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Fixed point theorems in *b*-metric and extended *b*-metric spaces

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Abstract - A necessary and sufficient condition is established for a Banach contraction *f* on a *b*-metric space (X,ρ_s) to have a fixed point using the greatest lower bound property of non-negative real numbers. The obtained fixed point is unique and hence a *b*-contractive fixed point to which each sequence of *f*-iterates converges. Further, fixed point theorems are proved obtained for some contraction type conditions in an extended *b*-metric space.

Keywords: *b*-metric space, Banach contraction, *b*-Cauchy sequence, unique fixed point, *b*-contractive fixed point, Extended *b*-metric space, Contraction types.

1. INTRODUCTION

Let $s \geq 1$. Suppose that X is a nonempty set and $\rho s : X \times X \to [0,\infty)$ is such that

(b1) $\rho_s(x,y) = 0$ if and only if $x = y$ for all $x,y \in X$

(b2) $\rho_s(x,y) = \rho_s(y,x)$ for all $x,y \in X$

(b3) $\rho_s(x,y) \leq s[\rho_s(x,z) + \rho_s(z,y)]$ for all $x,y \in X$.

Then ρ_s is called a *b*-metric on X, and the pair (X,ρ_s) denotes a *b*-metric space with parameter (or coefficient) s. The notion of a *b*-metric space was introduced by Bakthin [**1**] in 1989. Note that every *b*-metric space is a metric space with $s = 1$. A *b*-metric need not be continuous though a metric d is known to be continuous (See Example 2.13, [**3**]).

Definition 1.1. Let (X,ρ_s) be a *b*-metric space. A *b*-ball in X is defined by

$$B_{\rho_s}(x,r) = \{y \in X : \rho_s(x,y) < r\}.$$

The family of all *b*-balls forms a base topology, called the *b*-metric topology $\tau(\rho_s)$ on X.

Definition 1.2. Let (X,ρ_s) be a *b*-metric space with parameter s. A sequence $\langle x_n\rangle_{n=1}^{\infty}$ in X is said to be

(a) *b*-convergent with limit $p \in X$, if it converges to p in the *b*-metric topology $\tau(\rho_s)$

(b) *b*-Cauchy, if $\lim_{n,m\to\infty} \rho_s(x_n,x_m) = 0$.

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A b-metric space X is said to be b-complete, if every b-Cauchy sequence in X is b-convergent in it. As in a metric space, a b-convergent sequence has a unique limit, and is necessarily b-Cauchy.

In 2013, Kir and Kiziltunc [**2**] proved the following Banach contraction mapping theorem in a b-metric space:

Theorem 1.1. *Let (X, ρ_s) be a complete b-metric space with constant s, where ρ_s is continuous. Suppose that f is a self-map on X satisfying the condition*

(1.1) $\rho_s(f_x, f_y) \leq \alpha \rho_s(x, y)$ *for all* $x, y \in X$,

where $0 < \alpha < 1/s$. *Then f has a unique fixed point p.*

In this paper, a necessary and sufficient condition is obtained for the Banach contraction (1.1) to have a fixed point through the elegant greatest lower bound property of non-negative real numbers. Finally, we establish fixed point theorems through some more contraction type conditions in extended b-metric space.

2. FIXED POINT OF BANACH CONTRACTION

The greatest lower bound property, also known as the infimum property of real numbers, states that a nonempty subset of real numbers set, which is bounded below, has an in infimum. As an immediate consequence, we have

Lemma 2.1. Let α be the infimum of $S \subset \mathbb{R}$. Then there is a sequence $\langle p_n \rangle_{n=1}^{\infty}$ in S with $\lim_{n\to\infty} x_n = \alpha$.

