma \in [m, n]$ then using (2.2), we get

$$a^{\varsigma} e^{ax} = \alpha_{m,n}(\varsigma, x)a^n e^{ax} + (1 - \alpha_{m,n}(\varsigma, x))a^m e^{ax},$$

which further gives

$$a^{\varsigma} - a^m = \alpha_{m,n}(\varsigma, x)\,(a^n - a^m)\,,$$

and this leads to the desired expression. □

For $g(x) = e^{2x}$, the geometry of the fractional derivative $g^{(\varsigma)}(x)$ is presented in Figure 1.

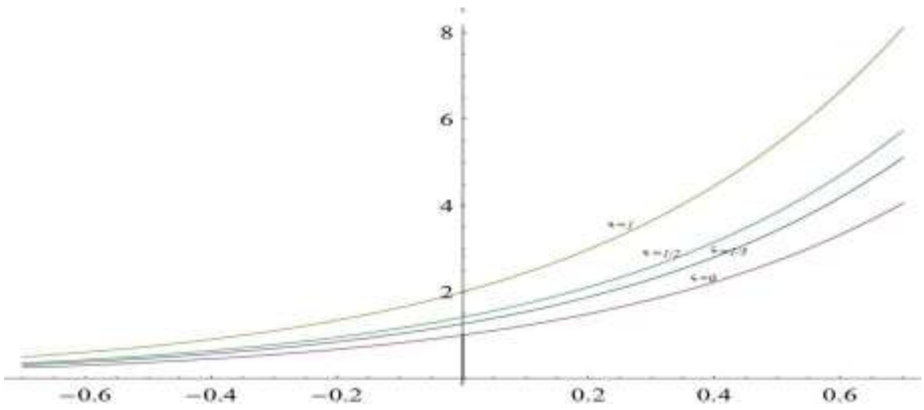

Figure 1: Graph of the derivatives $g^{(\varsigma)}(x)$, for fractional numbers $\varsigma = \frac{1}{3}$ and $\varsigma = \frac{1}{2}$, where $m = 0$ and $n = 1$, of the differentiable function $g(x) = e^{2x}$ on the interval $[-0.8, 0.8]$.

Definition 2.1. *Any function $g(x)$ is fractional differentiable of order ς, if it can be expressed as (2.1).*

2.1 Some Properties of the Proposed Fractional Derivatives

In this section, we analyze some rules and properties of the fractional derivative linked with the equation (2.1). For example

(i) If $g(x) = constant$, then (2.1) gives $g^{(\varsigma)} = 0$ for all $\varsigma \geq 1$. In case $0 < \varsigma < 1$ we use our assumption on $\alpha_{0,1}(\varsigma, x)$ to get $g^{(\varsigma)} = 0$.

(ii) To show the rule $\left(g^{(\varsigma)}\right)^{(\varrho)}(x) = g^{(\varsigma+\varrho)}(x)$, for any $\varsigma, \varrho \in [m, n]$, we specify the function $g(x)$ as we have not extensively analysed the complicated function $\alpha_{m,n}(\varsigma, x)$ for arbitrary function $g(x)$. For the result, we use Theorem 2.1 and the function $g(x) = e^{ax}$ there in.

Consider

$$\begin{aligned}
\left(g^{(\varsigma)}\right)^{(\varrho)}(x) &= \left((1-\alpha_{m,n}(\varsigma,x))g^{(m)}(x) + \alpha_{m,n}(\varsigma,x)g^{(n)}(x)\right)^{(\varrho)} \\
&= \alpha_{m,n}(\varrho,x)\left((1-\alpha_{m,n}(\varsigma,x))g^{(m)}(x) + \alpha_{m,n}(\varsigma,x)g^{(n)}(x)\right)^{(n)} \\
&\quad +(1-\alpha_{m,n}(\varrho,x))\left((1-\alpha_{m,n}(\varsigma,x))g^{(m)}(x) + \alpha_{m,n}(\varsigma,x)g^{(n)}(x)\right)^{(m)} \\
&= \frac{a^{\varrho}-a^m}{a^n-a^m}\left(\frac{a^{\varsigma}-a^m}{a^n-a^m}g^{(2n)}(x) + \left(1-\frac{a^{\varsigma}-a^m}{a^n-a^m}\right)g^{(m+n)}(x)\right) \\
&\quad + \left(1-\frac{a^{\varrho}-a^m}{a^n-a^m}\right)\left(\frac{a^{\varsigma}-a^m}{a^n-a^m}g^{(m+n)}(x) + \left(1-\frac{a^{\varsigma}-a^m}{a^n-a^m}\right)g^{(2m)}(x)\right),
\end{aligned}$$

