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New Results on Abstract Differential Equations of Elliptic Type with General Robin Boundary Conditions in Hölder Spaces and Noncommutative Case

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ABSTRACT

In this article, we prove new results on second order operational differential equations of elliptic type with Robin type operator coefficient conditions, in a non-commutative case. The study is performed when the second member f belongs to $C^{\alpha}([0,1];E)$, with $0<\alpha<1$, E being a complex Banach space. The existence, uniqueness and maximum regularity of the strict solution are proven. This article improves in a certain sense and naturally complements our last three works on this problem.

Key works second order operational differential equations, general Robin boundary conditions, interpolation spaces, analytic semi group.

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1. Introduction and hypotheses

This paper is devoted to study the following general problem

$$\begin{cases} u''(x) + Au(x) - \omega u(x) = f(x),\ x \in]0,1[\\ u'(0) - Hu(0) - \mu u(0) = r_0, \\ u(1) = d_1, \end{cases} \qquad (P)$$

with $f \in C^{\theta}([0,1]; E), 0 < \theta < 1$, where E is a complex Banach space, h_0, d_1 are given elements in E and A is a closed linear operator of domain $D(A)$ are not necessarily dense in $E. H$ is a closed linear operator in E of domain $D(A), \omega$ and μ complex parameters. The objective being to find necessary and sufficient conditions on the data to have the existence, uniqueness and maximum regularity of the solution, we use the Krein order reduction method, we prove that such a solution u Problem (P) can be represented explicitly like the sum $u = u_r + u_s$ of a regular part and a singular part in some natural form assumptions about the data. A good analysis of the terms of this representation makes it possible to find the regularity result. fact that large classes of problems governed by partial differential equations can be presented in the form of abstract differential equations (abbreviated EDAs). problems with spectral parameter in the equation and in the boundary conditions that appear in different problems concrete problems. As example, I will consider linearized static problems of certain equations of nonlinear evolution with integral conditions. We will now cite some interesting articles related to this research. In one of the latter works, In [5], the author considers some problems of second order elliptic boundary value problems on bounded domains with boundary conditions depending nonlinearly on the spectral parameter, see also [4], the paper considers a class of boundary problems with a spectral parameter in boundary conditions. In [2], we find a study, in a separable Hilbert space, of the following boundary-value second-order elliptic differential-operator equation:

$$u''(x) + Au(x) - \lambda u(x) = f(x), \qquad x \in (0,1)$$

with the following boundary conditions

$$\lambda u'(0) - \alpha u(1) = f_1, \qquad u(1) = f_2,$$

where α is a complex number with $Re(\alpha) \geq 0$ and A is a linear self-adjoint operator guaranteeing the ellipticity of the equation. Note that here, the parameter λ appears in the nonlocal boundary condition. Recently, in [1], the authors consider the following boundary- value problem for an elliptic differential-operator equation of second order

$$\lambda^2 u(x) - u''(x) + Au(x) = f(x), \qquad x \in (0,1)$$

with the following spectrals boundary conditions

$$u'(0) + \lambda u(1) = f_1, \qquad \beta u'(1) + \lambda u(0) = f_2,$$

where the same spectral parameter appears in the equation quadratic ally, here A is a closed positive linear operator in a separable complex Hilbert space. In [7], the authors consider Problem (P) in a complex Banach space E, where ω is a positive spectral parameter and $\mu = 0$. For ω large enough, under some geometrical assumptions on the space E and hypotheses on operators $A - \omega I$ and H, including the fact that they commute in the resolvent sense, the authors furnish necessary and sufficient conditions on the data r_0, d_1 to obtain the existence and uniqueness of a solution u of $(P)$with maximal regularity. Recently, in [8], the authors develop an interesting new approach in a non-commutative framework, concerning some general Sturm-Liouville problems with the same Robin boundary condition in 0.