Theorem 2.1. *Let $x_0 \in X$ be arbitrary and f be a Banach contraction satisfying* (1.1) *on a b-metric space (X, ρ_s) with constant s, where ρ_s is continuous. Then point $p \in X$ is a fixed point of f if and only if the sequence $\langle x_n \rangle_{n=1}^{\infty}$ of f-iterates given by*

(2.1) $$x_n = f x_{n-1} \text{ for } n \geq 1$$

converges to p.

Proof. Suppose that p is a fixed point of f. Writing $x = f^{n-1}x_0$ and $y = p$ in (1.1), we get

(2.2) $$\begin{aligned} \rho_s(f^n x_0, p) &= \rho_s(f^n x_0, f^n p) \\ &\leq \alpha \rho_s(f^{n-1} x_0, f^{n-1} p) \\ &\leq \alpha^2 \rho_s(f^{n-2} x_0, f^{n-2} p) \\ &\dots \\ &\leq \alpha^n \rho_s(x_0, p). \end{aligned}$$

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Now, by (b3) and (1.1), we see that

$$\rho_s(x_0, p) \le s[\rho_s(x_0, fx_0) + \rho_s(fx_0, f_p)] \le s[\rho_s(x_0, fx_0) + \alpha\rho_s(x_0, p)]$$

so that

(2.3) $$\rho_s(x_0, p) \le \frac{s}{1-\alpha s} . \rho_s(x_0, fx_0).$$

Substituting (2.3) in (2.2), we get

(2.4) $$\rho_s(f^n x_0, p) = \frac{\alpha^n s}{1-\alpha s} . \rho_s(x_0, fx_0) \ \ for\ all\ x_0 \in X, n = 1, 2, 3, ...$$

Since $\lim_{n\to\infty} \alpha^n = 0$ from (2.4), it follows that $\rho_s(f^n x_0, p) \to 0$ as $n \to \infty$ for all $x_0 \in X$.

Conversely, suppose that

(2.5) $$\lim_{n\to\infty} x_n = p.$$

Now, with $x = x_{n-1}$ and $y = z = p$, the inequality (1.1) and repeated use of (b2), give

(2.6) $\rho_s(fp, p) \le s[\rho_s(fp, fx_n) + \rho_s(fx_n, p)] \le s\alpha\rho_s(p, x_n) + s^2[\rho_s(fx_n, x_n) + \rho_s(x_n, p)]$

$= (s^2 + s\alpha)\rho_s(p, x_n) + s^2\rho_s(fx_n, x_n) = (s^2 + s\alpha)\rho_s(p, x_n) + s^2\rho_s(fx_{n+1}, x_n)$ for all n.

In the limiting case as $n \to \infty$, (2.5) and (2.6) impliy that $\rho_s(fp, p) = 0$ or $fp = p$. Thus p is a fixed point of f.

Theorem 2.2. *Let* $x_0 \in X$ *be arbitrary and f be a Banach contraction satisfying* (1.1) *on a b-metric space* (X, ρ_s) *with constant s, where* ρ_s *is continuous. Then the sequence given by* (2.1) *is b-Cauchy.*

Proof. The proof is organized into various steps as follows:

Step 1 – Existence of the infimum: Define $\mathcal{A} = \{\rho_s(x, fx): x \in X\}$. If $\rho_s(x, fx) = 0$ for some $x \in X$, then by (b1), we see that $fx = x$. That is x is a fixed point of f. Therefore, we assume that every element $\rho_s(x, fx)$ of $\mathcal{A}$ is positive. *Let* $\mu \ge 0$ be the infimum of $\mathcal{A}$.

Step 2 – Vanishing infimum: If possible suppose that $\mu > 0$. Writing $y = fx$ in (1.1),

(2.7) $$\rho_s(fx, f^2x) \le \alpha\rho_s(x, fx).$$

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Now $s \geq 1$ implies that $1/s \leq 1$. Therefore, $0 < \alpha < 1/s \leq 1$ and hence $\mu/\alpha > \mu$. Thus μ/α cannot be a lower bound of $\mathcal{A}$. In other words,

(2.8) $\rho_s(x, fx) < \mu/\alpha$ or $\alpha\rho_s(x, fx) < \mu$ for some $x \epsilon X$.