where we have used Theorem 2.1. Further, we get

$$
\begin{aligned}
g^{(\varsigma)\,(\varrho)}(x) &= \frac{a^{\varrho}-a^{m}}{a^{n}-a^{m}}\left(\frac{a^{\varsigma}-a^{m}}{a^{n}-a^{m}}a^{2n} + \left(1-\frac{a^{\varsigma}-a^{m}}{a^{n}-a^{m}}\right)a^{m+n}\right)e^{ax} \\
&\quad + \left(1-\frac{a^{\varrho}-a^{m}}{a^{n}-a^{m}}\right)\left(\frac{a^{\varsigma}-a^{m}}{a^{n}-a^{m}}a^{m+n} + \left(1-\frac{a^{\varsigma}-a^{m}}{a^{n}-a^{m}}\right)a^{2m}\right)e^{ax} \\
&= \frac{a^{\varrho}-a^{m}}{a^{n}-a^{m}}\left(\frac{a^{\varsigma}-a^{m}}{a^{n}-a^{m}}a^{2n} + \frac{a^{n}-a^{\varsigma}}{a^{n}-a^{m}}a^{m+n}\right)e^{ax} \\
&\quad + \frac{a^{n}-a^{\varrho}}{a^{n}-a^{m}}\left(\frac{a^{\varsigma}-a^{m}}{a^{n}-a^{m}}a^{m+n} + \frac{a^{n}-a^{\varsigma}}{a^{n}-a^{m}}a^{2m}\right)e^{ax} \\
&= \frac{1}{(a^{n}-a^{m})^{2}}\left(a^{2n+\varrho+\varsigma} - 2a^{m+n+\varrho+\varsigma} + a^{2m+\varrho+\varsigma}\right)e^{ax} \\
&= a^{\varrho+\varsigma}e^{ax} \\
&= g^{(\varrho+\varsigma)}(x).
\end{aligned}
$$

Proof of $g^{(\varsigma)\,(\varrho)}(x) = g^{(\varrho+\varsigma)}(x)$ for arbitrary $g(x)$ will appear in our next work, when we will analyse the function $\alpha_{m,n}(\varsigma, x)$. We claim that our definition satisfies almost all the properties of the classical derivative.

In the following results, we discuss differentiability and convexity properties:

Theorem 2.2. *If $g^{(n)}(x)$ is differentiable then $g(x)$ is differentiable of order ς, for any $n < \varsigma < n+1$.*

Proof. If $g^{(n)}(x)$ is differentiable then $g^{(n+1)}(x)$ exists. Thus by (2.1) we can express

$$g^{(\varsigma)}(x) = \alpha_{n,n+1}(\varsigma, x)g^{(n+1)}(x) + (1-\alpha_{n,n+1}(\varsigma, x))g^{(n)}(x), \tag{2.4}$$

this expression gives the result. □

Theorem 2.3. *If a smooth function $g(x)$ is convex and increasing on a closed interval $[a, b]$ then $g^{(\varsigma)}(x)$, for all $\varsigma \in (2, 3)$, is convex on the open interval (a, b), for any $a, b \in \mathbb{R}$.*

Proof. If $g(x)$ is convex then (see Royden [12]) $g^{(2)}(x) \geq 0$, moreover if $g(x)$ is smooth and increasing then $g^{(3)}(x)$ exists and positive. Using these results and the expression

$$
\begin{aligned}
g^{(\varsigma)\,(2)}(x) &= g^{(\varsigma+2)}(x) \\
&= \alpha_{2,3}(\varsigma, x)g^{(3)}(x) + (1-\alpha_{2,3}(\varsigma, x))g^{(2)}(x) \\
&\geq 0,
\end{aligned}
\tag{2.5}
$$

this shows $g^{(\varsigma)}(x)$ is convex. □

3 Main Result and its Applications

In this section we show that the link (2.1) converts any fractional differential equation to an ordinary differential equation and vice versa if the function $\alpha_{m,n}(\varsigma, x)$ does not depend on the fraction order.

Theorem 3.1. *If $\alpha_{m,n}(\varsigma, w)$ is defined as in (2.1), then the proposed fractional derivative (2.1) expresses the fractional differential equation*

$$b_n(w, y)\frac{d^{\gamma}y}{dw^{\gamma}} + b_{n-1}(w, y)\frac{d^{\delta}y}{dw^{\delta}} + ... + b_1(w, y)\frac{d^{\beta}y}{dw^{\beta}} + b_0(w, y)y = a_0(w, y), \quad (3.1)$$

where $\gamma \in [n-1, n]$, $\delta \in [n-2, n-1]$, ..., $\beta \in [0, 1]$ with initial/boundary conditions of the form $y^{(\rho)}(c) = d$, $\rho \in [m, n]$, $y(e) = f$, to the following differential form