1.1. problem position:

Here in this work, we assume that $f \in C^{\theta}([0,1];E), 0 < \theta < 1$, and we study the equation

$$u''(x) + A\,u(x) - \omega u(x) = f(x), \qquad x \in]0,1[, \tag{1}$$

with the generals Robin boundary conditions

$$\begin{cases} u'(0) - Hu(0) - \mu u(0) = r_0 \\ u(1) = d_1, \end{cases} \tag{2}$$

where μ is a spectral parameter and ω is a fairly large positive real number, r_0 and d_1 are data in E. Here A and H the problem (1)-(2) that is to say a function u such that

$$u \in C^2([0,1]\,;\,E) \cap C([0,1]\,;\,D(A)),$$

$u(0) \in D(H_\mu)$, and u satisfying Problem (1)-(2) as well as the maximum regularity of the strict solution u, that is to say u'', $Au \in C^\theta([0,1];E)$ with $\theta \in]0,1[$, general, the condition $f \in C([0,1]\,;\,E)$, is not sufficient, to be able to solve the problem *(1)-(2),* This is why we will assume that $f \in C^\theta([0,1]\,;\,E)$, $0 < \theta < 1$, with $C^\theta([0,1]\,;\,E)$ is the Hölder space, defined by

$$C^\theta([0,1];\,E) = \left\{ f \in C([0,1];\,E) : \underbrace{sup}_{t-s\neq 0,\ t,s\in[0,1]} \frac{\|f(s) - f(t)\|_E}{|s-t|^\theta} < +\infty \right\}.$$

It is also an interpolation space.

We have the following two propositions:

Let $\theta \in]0;\ 1[$. *So*

$$C^1([0,1]\,;\,E) \subset C^\theta([0,1]\,;\,E) \subset C([0,1]\,;\,E), \tag{3}$$

and $C^\theta([0,1]\,;\,E)$ with $\theta \in]0,1[$ provided with the standard

$$\|f\|_{C^\theta([0,1]\,;\,E)} = \|f\|_{C([0,1]\,;\,E)} + \underbrace{sup}_{t-s\neq 0,\ t,s\in[0,1]} \frac{\|f(s) - f(t)\|_E}{|s-t|^\theta},$$

is a Banach space, see [17], page 12.

For reasons of convenience, we will treat the problem (1)-(2) with the notation $A_\omega = A - \omega I,\ \omega > 0$, the ellipticity of the equation is guaranteed by hypothesis (5) below; this assumption allows us to consider, for suitable ω, μ, the operators

$$\begin{cases} D\left(\Lambda_{\omega,\mu}\right) = D(Q_\omega) \cap D\left(H_\mu\right) \\ \Lambda_{\omega,\mu} = \left(Q_\omega - H_\mu\right) + e^{2Q_\omega}\left(Q_\omega + H_\mu\right), \end{cases} \tag{4}$$

with $Q_\omega = -\sqrt{-A+\omega I}$, $H_\mu = H + \mu I$, $D(Q_\omega) = D(Q)$, and $\mathrm{D}\left(H_\mu\right) = \mathrm{D}\ (\mathrm{H})$.

In all the sequel, for any closed linear operator B on E, $D(B\ )$ denotes the domain of B and $\rho(B\ )$ the resolvent set of . The key point will be to obtain the invertibility of the determinant $\Lambda_{\omega,\mu}$ of system $(1)-(2)$. Note that $\Lambda_{\omega,\mu}$ will be, in some sense, the abstract determinant of our problem$(1)-(2)$. We give some new results by using semigroups and interpolation theory. The originality of this article lies in the fact that we will consider operator, $Q = -\sqrt{-A}$ principal, see assumptions (6), with $\overline{D(Q)} = \overline{D(A)}$, and also, we do not assume the commutativity between *H* and *A* or *H* and *Q*.