From (2.7) and (2.8), we find that

(2.9) $\rho_s(fx, f^2x) < \mu$ for some $x \epsilon X$,

where $\rho_s(fx, f^2x) \epsilon \mathcal{A}$. This reveals that μ cannot be the greatest lower bound of $\mathcal{A}$, which is a contradiction. Hence $\mu = 0$.

Step 3 – Existence of a sequence: By Lemma 2.1, there exists a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in X such that

(2.10) $\rho_s(fx_n, x_n) \epsilon \mathcal{A}\ for\ all\ n = 1, 2, 3, \dots$ and $\lim_{n\to\infty} \rho_s(fx_n, x_n) = 0$.

Step 4 – $\langle x_n \rangle_{n=1}^{\infty}$ is S-Cauchy: In fact, by (b2) and (1.1), we have

$$\begin{aligned}\rho_s(x_n, x_m) &\leq s[\rho_s(x_n, fx_n) + \rho_s(fx_n, x_m)]\\ &\leq s\rho_s(x_n, fx_n) + s^2\rho_s(fx_n, fx_m) + s^2\rho_s(fx_m, x_m)\\ &\leq s\rho_s(x_n, fx_n) + s^2\alpha\rho_s(x_n, x_m) + s^2\rho_s(fx_m, x_m)\end{aligned}$$

Rearranging the terms and simplifying this, it follows that

(2.11) $\rho_s(x_n, x_m) \leq \frac{s}{1-\alpha s^2} \cdot \rho_s(x_n, fx_n) + \frac{s^2}{1-\alpha s^2} \cdot \rho_s(fx_m, x_m)$.

Applying that the limit as $m, n \to \infty$ in this and using (2.10), we see that $\rho_s(x_n, x_m) \to 0$. That is $\langle x_n \rangle_{n=1}^{\infty}$ is a b-cauchy sequence in X.

3. UNIQUE FIXED POINT AS A CONTRACTIVE FIXED POINT

Definition 3.1. Let *f* be a self-map on a *b*-metric space (X, ρ_s) and $x_0 \in X$. A fixed point *p* of *f* is said to be a *b*-contractive fixed point, if for every $x_0 \in X$, the sequence (2.1) of iterates converges to *p*.

Since a convergent sequence in a *b*-metric space has a unique limit, a contractive fixed point must be a unique fixed point. We now prove that the unique fixed point of (1.1) is *b*-contractive fixed point.

Theorem 3.1. *Let f be a self-map on a complete b-metric space* (X, ρ_s) *with constant s, where* ρ_s *is continuous. Then f has a unique fixed point, and hence a b-contractive fixed point.*

Proof. By Theorem 2.2, $\langle x_n \rangle_{n=1}^{\infty} \subset X$ is *b*-Cauchy. Since *X* is *b*-Complete, we can find a point $p \in X$ such that (2.5) holds good, which is a fixed point of *f*, in view of Theorem 2.1.

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To establish the uniqueness of the fixed point, suppose q is another fixed point of f . That is $fq = q$.

From (1.1) with $x = p$ and $y = q$ gives

$$\rho_s(p,q) = \rho_s(fp,fq) \leq \alpha\rho_s(p,q) \text{ or } (1-\alpha)\rho_s(p,q) \leq 0$$

so that $p = q$. That is, p is the unique fixed point of f , and hence is a b-contractive fixed point.

Kamran et al. [**2**] introduced an extended b-metric space as follows:

Definition 2.1. Let $s \geq 1$, X be a nonempty set and $\theta : X \times X \rightarrow [1,\infty)$. Consider $\rho_\theta : X \times X \rightarrow [0,\infty)$ such that

(eb1) $\rho_\theta(x,y) = 0$ for all $x,y \in X$

(eb2) $\rho_\theta(x,y) = 0$ implies that $x = y$ for all $x,y \in X$

(eb3) $\rho_\theta(x,y) = \rho_\theta(y,x)$ for all $x,y \in X$

(eb4) $\rho_\theta(x,y) \leq \theta(x,y)[\rho_\theta(x,z) + \rho_\theta(z,y)]$ for all $x,y,z \in X$.

