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Banach Contraction Principle on Cone Octagonal Metric Space

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Abstract

In this paper we introduce cone octagonal metric space, a generalization of cone polygonal metric spaces and we prove the Banach contraction principle with supporting example.

1.Introduction:

The concept of cone metric spaces, introduced by Huang and Zhang [1], generalizes classical metric spaces by replacing the set of real numbers with elements from an ordered Banach space. The Banach contraction mapping principle, a cornerstone of fixed point theory, was also adapted to this context by Huang and Zhang. Due to its wide range of applications, the study of fixed points—specifically, the existence and uniqueness of such points for various mappings—has become a central topic in analysis. Numerous researchers have since extended the Banach contraction principle to various generalized metric spaces [3–5].

Huang and Zhang formalized the notion of cone metric spaces and established fixed point theorems for contractive-type conditions in these spaces. Subsequent studies have proved fixed point theorems for different contractive conditions in cone metric spaces [3–5]. Azam et al. introduced cone rectangular metric spaces and proved the Banach contraction principle in this context. Garg and Agarwal further generalized this framework to cone pentagonal and hexagonal metric spaces, respectively, establishing analogous fixed point results. More recently, Ampadu introduced cone heptagonal metric spaces and proved the Chatterjea contraction mapping principle in this setting. This paper aims to present a fixed point theorem in cone octagonal metric spaces, thereby extending and improving upon the results of [6–8].

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2. Preliminaries

Definition 2.1. [2] Let E be a real Banach space and P a subset of E. P is called a cone, if and only if:

(1) P is closed, nonempty, and $P \neq \{0\}$;

(2) $a, b \in R, a, b \geq 0$ and $x, y \in P$ implies $ax + by \in P$;

(3) $x \in P$ and $-x \in P$ imply $x = 0$.

Given cone $P \subseteq E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ stands for $y - x \in \operatorname{int}(P)$, where $\operatorname{int}(P)$ denotes the interior of P.

Definition 2.2. [2] A cone P is called normal if there is a number $K^* > 0$ such that for all $x, y \in E$, the inequality

$0 \leq x \leq y$ implies $\|x\| \leq K^* \|y\|$.

(2.1)

The least positive number K^* satisfying (2.1) is called the normal constant of P.

In this paper we denote that E is a real Banach space and P is a cone in E with $\operatorname{int}(P) \neq \phi$ and $\leq$ is a partial ordering with respect to P.

Definition 2.3. [2] Let X be a nonempty set. Suppose the mapping $\rho: X \times X \to E$ satisfies:

(1) $0 < \rho(x, y)$ for all $x, y \in X$ and $\rho(x, y) = 0$ if and only if $x = y$;

(2) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$;

(3) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for all $x, y, z \in X$.

Then ρ is called a cone metric on X, and (X, ρ) is called a cone metric space.

Remark 2.4. The concept of a cone metric space is more general than a metric space, because each metric space is a cone metric space where $E = R$ and $P = [0. \infty)$ (cf. [2])

Definition 2.5. [6] Let X be a nonempty set. Suppose the mapping $\rho: X \times X \to E$ satisfies:

(1) $0 < \rho(x, y)$ for all $x, y \in X$ and $\rho(x, y) = 0$ if and only if $x = y$;

(2) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$;

(3) $\rho(x,y) \le \rho(x,w) + \rho(w,z) + \rho(z,y)$ for all $x,y,z \in X$ and for all distinct points $w,z \in X - \{x,y\}$ (Rectangular Property).

Then ρ is called a cone rectangular metric on X, and (X,ρ) is called a cone rectangular metric space.

Remark 2.6. Every cone metric space is cone rectangular metric space. The converse is not true [6].

Definition 2.7. [7] Let X be a nonempty set. Suppose the mapping $\rho: X \times X \to E$ satisfies:

(1) $0 < \rho(x,y)$ for all $x,y \in X$ and $\rho(x,y) = 0$ if and only if $x = y$;

(2) $\rho(x,y) = \rho(y,x)$ for all $x,y \in X$;

(3) $\rho(x,y) \le \rho(x,z) + \rho(z,w) + \rho(w,u) + \rho(u,y)$ for all $x,y,z,w,u \in X$ and for all distinct points $z,w,u \in X - \{x,y\}$ (Pentagonal Property).

Then ρ is called a cone pentagonal metric on X, and (X,ρ) is called a cone pentagonal metric space.

Remark 2.8. Every cone rectangular metric space and so cone metric space is cone pentagonal metric space. The converse is not true [7].

Definition 2.9. [8] Let X be a nonempty set. Suppose the mapping $\rho: X \times X \to E$ satisfies:

(1) $0 < \rho(x,y)$ for all $x,y \in X$ and $\rho(x,y) = 0$ if and only if $x = y$;

(2) $\rho(x,y) = \rho(y,x)$ for all $x,y \in X$;

(3) $\rho(x,y) \le \rho(x,z) + \rho(z,w) + \rho(w,u) + \rho(u,v) + \rho(v,y)$ for all $x,y,z,w,u,v \in X$ and for all distinct points $z,w,u,v \in X - \{x,y\}$ (Hexagonal Property).