Consider some fixed $\omega_0 >\ 0$ and for $\omega > \omega_0$. Our main ellipticity assumption is the following:

$$\begin{cases}]0,+\infty[\subset\ \rho(A_\omega), ker(A_\omega) = \{0\}, \overline{Im(A_\omega)} = E \text{ and} \\ \qquad \lambda \geq 0: sup\|\lambda(A-\lambda I)^{-1}\|_{\mathcal{L}(E)} < +\infty. \end{cases} \tag{5}$$

Here we do not assume the density of $D(A)$ in semi-groups $(e^{xQ})_{x\geq 0}$, $(e^{xQ_\omega})_{x\geq 0}$, on E, not necessarily strongly continuous in 0, see for instance Balakrishnan [1] for densely defined operators and C. Martinez and M. Sanz [16].

We will assume that Q is principal in the since that:

$$D\ (Q_\omega) \subset\ D\ (H), \tag{6}$$

and

$\exists\ \varepsilon \in]0, \frac{1}{2}[, \exists C > 0$:

$$\underbrace{\sup}_{\mu \geq 0} (1 + |\mu|)^{\varepsilon} \left\| H_{\mu} \right\|_{\mathcal{L}(E)}^{-1} \leq C. \tag{7}$$

We will prove that the operator $\Lambda_{\omega,\mu}$, is closed and bounded invertible for a large omega we also assume that $\Lambda_{\omega,\mu}^{-1}$, is regularizing, that is:

$\forall \xi \in D(Q_{\omega}), Q_{\omega}\Lambda_{\omega,\mu}^{-1}\xi \in D(Q_{\omega})$, and

$$Q_{\omega}^{2}\Lambda_{\omega,\mu}^{-1}\xi \in (D(A), E)_{1-\frac{\theta}{2},\infty}. \tag{8}$$

With the following notation $(D(A), E)_{1-\frac{\theta}{2},\infty} = D_Q(\theta, +\infty)$, with $D_Q(\theta, +\infty)$is the interpolation space defined by

$$D_Q(\theta, +\infty) = \left\{ x \in E : \underbrace{sup}_{t>0} \|tQ(Q - tI)^{-1}x\|_E < +\infty \right\},$$

and

$$\|x\|_{D_Q(\theta,+\infty)} = \|x\|_E + \underbrace{sup}_{t>0} \|tQ(Q - tI)^{-1}x\|_E.$$

See [5].

The description of the main sections and results of this article are described as follows.

In section 2, we need demonstrations of some technical lemmas which will be useful in the study of the problem (1)-(2), Also we need, (without demonstrations), to some technical results for Sinestrari, which will be useful later. In section 3, using the S. G method, we give us a formula for representing the strict solution of problem (1)-(2). In section 4, we study the regularity of the solution and give our main results, in the last section, we explain our abstract theory with a concrete example application in EDP.

2. Technical lemmas

Let$\omega, \mu \in C$, in a very specific sector, see $[12], P. 30$. Remark 6.2. In the following K denotes various Constants, independent of $\omega, \mu \in C$, which can vary from one line to another.

Lemma 1 Assume (5) ~ (7). Then there exists $\omega_1 \geq \omega_0$ ($\omega_0$fix), such that, for all $\omega \geq \omega_1$, operator $\Lambda_{\omega,\mu}$, is invertible and has a bounded, moreover $\Lambda_{\omega,\mu}^{-1}$, can also be written in the form

$$\Lambda_{\omega,\mu}^{-1} = (Q_\omega - H_\mu)^{-1}(I + e^{2Q_\omega})W_{\omega,\mu},$$

with $W_{\omega,\mu}$ is a bounded linear operator in E, moreover, there exists a constant $K > 0$ independent of ω and μ such that

$$\left\|W_{\omega,\mu}\right\|_{\mathcal{L}(E)} \leq K.$$

Proof. see $[12]$, p. 29, lemma 6.3, estimate 60, and lemma 6.4, the assertion, 3, a similar proof.

Lemma 2 Let $\omega, \mu \in \mathbb{C}$. Assume (5) ~ (7). There exists $\mathrm{K} > 0$ such that for all $\omega \geq \omega_2$,

$$\left\|Q_\omega^2 \Lambda_{\omega,\mu}^{-1} Q_\omega{}^{-1}\right\|_{\mathcal{L}(E)} \leq K.$$

Proof. For all, $\omega = \max(\omega_1, \ \omega_2)$, this estimate is a direct consequence of Lemma 6.4, statement 3, see [12]. Below, the technical results useful for treating EDAs in Hölder spaces see [17].