Then ρ_θ is called an extended b-metric on X, and (X,ρ_θ) an extended b-metric space. If $\theta(x,y) = s \geq 1$ for all $x,y \in X$, then ρ_θ reduces to a b-metric ρ_s. In this paper, (X,ρ_θ) denotes an extended b-metric space. Convergence and completeness in an extended b-metric space are similar to that in a b-metric space.

Definition 2.2. A sequence $\langle x_n\rangle_{n=1}^{\infty} \subset X$ is said to converge to $z \in X$, written as $\lim_{n\to\infty} x_n = z$, if for every $\epsilon > 0$ there exists a natural number N such that $\rho_\theta(x_n,z) < \epsilon$ for all $n \geq N$.

If ρ_θ is continuous, then every convergent sequence in X has a unique limit in it.

Definition 2.3. A sequence $\langle x_n\rangle_{n=1}^{\infty} \subset X$ is said to be Cauchy, if for every $\epsilon > 0$ here exists a natural number N such that $\rho_\theta(x_n,z) < \epsilon$ for all m, $n \geq N$.

A Cauchy sequence in X need not be convergent in it. But, if Cauchy sequence in X is convergent in it, we say that X is complete. Banach contraction mapping theorem in an extended b-metric space was proved in [**2**].

We establish fixed point theorems for some contraction types other than Banach's, in an extended b-metric space. In this sequel, we employ the following notion:

Definition 2.4. Let f be a self-map on an extended b-metric space (X,ρ_θ) and $x_0 \in X$. The orbit $O_f(x_0)$ at x_0 is the sequence of f-iterates $x_0, fx_0, \dots, f^n x_0, \dots$.

We need the following:

Lemma 3.1 (Theorem 3.22, [**3**], p. 59). The infinite series $\sum_{n=1}^{\infty} u_n$ of positive terms converges if and only if, given $\in > 0$, there is a natural number n_0 such that $\sum_{j=n}^{m} u_n \leq \epsilon$ for all $m \geq n \geq n_0$.

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Lemma 3.2 (Theorem 3.34, [**3**], p. 66). The infinite series $\sum_{n=1}^{\infty} u_n$ of positive terms is convergent, provided $\limsup_{n\to\infty} u_{n+1}/u_n < 1$.

Theorem 3.1. *Let (X, ρ_θ) be a complete extended b-metric space, where ρ_θ is continuous. Suppose that $f: X \to X$ satisfies the condition*

(3.1) $\rho_\theta(fx, fy) \leq \beta[\rho_\theta(x, fx) + \rho_\theta(y, fy)]$ *for all $x, y \in X$,*

where $0 < \beta < 1/2$ is such that for each $x_0 \in X$,

(3.2) $\lim_{n,m\to\infty} \theta(f^n x_0, f^m x_0) < \frac{1-\beta}{\beta}$.

Then f will have a unique fixed point z, and $\lim_{n\to\infty} f^n \xi = z$ for each $\xi \in X$.

Proof. Let $x_0 \in X$ be arbitrary. Define $\langle x_n \rangle_{n=1}^{\infty} \subset X$ by

(3.3) $x_n = f x_{n-1}$ for $n \geq 1$.