Then ρ is called a cone hexagonal metric on X, and (X,ρ) is called a cone hexagonal metric space.

Remark 2.10. Every cone pentagonal metric space and so cone rectangular metric space is cone hexagonal metric space. The converse not necessarily true (e.g. see[8]).

Definition 2.11. [9] Let X be a nonempty set. Suppose the mapping $\rho: X \times X \to E$ satisfies:

(1) $0 < \rho(x,y)$ for all $x,y \in X$ and $\rho(x,y) = 0$ if and only if $x = y$;

(2) $\rho(x,y) = \rho(y,x)$ for all $x,y \in X$;

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(3) $\rho(x,y) \leq \rho(x,z) + \rho(z,w) + \rho(w,u) + \rho(u,v) + \rho(v,t) + \rho(t,y)$ for all $x,y,z,w,u,v,t \in X$ and for all distinct points $z,w,u,v,t \in X - \{x,y\}$ (Heptagonal Property).

Then ρ is called a cone heptagonal metric on X, and (X,ρ) is called a cone heptagonal metric space.

Remark 2.12. Every cone hexagonal metric space, cone pentagonal metric space and cone rectangular metric space are cone heptagonal metric spaces. The converse is not true [9].

Definition 2.12. Let X be a nonempty set. Suppose the mapping $\rho: X \times X \to E$ satisfies:

(1) $0 < \rho(x,y)$ for all $x,y \in X$ and $\rho(x,y) = 0$ if and only if $x = y$;

(2) $\rho(x,y) = \rho(y,x)$ for all $x,y \in X$;

(3) $\rho(x,y) \leq \rho(x,z) + \rho(z,w) + \rho(w,u) + \rho(u,v) + \rho(v,t) + \rho(t,s) + \rho(s,y)$ for all $x,y,z,w,u,v,t,s \in X$ and for all distinct points $z,w,u,v,t,s \in X - \{x,y\}$ [octagonal Property].

Then ρ is called a cone octagonal metric on X, and (X,ρ) is called a cone octagonal metric space.

Example 2.13. $X = [0,1]$, $E = R^2$,$P = \{(a,b): a \geq 0, b \geq 0\}$, Define metric $\rho: X \times X \to E$ by $\rho(x,y) = (|x-y|, k|x-y|)$, where $k \geq 1$.

3. Main Result

In this section we prove Banach contraction principle in cone octagonal metric space.

Theorem 3.1. Let (X,ρ) be a complete cone octagonal metric space with normal cone P. Suppose that $T: X \to X$ satisfies

$$\rho(Tx,Ty) \preccurlyeq K\rho(x,y) \text{ for all } x,y \in X,$$

where $0 \leq K \leq 1$. Then T has a unique point $u \in X$, and for any $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to u.

Proof. Let $x_0 \in X$ be arbitrary. Define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \geq 0$.

By the contraction property

$$\rho(x_n, x_{n+1}) = \rho(Tx_{n-1}, Tx_n) \leq K\rho(x_{n-1}, x_n).$$

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Repeating this inequality

$$\rho(x_n, x_{n+1}) \leq K^n \rho(x_0, x_1) \text{ for all } n.$$

For $m > n$, apply the octagonal inequality with intermediate points $x_{n+1}, x_{n+2}, \dots, x_{n+6}$.

$$\rho(x_n, x_m) \leq \rho(x_n, x_{n+1}) + \textstyle\sum_{k=1}^{6} \rho(x_{n+k}, x_{n+k+1}) + \rho(x_{n+7}, x_m).$$

$$\leq K^n \rho(x_0, x_1)(1 + K + K^2 + \cdots + K^{m-n-1}).$$

For $m \to \infty$, the geometric series converges and

$$\rho(x_n, x_m) \leq \frac{K^n}{1-K} \rho(x_0, x_1).$$

Since P is normal, there exists $C > 0$ such that

$$\|\rho(x_n, x_m)\| \leq C \frac{K^n}{1-K} \|\rho(x_0, x_1)\| \to 0 \text{ as } n \to \infty.$$

Thus $\{x_n\}$ is a Cauchy sequence. By completeness, $x_n \to u \in X$.

Using continuity of T

$$\rho(u, Tu) \leq \rho(u, x_{n+1}) + \rho(Tx_n, Tu) \leq \rho(u, x_{n+1}) + K\rho(x_n, u).$$

As $n \to \infty$, this inplies that $\rho(u, Tu) = 0$ or $Tu = u$.

Suppose v is another fixed point. Then $\rho(u, v) = \rho(Tu, Tv) \leq K\rho(u, v)$. Since $K < 1$, this implies $\rho(u, v) = 0$ implies $u = v$.

Example $X = R, E = R^2, P = \{(a, b): a \geq 0, b \geq 0\}$. Define $\rho: X \times X \to E$ by

$$\rho(x, y) = (|x - y|, k|x - y|), 0 < k < 1.$$

For $T: x \to \frac{kx}{2}$, we see that $\rho(Tx, Ty) = \left(\frac{k}{2}|x - y|, \frac{k^2}{2}|x - y|\right) \leq \frac{k}{2}\rho(x, y)$.

Here, $K = \frac{k}{2} < 1$, and 0 is the unique fixed point.

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