Lemma 3 Let Q is the infinitesimal generator of a generalized analytic semi- group on E, $(e^{\xi Q})_{\xi \geq 0}$, not necessarily Continuous at 0, $\xi \in E$ and $\theta \in]0,1[$. Then, the following two assertions are equivalent:

1. $e^{\cdot Q}\xi \in C([0,1];\ E)$.
2. $\xi \in \overline{D(Q)}$.

Likewise, the following two assertions are equivalent:

1. $e^{\cdot Q}\xi \in C^{\theta}([0,1]\ ;\ E)$.
2. $\xi \in D_Q(\theta, +\infty)$.

Lemma 4 Let $\varphi \in C^{\theta}([0,1]\ ;\ E)$ with $\theta \in]0,1[$. Then

$$Q\int_0^1 e^{sQ_\omega}\big(\varphi(s)-\varphi(0)\big)ds \in D_Q(\theta, +\infty),$$

and

$$x \mapsto \int_0^x e^{(x-s)Q_\omega}\big(\varphi(s)-\varphi(0)\big)ds \in C^{1,\theta}([0,1];\ E) \cap C^{\theta}\big([0,1];\ D(Q)\big).$$

3. Representation of the solution

In order to solve problems (1)-(2), we use Krein's famous method of reducing the order of equation (1), see [15]. Under assumptions (5) ~ (7), suppose for everything $\omega \geq \omega_1$, the problem has a classical solution u; that is to say

$$u \in C^2([0,1]\,;\, E) \cap C([0,1]\,;\, D(A)),$$

$u(0) \in D(H\mu)$, and (1)-(2) are satisfied. So you admit exactly a representation formula (24), p. 10, as in [12], we apply this formula, with A, H, Q and Λ replaced by $A - \omega I$, $H + \mu I$, Q_ω and $\Lambda_{\omega,\mu}$.

Now, To obtain the necessary and sufficient conditions of existence and uniqueness of the strict solution u of Problem (1)-(2), we will study the regularity of this formula, which can be written in the form

$$u(x) = S(.,r_0,d_1,f) + \mathrm{R}(.,\mathrm{r}_0,\mathrm{d}_1,\mathrm{f}), \tag{9}$$

where $S(.,r_0,d_1,f)$ is the singular part given by

$$S(.,r_0,d_1,f) = S_1(.,r_0,d_1) + S_2(.,\mathrm{f}),$$

with, for all $x \in (0,1)$,

$$S_1(x,r_0,d_1) = e^{xQ_\omega}\left[\Lambda_{\omega,\mu}^{-1}(r_0 - Q_\omega{}^{-1}f(0))\right] + Q_\omega{}^{-2}f(0) + e^{(1-x)Q_\omega}\left[d_1 + Q_\omega{}^{-2}f(1)\right], \tag{10}$$

and

$$S_2(x,f) = e^{xQ_\omega}\Lambda_{\omega,\mu}^{-1}\int_0^1 e^{sQ_\omega}\big(f(s)-f(0)\big)ds - e^{xQ_\omega}\Lambda_{\omega,\mu}^{-1}e^{Q_\omega}\int_0^1 e^{(1-s)Q_\omega}\big(f(s)-f(1)\big)ds - \frac{1}{2}e^{xQ_\omega}{Q_\omega}^{-1}\int_0^1 e^{sQ_\omega}\big(f(s)-f(0)\big)ds + e^{xQ_\omega}\Lambda_{\omega,\mu}^{-1}{Q_\omega}^{-1}e^{Q_\omega}f(0) - \frac{1}{2}e^{(1-x)Q_\omega}{Q_\omega}^{-1}\int_0^1 e^{(1-s)Q_\omega}\big(f(s)-f(1)\big)ds + \frac{1}{2}{Q_\omega}^{-1}\int_0^x e^{(x-s)Q_\omega}\big(f(s)-f(0)\big)ds + \frac{1}{2}{Q_\omega}^{-1}\int_x^1 e^{(s-x)Q_\omega}\big(f(s)-f(1)\big)ds + e^{xQ_\omega}\Lambda_{\omega,\mu}^{-1}e^{Q_\omega}(I-e^{Q_\omega}){Q_\omega}^{-1}f(1), \tag{11}$$