By induction, (3.3) gives

(3.4) $x_1 = f x_{0,}\ x_2 = f^2 x_{0,} \ldots\ x_n = f^n x_{0,} \ldots n \geq 1.$

Now writing $x = x_{n-1}$ and $y = x_n$ in (3.1) and using (3.4), we find that

$$\rho_\theta(f x_{n-1}, f x_n) \leq \beta[\rho_\theta(x_{n-1}, f x_{n-1}) + \rho_\theta(x_n, f x_n)]$$

or $\rho_\theta(x_n, x_{n+1}) \leq \beta[\rho_\theta(x_{n-1}, x_n) + \rho_\theta(x_n, x_{n+1})] \leq \frac{\beta}{1-\beta} \cdot\ \rho_\theta(x_{n-1}, x_n).$

By induction, it follows that

(3.5) $\rho_\theta(x_n,\ x_{n+1}) \leq \left(\frac{\beta}{1-\beta}\right)^n \rho_\theta(x_0, x_1)$ for $n \geq 1$.

Now for $m > n$, by (eb4), we find that

(3.6) $\rho_\theta(x_n, x_m) \leq \theta(x_n, x_m)[\rho_\theta(x_n,\ x_{n+1}) + \rho_\theta(x_{n+1},\ x_m)]$

$$\leq \theta(x_n, x_m)\left[\left(\frac{\beta}{1-\beta}\right)^n . \rho_\theta(x_0, x_1) + \rho_\theta(x_{n+1}, x_m)\right].$$

But again by (eb4),

(3.7) $\rho_\theta(x_{n+1}, x_m) \leq\ \theta(x_{n+1}, x_m)[\rho_\theta(x_{n+1},\ x_{n+2}) + \rho_\theta(x_{n+2}, x_m)]$

$$\leq \theta(x_{n+1}, x_m)\left[\left(\frac{\beta}{1-\beta}\right)^{n+1} \rho_\theta(x_0, x_1) + \rho_\theta(x_{n+2}, x_m)\right].$$

Inserting (3.7) in (3.6),

(3.8) $\rho_\theta(x_n, x_m) \leq\ \theta(x_n, x_m)\left[\left(\frac{\beta}{1-\beta}\right)^n \rho_\theta(x_0, x_1) + \theta(x_{n+1}, x_m)\left\{\left(\frac{\beta}{1-\beta}\right)^{n+1} \rho_\theta(x_0, x_1) + \rho_\theta(x_{n+2}, x_m)\right\}\right]$

$$= \theta(x_n, x_m)\left(\frac{\beta}{1-\beta}\right)^n \rho_\theta(x_0, x_1) + \theta(x_n, x_m)\theta(x_{n+1}, x_m)\left(\frac{\beta}{1-\beta}\right)^{n+1} \rho_\theta(x_0, x_1)$$

$$+ \theta(x_n, x_m)\theta(x_{n+1}, x_m)\rho_\theta(x_{n+2}, x_m)$$

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$$\vdots$$
$$\le \rho_\theta(x_0,x_1)\theta(x_n,x_m)\left[\left(\frac{\beta}{1-\beta}\right)^n + \theta(x_{n+1},x_m)\left(\frac{\beta}{1-\beta}\right)^{n+1} + \cdots + \theta(x_{n+1},x_m)\theta(x_{n+2},x_m)\cdots\theta(x_{m-1},x_m)\left(\frac{\beta}{1-\beta}\right)^{m-1}\right].$$

Note that the right hand side of (3.8) has $m-n$ terms. Since $\theta(x,y)\ge 1$ for all x and y, this is written as

(3.9) $$\rho_\theta(x_n,x_m) \le \rho_\theta(x_0,x_1)[\theta(x_1,x_m)\theta(x_2,x_m)\cdots\theta(x_n,x_m)\xi^n + [\theta(x_1,x_m)\theta(x_2,x_m)\cdots\theta(x_n,x_m)\theta(x_{n+1},x_m)\xi^{n+1} + \cdots + [\theta(x_1,x_m)\theta(x_2,x_m)\cdots\theta(x_n,x_m)\cdot \theta(x_{n+1},x_m)\theta(x_{n+2},x_m)\cdots\theta(x_{m-1},x_m)\xi^{m-1}],$$

where $0<\xi=\beta/(1-\beta)<1$, in view of the choice of β.