$$b_n(w, y)\alpha_{n-1,n}(\gamma, w)\frac{d^ny}{dw^n} + [b_n(w, y)(1 - \alpha_{n-1,n}(\gamma, w)) + b_{n-1}(w, y)\alpha_{n-2,n-1}(\delta, w)]\frac{d^{n-1}y}{dw^{n-1}} + ... + [b_1(w, y)(1 - \alpha_{0,1}(\beta, w)) + b_0(w, y)]\, y = a_0(w, y), \quad (3.2)$$

with initial/boundary conditions $\alpha_{m,n}(\rho, w)y^{(n)}(c) + (1 - \alpha_{m,n}(\rho, w))y^{(m)}(c) = d$ and $y(e) = f$.

Moreover, If $\alpha_{m,n}(\varsigma, w)$ does not depend on the variable w, for all m and n, then (3.1) reduces to the following ordinary differential equation

$$b_n(w, y)\alpha_{n-1,n}(\gamma)\frac{d^ny}{dw^n} + [b_n(w, y)(1 - \alpha_{n-1,n}(\gamma)) + b_{n-1}(w, y)\alpha_{n-2,n-1}(\delta)]\frac{d^{n-1}y}{dw^{n-1}} + ... + [b_1(w, y)(1 - \alpha_{0,1}(\beta)) + b_0(w, y)]\, y = a_0(w, y), \quad (3.3)$$

with initial/boundary conditions $\alpha_{m,n}(\rho)y^{(n)}(c)+(1 - \alpha_{m,n}(\rho))y^{(m)}(c) = d$ and $y(e) = f$.

Proof. Using (2.1) in the fractional differential equation (3.1) we write

$$b_n(w, y)\left(\alpha_{n-1,n}(\gamma, w)\frac{d^ny}{dw^n} + (1 - \alpha_{n-1,n}(\gamma, w))\frac{d^{n-1}y}{dw^{n-1}}\right) + b_{n-1}(w, y)\left(\alpha_{n-2,n-1}(\delta, w)\frac{d^{n-1}y}{dw^{n-1}} + (1 - \alpha_{n-2,n-1}(\delta, w))\frac{d^{n-2}y}{dw^{n-2}}\right) + ... + b_1(w, y)\left(\alpha_{0,1}(\beta, w)\frac{dy}{dw} + (1 - \alpha_{0,1}(\beta, w))y\right) + b_0(w, y)y = a_0(w, y), \quad (3.4)$$

while the initial/boundary condition takes the form $\alpha_{m,n}(\rho, w)y^{(n)}(c)+(1 - \alpha_{m,n}(\rho, w))y^{(m)}(c) = d$ and $y(e) = f$.

Arranging, we get the result.

For the second part, we let $\alpha_{m,n}(\varsigma, w) = \alpha_{m,n}(\varsigma,)$ in (3.4) and arranging we complete the proof. □

Next we solve and verify the fractional differential equation introduced in Khalil et al. [10].

Example 3.2. *The fractional differential equation in Khalil et al. [10] is given as:*

$$y^{(\frac{1}{2})} + y = x^{\frac{1}{2}} + 2x^{\frac{3}{2}}; \; y(0) = 0. \tag{3.5}$$

Solution: Let us choose $m = 0$, $n = 1$ and $\alpha_{0,1}\left(\frac{1}{2}, x\right) = \frac{1}{2}$ then, using Theorem 2.5, we express (3.5) as

$$\frac{1}{2}y + \frac{1}{2}y^{(1)} + y = x^{\frac{1}{2}} + 2x^{\frac{3}{2}}; \; y(0) = 0,$$

which further reduces to

$$y^{(1)} + 3y = 2x^{\frac{1}{2}} + 4x^{\frac{3}{2}}; \; y(0) = 0,$$

which is a non-fractional ordinary differential equation with integrating factor e^{3x}. This gives

$$\frac{d}{dx}\left(ye^{3x}\right) = 2x^{\frac{1}{2}}e^{3x} + 4x^{\frac{3}{2}}e^{3x}.$$

Integrating both sides we obtain

$$y = \frac{4}{3}x^{\frac{3}{2}} + A_1e^{-3x},$$

where A_1 is some constant.

After using $y(0) = 0$, we finally obtain $y = \frac{4}{3}x^{\frac{3}{2}}$.