and $R\,(.,r_0,d_1,f\,)$is the regular part given, for almost all $x \in (0,1)$, by

$$R\,(x,r_0,d_1,f\,) = -\frac{1}{2}e^{xQ_\omega}{Q_\omega}^{-2}e^{Q_\omega}f(0) - \frac{1}{2}e^{(1-x)Q_\omega}{Q_\omega}^{-2}e^{Q_\omega}f(1) - \frac{1}{2}{Q_\omega}^{-2}\big(f(0)+f(1)\big) + \bar{R}(.,r_0,d_1,f\,). \tag{12}$$

With

$$\bar{R}(x,r_0,d_1,f\,) = -e^{(2+x)Q}(I-e^{2Q})^{-1}\Big(2\Pi^{-1}Q\big[e^{Q}I_f(1)-J_f(0)\big]+J_f(0)\Big) + e^{(2-x)Q}(I+e^{2Q}(I-e^{2Q})^{-1})\Big(2\Pi^{-1}Q\big[e^{Q}I_f(1)-J_f(0)\big]+J_f(0)\Big) - e^{(3-x)Q}(I-e^{2Q})^{-1}I_f(1) + e^{(1+x)Q}[I+e^{2Q}(I-e^{2Q})^{-1}]I_f(1). \tag{13}$$

4. Regularity results

Proposition 7 Let $(E;\ \|\cdot\|_E)$ be a Banach space, suppose that (5) ~ (7), let $f \in C^\theta([0,1];E)$, $0<\theta<1$ and $r_0, d_1 \in E$. Then for all $\omega \geq \omega_1$we have

$$R\,(.,r_0,d_1,f\,) \in C^\infty([0,1];E) \text{ and } A_\omega R\,(.,r_0,d_1,f\,) \in C^\infty([0,1];E).$$

Proof. Since for all $\geq \omega_0$, ($\omega_0 > 0$, fixed), the operator Q_ω generates a generalized analytical semi-group, we have for all $n \in \mathbb{N}$

$$\mathrm{e}^{\mathrm{Q}_\omega} \in \mathcal{L}(E, D((Q_\omega)^n)),$$

so we have for all $\varphi \in \mathrm{E}$,

$$\mathrm{A}_\omega\, \mathrm{e}^{\cdot \mathrm{Q}_\omega} \mathrm{e}^{\mathrm{Q}_\omega} \varphi = -\, \mathrm{e}^{\cdot \mathrm{Q}_\omega} \mathrm{Q}_\omega^2 \mathrm{e}^{\mathrm{Q}_\omega} \varphi \in \mathrm{C}^\infty([0,1]; \mathrm{E}),$$

using (12) and (13) , we find for $x \in [0, 1]$

$$A_\omega R\,(x, r_0, d_1, \mathrm{f}) = \frac{1}{2} e^{Q_\omega}\left[e^{xQ_\omega} f(0) + e^{(1-x)Q_\omega} f(1)\right] + \frac{1}{2}\left(\mathrm{f}(0) + \mathrm{f}(1)\right) - \mathrm{Q}_\omega^2 \mathrm{R}\,(\mathrm{x}, \mathrm{r}_0, \mathrm{d}_1, \mathrm{f}),$$

as

$$Q_\omega^2 \bar{R}(.\,, r_0, d_1, \mathrm{f}) \in C^\infty([0,1]; E), \text{ so } \ A_\omega R(.\,, r_0, d_1, \mathrm{f}) \in C^\infty([0,1]; E).$$