Consider the series $S=\sum_{n=1}^{\infty}\xi^n\prod_{i=1}^{n}\theta(x_i,x_m)$ for each $m\ge 1$. write $u_n = \xi^n\prod_{i=1}^{n}\theta(x_i,x_m)$ for each m *and* $n\ge 1$. Then from (3.2), we see that

(3.10) $$\limsup_{n\to\infty}\frac{u_{n+1}}{u_n} = \limsup_{n\to\infty}\frac{\xi^{n+1}\prod_{i=1}^{n+1}\theta(x_i,x_m)}{\xi^n\prod_{i=1}^{n}\theta(x_i,x_m)} = \limsup_{n\to\infty}\xi\,\theta(x_{n+1},x_m) < 1 \text{ for each } m.$$

In view of Lemma 3.2, the series S converges. Also, the partial sums of S, given by

(3.11) $$S_n=\sum_{j=1}^{n}\xi^j\prod_{i=1}^{j}\theta(x_i,x_m),\ n=1,2,\cdots, \text{ for each } m,$$

are bounded.

Using (3.11) in (3.9), it follows that

(3.12) $$\rho_\theta(x_n,x_m)\le\rho_\theta(x_0,x_1)[S_{m-1}-S_n] \text{ for } m>n.$$

Given $\epsilon>0$, using the convergence of S and Lemma 3.1, (3.12) imply that

(3.13) $$\rho_\theta(x_n,x_m)\le\rho_\theta(x_0,x_1)\epsilon \text{ for } m>n\ge n_0 \text{ for some natural number } n_0.$$

Thus $\langle x_n\rangle_{n=1}^{\infty}$ is Cauchy sequence in X. Since X is b-complete, we can find a point $z\in X$ such that

(3.14) $$\lim_{n\to\infty}x_n=\lim_{n\to\infty}fx_{n-1}=z.$$

Now we establish that z is a fixed point of f. In fact, writing $x=x_{n-1}$ and $y=z$, the inequality (3.1) gives

$$\rho_\theta(fx_{n-1},fz)\le\beta[\rho_\theta(x_{n-1},fx_{n-1})+\rho_\theta(z,fz)]$$

or

$$\rho_\theta(x_n,fz)\le\beta[\rho_\theta(x_{n-1},x_n)+\rho_\theta(z,fz)].$$

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Applying the limit as $n \to \infty$ and using (3.14) and the continuity of ρ_θ, we obtain

$$\rho_\theta(z, fz) \leq \beta[\rho_\theta(z,z) + \rho_\theta(z, fz)] \text{ so that } \rho_\theta(z, fz) \leq \beta\rho_\theta(z, fz) \text{ or } \rho_\theta(z, fz) = 0.$$

That is, $fz = z$.

To establish the uniqueness of the fixed point, let $q \neq z$ be also a fixed point of *f*. With $x = z$ and $y = z$ in (3.1),

$$0 < \rho_\theta(z, q) = \rho_\theta(fz, fq) \leq \beta[\rho_\theta(z, fz) + \rho_\theta(q, fq)] = 0$$

which is a contradiction. Hence $z = q$, and the fixed point is unique.

□

Writing $\theta(x, y) = s \geq 1$ for all $x, y \in X$ in Theorem 3.1, we see that

(3.15) $$\lim_{n,m\to\infty} \theta(f^n x_0, f^m x_0) \frac{\beta}{1-\beta} = \frac{s\beta}{1-\beta} < 1,$$

which implies that $0 < \beta < 1/(s+1) \leq 1/2$. Hence, we have

Corollary 3.1. *Let* (X, ρ_s) *be a complete b-metric space with constant s, where* ρ_s *is continuous, and* $f: X \to X$
satisfy the condition