To verify, consider the left side of (2.4) and use (2.2) one gets

$$\begin{aligned} y^{(\frac{1}{2})} + y &= \left(\frac{4}{3}x^{\frac{3}{2}}\right)^{(\frac{1}{2})} + \frac{4}{3}x^{\frac{3}{2}} \\ &= \frac{1}{2}\left(\frac{4}{3}x^{\frac{3}{2}}\right)^{(1)} + \frac{1}{2}\left(\frac{4}{3}x^{\frac{3}{2}}\right) + \frac{4}{3}x^{\frac{3}{2}} \\ &= x^{\frac{1}{2}} + \frac{2}{3}x^{\frac{3}{2}} + \frac{4}{3}x^{\frac{3}{2}} \\ &= x^{\frac{1}{2}} + 2x^{\frac{3}{2}}, \end{aligned}$$

which is the right side of (3.5).

Example 3.3. *Let's study the well known fractional differential equation introduced in [2, 8, 9, 11], that is,*

$$\frac{d^{\beta}}{dx^{\beta}}y = \beta y, \;\; \beta \in (0, 1) \tag{3.6}$$

to an ordinary differential equation.

For $\beta \in (0,1)$ we choose $m = 0$, $n = 1$ and $\alpha_{0,1}(\beta, x) = \beta$ then (3.6) can be expressed as

$$\beta \frac{dy}{dx} + (1-\beta)y = \beta y,$$

this is an ordinary linear differential equation with integrating factor $e^{\frac{1-2\beta}{\beta}x}$. Using this, we obtain the solution of (3.6) as $y = De^{\frac{2\beta-1}{\beta}x}$, where D is some constant.

In the next, using Theorem 2.5, we convert all the fractional differential equations given in Khalil et al. [10] to the form (3.2).

Example 3.4.

$$y^{(0.5)} + \sqrt{x}y = xe^{-x}. \tag{3.7}$$

As $0.5 \in [0,1]$ therefore, (3.7) can be expressed as

$$1-\alpha_{0,1}(0.5,x) + \sqrt{x}\, y + \alpha_{0,1}(0.5,x)y^{(1)} = xe^{-x}.$$

Example 3.5.

$$y^{(0.5)} = \frac{x^{3/2} + \sqrt{x}y}{2x+3y}. \tag{3.8}$$

Through (2.1) we easily express (3.8) as

$$(1-\alpha_{0,1}(0.5,x))y + \alpha_{0,1}(0.5,x)y^{(1)} = \frac{x^{3/2} + \sqrt{x}y}{2x+3y}.$$

Next, we come to the well known Riccati fractional differential equation (given in Syam and Al-Refai [13]) to the form (3.2).

Example 3.6. *Riccati Fractional differential equation in [13] is of the form*

$$T_\varsigma(g)(x) = g(x) + b_0(x) - g^2(x), \tag{3.9}$$

with fractional part

$$b_0(x) = -\frac{\varsigma+1}{\varsigma\Gamma(\varsigma)} x\, E_{\varsigma,2}\left(-\frac{\upsilon}{1-\varsigma}x^\varsigma - 1\right) - \left(x^{\varsigma+1}+1\right) + \left(x^{\varsigma+1}+1\right)^2, \tag{3.10}$$

where $g(0) = 1$ *and* $\varsigma \in (0,1)$.

As $\varsigma \in (0,1)$ leads to $\varsigma + 1 \in (1,2)$. Inserting in (2.1) we find

$$x^\varsigma = (1-\alpha_{0,1}(\varsigma,x))x + \alpha_{0,1}(\varsigma,x)x^{(1)} = \alpha_{0,1}(\varsigma,x) + (1-\alpha_{0,1}(\varsigma,x))x \tag{3.11}$$

while

$$x^{\varsigma+1} = (1-\alpha_{1,2}(\varsigma,x))x^{(1)} + \alpha_{1,2}(\varsigma,x)x^{(2)} = 1-\alpha_{1,2}(\varsigma,x). \tag{3.12}$$

Inserting (3.11) and (3.12) in the fractional term (3.10) and simplifying, we get

$$b_0(x) = -\frac{\varsigma+1}{\varsigma\Gamma(\varsigma)} x\, E_{\varsigma,2}\left(-\frac{(1-\alpha_{0,1}(\varsigma,x))x + \upsilon(\alpha_{0,1}(\varsigma,x)}{1-\varsigma} - 1\right) + 2 - 3\varsigma + \varsigma^2. \tag{3.13}$$

Combining (3.9) with non-fractional term (3.13), we get Riccati ordinary differential equation solution of which gives solution of the given Riccati fractional differential equation.

4 Conclusion

In this work we have shown that fractional derivative of any positive order can be linked with ordinary derivative of differentiable functions by using a weighted linear combination model. Many real world problems are best modelled by using the fractional differential equations so this link establishes several generalizations about the properties of the ordinary derivatives. We obtain a dual form in the ordinary differential equations by converting all types of fractional differential equations and vice versa.

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