Proposition 8 Let $(\mathrm{E}; \|\cdot\|)$ be a Banach space, assume (5) ~ (8), let $\mathrm{f} \in \mathrm{C}^\theta([0,1]; \mathrm{E}), 0 < \theta < 1$ and

$\mathrm{r}_0, \mathrm{d}_1 \in \mathrm{E}$. Then for all $\omega \geq \max(\omega_1,\ \omega_2)$,we have

$$A_\omega S_2(.\,, \mathrm{f}) \in C^\theta([0,1]; E).$$

Proof. Thanks to formula (11), we obtain

$$
\begin{aligned}
A_\omega S_2(x,\mathrm{f}) = {} & -Q_\omega^2 e^{xQ_\omega}\Lambda_{\omega,\mu}^{-1}\int_0^1 e^{sQ_\omega}\big(f(s)-f(0)\big)ds + Q_\omega^2 e^{xQ_\omega}\Lambda_{\omega,\mu}^{-1}e^{Q_\omega}\int_0^1 e^{(1-s)Q_\omega}\big(f(s)-f(1)\big)ds \\
& +\frac{1}{2}Q_\omega e^{xQ_\omega}\int_0^1 e^{sQ_\omega}\big(f(s)-f(0)\big)ds{Q_\omega}^2 e^{xQ_\omega}\Lambda_{\omega,\mu}^{-1}{Q_\omega}^{-1}e^{Q_\omega}f(0) \\
& -\frac{1}{2}Q_\omega e^{(1-x)Q_\omega}\int_0^1 e^{(1-s)Q_\omega}\big(f(s)-f(1)\big)ds - \frac{1}{2}Q_\omega\int_0^x e^{(x-s)Q_\omega}\big(f(s)-f(0)\big)ds \\
& -\frac{1}{2}Q_\omega\int_x^1 e^{(s-x)Q_\omega}\big(f(s)-f(1)\big)ds - {Q_\omega}^2 e^{xQ_\omega}\Lambda_{\omega,\mu}^{-1}e^{Q_\omega}(I-e^{Q_\omega}){Q_\omega}^{-1}f(1) \\
& = -{Q_\omega}^2 e^{xQ_\omega}\Lambda_{\omega,\mu}^{-1}e^{Q_\omega}(I-e^{Q_\omega}){Q_\omega}^{-1}f(1) - {Q_\omega}^2 e^{xQ_\omega}\Lambda_{\omega,\mu}^{-1}{Q_\omega}^{-1}e^{Q_\omega}f(0) + \sum_{i=1}^{6} I_i(x).
\end{aligned}
$$

We use Lemma 4, we obtain $\mathrm{I}_5(\mathrm{x}), \mathrm{I}_6(\mathrm{x}) \in \mathrm{C}^\theta([0,1];\mathrm{E})$, then we use the Lemma 3, we obtain also $\mathrm{I}_3(\mathrm{x})$, $\mathrm{I}_4(\mathrm{x}) \in \mathrm{C}^\theta([0,1];\mathrm{E})$.

Now, taking into account to hypothesis (8), we have

$$Q_\omega^2 \Lambda_{\omega,\mu}^{-1} {Q_\omega}^{-1} e^{Q_\omega}(I - e^{Q_\omega}) f(1) \ \in D_Q(\theta, +\infty) \text{ and } Q_\omega^2 \Lambda_{\omega,\mu}^{-1} {Q_\omega}^{-1} e^{Q_\omega} f(0) \in D_Q(\theta, +\infty),$$

we have

$$I_1(x) = -e^{xQ_\omega} Q_\omega^2 \Lambda_{\omega,\mu}^{-1} {Q_\omega}^{-1} Q_\omega \int_0^1 e^{sQ_\omega}\big(f(s) - f(0)\big) ds \in C^\theta([0,1]; E),$$

due to Lemma 4, we conclude that

$$e^{xQ_\omega} Q_\omega^2 \Lambda_{\omega,\mu}^{-1} {Q_\omega}^{-1} e^{Q_\omega}(\mathrm{I} - e^{Q_\omega}) f(1) \in C^\theta([0,1]; E), \text{and } e^{xQ_\omega} Q_\omega^2 \Lambda_{\omega,\mu}^{-1} {Q_\omega}^{-1} e^{Q_\omega} \ \in C^\theta([0,1]; E),$$