(3.15) $$\rho_s(fx, fy) \leq \beta[\rho_s(x, fx) + \rho_s(y, fy)] \text{ for all } x, y \in X,$$

where $0 < \beta < 1/2$. *Then f will have a unique fixed point z.*

Theorem 3.2. *Let* (X, ρ_θ) *be a complete extended b-metric space, where* ρ_θ *is continuous. Suppose that* $f: X \to X$ *satisfy the condition*

(3.17) $$\rho_\theta(fx, fy) \leq \gamma[\rho_\theta(x, fy) + \rho_\theta(y, fx)] \text{ for all } x, y \in X,$$

where $0 < \gamma < 1/2$ *is such that for each* $x_0 \in X$,

(3.18) $$\lim_{n,m\to\infty} \frac{\gamma\theta(f^n x_0, f^{n+2} x_0)\theta(f^{n+1} x_0, f^m x_0)}{1 - \gamma\theta(f^n x_0, f^{n+2} x_0)} < 1.$$

Then f will have a unique fixed point z.

Proof. Let $x_0 \in X$ be arbitrary. Define $\langle x_n \rangle_{n=1}^{\infty} \subset X$ as in (3.3) so that (3.4) holds. Now writing $x = x_{n-1}$ and $y = x_n$ in (3.17) and using (3.4), we find that

$$\begin{aligned} \rho_\theta(x_n, x_{n+1}) &= \rho_\theta(fx_{n-1}, fx_n) \\ &\leq \gamma[\rho_\theta(x_{n-1}, fx_n) + \rho_\theta(x_n, fx_{n-1})] \\ &\leq \gamma\, \theta(x_{n-1}, x_{n+1})[\rho_\theta(x_{n-1}, x_n) + \rho_\theta(x_n, x_{n+1})] \\ &\leq \frac{\gamma\, \theta(x_{n-1}, x_{n+1})}{1 - \gamma\, \theta(x_{n-1}, x_{n+1})} \cdot \rho_\theta(x_{n-1}, x_n). \end{aligned}$$

By induction, it follows that

(3.19) $$\rho_\theta(x_n, x_{n+1}) \leq \psi_n . \rho_\theta(x_0, x_1), n = 1, 2, \cdots,$$

where

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(3.20) $\psi_n = \prod_{j=1}^{n}\left\{\frac{\gamma\theta(x_{j-1},\ x_{j+1})}{1-\gamma\theta(x_{j-1},\ x_{j+1})}\right\}$ for all n.

Now for $m > n$, by using (eb4) repeatedly and (3.20), we obtain

$$\begin{aligned}\rho_\theta(x_n, x_m) &\le \theta(x_n, x_m)[\rho_\theta(x_n, x_{n+1}) + \rho_\theta(x_{n+1}, x_m)]\\ &\le \theta(x_n, x_m)[\psi_n\, \rho_\theta(x_0, x_1) + \rho_\theta(x_{n+1}, x_m)]\\ &\vdots\\ &\le \rho_\theta(x_0, x_1)\theta(x_n, x_m)[\psi_n + \psi_{n+1}\theta(x_{n+1}, x_m)\\ &\qquad + \cdots + \psi_{m-1}\theta(x_{n+1}, x_m)\theta(x_{n+2}, x_m)\cdots\theta(x_{m-1}, x_m)].\end{aligned}$$

Since, $\theta(x, y) \ge 1$ for all *x* and *y*, this can be written as

(3.21) $$\begin{aligned}\rho_\theta(x_n, x_m) \le\ &\rho_\theta(x_0, x_1)[\psi_n \cdot \theta(x_1, x_m)\theta(x_2, x_m)\cdots\theta(x_n, x_m)\\ &+[\psi_{n+1} \cdot \theta(x_1, x_m)\theta(x_2, x_m)\cdots\theta(x_n, x_m)\theta(x_{n+1}, x_m)\\ &+ \cdots + [\psi_{m-1}\theta(x_1, x_m)\theta(x_2, x_m)\cdots\theta(x_n, x_m)\end{aligned}$$

$\theta(x_{n+1}, x_m)\theta(x_{n+2}, x_m)\cdots\theta(x_{m-1}, x_m)]$.