As a conclusion

$$A_\omega S_2(.,\mathrm{f}) \in C^\theta([0,1]; E).$$

4.1 Main results

Theorem 9 Let $(\mathrm{E}; \|\cdot\|)$ be a Banach space, assume (5) ~ (8), let $\mathrm{f} \in \mathrm{C}^\theta([0,1]; \mathrm{E}), 0 < \theta < 1$ and $\mathrm{r}_0, \mathrm{d}_1 \in \mathrm{E}$.then

for all $\omega \geq \max(\omega_1,\ \omega_2)$,we have:

1. Problem (1)–(2) admits a single strict solution u defined by (9), if and only

$$\begin{cases}\wedge_{\omega,\mu}{}^{-1}\, r_0,\ d_1 \ \in\ D\,(A),\ Q_\omega^2 \Lambda_{\omega,\mu}^{-1}(r_0 - Q_\omega{}^{-1} f(0)) + f(0) \in \overline{D(A)} \\ A_\omega d_1 + f(1) \in \overline{D(A)}.\end{cases}$$

2. Problem (1)−(2) admits a strict solution u verifying the property of maximum regularity u'', $A_\omega u \in C^\theta([0,1]; E)$, if and only if

$$\begin{cases}\wedge_{\omega,\mu}{}^{-1}\, r_0,\ d_1 \ \in\ D\,(A),\ Q_\omega^2 \Lambda_{\omega,\mu}^{-1}(r_0 - Q_\omega{}^{-1} f(0)) + f(0) \in D_Q(\theta, +\infty) \\ A_\omega d_1 + f(1) \in D_Q(\theta, +\infty).\end{cases}$$

Moreover, (9) can formulate this unique solution.

Proof.

for the proof, we use Lemma 3 as well as propositions 7 and 8, we apply direct reasoning, we proceed as given [4], Proof of Theorem 4.1, page 11.

5. Application example

Consider the complex Banach space$E = L^p(\Omega)$, 1<p<+∞ and Ω a bounded domain in $\mathbb{R}^n$, n> 1 with a regular boundary $\partial\Omega$, let A, H be operators defined in $E = L^p(0,1)$ by

$$\begin{cases} D(A) = \{\varphi \ \in\ W^4(\Omega) : \varphi\} \\ A\varphi(y) = a\Delta^2\varphi, \qquad y \in (0,1),\end{cases}$$

and

$$\begin{cases} D(H) = W^{2,p}(0,1) \cap W_0^{2,p}(0,1) \\ (H\varphi)(y) = b\Delta\varphi, \ \ y \in (0,1).\end{cases}$$

with *a, b* < 0. By direct calculations, one proves the hypotheses (5) ~ (8), see Favini et al in [12]. Then all our results apply to the following concrete quasi- elliptic boundary value problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2}(x,y) + a\Delta^2 u(x,y) - \omega u(x,y) = f(x,y), & (x,y) \in (0,1) \times \Omega. \\ \dfrac{\partial u}{\partial x}(0,y) - a\Delta \dfrac{\partial u}{\partial y}(0,y) - \mu u(0,y) = r_0\,, & y \in (0,1). \\ u(x,\xi) = \Delta_y\, u(x,\xi) = 0, & (x,\xi) \in (0,1) \times \partial\Omega \\ u(x,y) = d_1(y), & y \in (0,1). \end{cases}$$

where $\omega \geq 0, \mu \geq 0$, and d_1, r_0 are some functions given in $E = L^p(0,1)$, suppose that ω is big enough, $f \in C^\theta\big([0,1]; L^p(\Omega)\big)$, with $0 < \theta < 1$ and $1 < p < +\infty$.

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