Consider the series $P = \sum_{n=1}^{\infty}\psi_n\prod_{i=1}^{n}\theta(x_i, x_m)$ for each $m \ge 1$. write $v_n = \psi_n\prod_{i=1}^{n}\theta(x_i, x_m)$ for each $m, n \ge 1$. Then

(3.22) $\lim\limits_{n\to\infty}\frac{v_{n+1}}{v_n} = \lim\limits_{n\to\infty}\frac{\psi_{n+1}\prod_{i=1}^{n}\theta(x_i,x_m)}{\psi_n\prod_{i=1}^{n}\theta(x_i,x_m)} = \lim\limits_{n\to\infty}\ \frac{\gamma\theta(x_n,x_{n+2})\theta(x_{n+1},x_m)}{1-\gamma\theta(x_n,x_{n+2})}$ for each m.

Now from (3.18) we find that $\lim\limits_{n\to\infty} v_{n+1}/v_n < 1$. Hence in view of Lemma 3.2, the series *P* converges.

Also, the partial sums of *P* , given by

(3.23) $P_n = \sum_{j=1}^{n}\psi_j\prod_{i=1}^{j}\theta(x_i, x_m)\,, n = 1, 2, \cdots,$ for each m.

are bounded. Using (3.23) in (3.21), it follows that

(3.24) $\rho_\theta(x_n, x_m) \le \rho_\theta(x_0, x_1)[P_{m-1} - P_n]$ for $m > n$.

Given $\epsilon > 0$, using the convergence of *P* and Lemma 3.1, (3.24) imply that

(3.25) $\rho_\theta(x_n, x_m) \le \rho_\theta(x_0, x_1)\epsilon$ for $m > n \ge n_0$ for some natural number n_0.

Thus $\langle x_n\rangle_{n=1}^{\infty}$ is Cauchy sequence in *X*.

Since *X* is complete, we can find a point $z \in X$ such that

(3.26) $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} fx_{n-1} = z$.

Now we establish that *z* is a fixed point of *f*. In fact, writing $x = x_{n-1}$ and $y = z$, the inequality (3.17) gives

$$\rho_\theta(fx_{n-1}, fz) \le \gamma[\rho_\theta(x_{n-1}, fz) + \rho_\theta(z, fx_{n-1})]$$

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or $$\rho_\theta(x_n, fz) \leq \gamma[\rho_\theta(x_{n-1}, fz) + \rho_\theta(z, x_n)].$$

Applying the limit as $n \to \infty$ and using (3.26) and the continuity of ρ_θ, we obtain $\rho_\theta(z, fz) \leq \gamma[\rho_\theta(z, fz) + 0]$ so that $\rho_\theta(z, fz) \leq \gamma\rho_\theta(z, fz)$ or $\rho_\theta(z, fz) = 0$. That is, $fz = z$.
To establish the uniqueness of the fixed point, let $q \neq z$ be also a fixed point of f. Then with $x = z$ and $y = z$
in (3.17),

$$0 < \rho_\theta(z, q) = \rho_\theta(fz, fq) \leq \gamma[\rho_\theta(z, fq) + \rho_\theta(q, fz)] = 2\gamma\rho_\theta(z, fq) < \rho_\theta(z, fq),$$

which is a contradiction. Hence $z = q$, and the fixed point is unique.

Corollary 3.2. Suppose that (X, ρ_s) *is a complete b-metric space with constant s, and f is a self-map on X such that*

(3.27) $$\rho_s(fx, fy) \leq \gamma[\rho_s(x, fy) + \rho_s(y, fx)] \text{ for all } x, y \in X,$$

where γ *is areal number such that* $0 < \gamma s < 1/2$. *Then f will have a unique fixed point z.*

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