

A Study on Intuitionistic Fuzzy Subgroups, their Translations in Invariant Space and Purpose of Topological Space

Md Raza Ansari (Research Scholar)

Department of Mathematics
Jai Prakash University
Chapra, Bihar, India
razaansari264@gmail.com

Prof. (Dr.) Md. Mushtaque Khan (Supervisor)

Department of Mathematics
Jai Prakash University
Chapra, Bihar, India
mushtaquemath@gmail.com

Abstract:

The study builds a topological approach for investigating invariant translational spaces of intuitionistic fuzzy subgroups with the goal to improve both theoretical understanding and practical usage in situations involving uncertainty. The research uses topological spaces to identify how intuitionistic fuzzy structures transform and remain continuous during translations for effective analysis of their translational behavior. Research in existing literature has presented theoretical ideas about fuzzy distance measures and class groupings yet studies remain insufficiently developed. The paper examines translational invariant homomorphism and isomorphism of intuitionistic fuzzy subgroups before introducing generalized concepts to describe these relationships. The developed model enables the analysis of imprecise data while finding direct application in secure network design and robotics path planning fields that need constant subgroup transformations. The paper contains groundwork which reveals the path toward applying the framework to higher-order groupoids and fuzzy logic-based integration with machine learning while establishing scalable and interpretable solutions.

Keywords: intuitionistic fuzzy subgroups, invariant translation, invariant space, homomorphism, isomorphism, homeomorphism

1. Introduction

Intuitionistic fuzzy sets emerged as an extended concept which took inspiration from Lotfi Zadeh's work (1965) on fuzzy logic and his recognition of membership definition based on human perception (Alharbi and Alghazzawi, 2022). Fuzzy logic introduced a different paradigm from classical set theory since its boundaries feature graded memberships through functions defining values within $[0, 1]$. The method used modeling techniques that fit to the vagueness and

uncertainty constantly found in human cognitive processes (Vidyadhar, 2020). In definition, it can be said that:

A fuzzy subgroup assigns a membership degree (a number between 0 and 1) to each member of a group as a measure to which the member belongs to the subgroup. The membership is defined through a mapping $\mu(A(x))$, where $\mu(A(x)) = 1$ indicates full membership, $\mu(A(x)) = 0$ indicates non-membership, and values between indicate partial membership. This concept generalizes the definition of a crisp subgroup, which is characterized through binary membership—either totally included (1) or totally excluded (0) (Razzaque and Razaq, 2022).

Atanassov took the research further by combining membership degrees with their corresponding non-membership degrees so intuitionistic fuzzy sets could be created (Sharma and Sharma, 2015). The methodology delivers exceptional benefits during situations which demand creative methods to determine exact group affiliation. The concept of intuitionistic fuzzy subgroups exists in algebraic structures through their foundation in previous theories.

The definition of subgroup receives its extension through intuitionistic fuzzy subgroups which permit membership and non-membership degrees for their elements. Such an alternative approach provides enhanced precision for analyzing uncertain subgroup-like structures that do not require absolute accuracy (Abbas, 2020).

Take an example. A university math club member can be 60% excited about algebra but 30% uninterested using intuitionistic fuzzy subgroup models. Thus, classical group theory is expanded to intuitive fuzzy subgroups, which allow group constituents to join or leave. Dual evaluation that measures involvement helps analysts understand team member relationships better than binary "in or out" assignments, making it suitable for loosely connected teams.

We introduce a topological space to study translational spaces of intuitionistic fuzzy subgroups because it creates an organized framework for their relationships and transformations. The main goal behind introducing topological spaces lies in converting intuitionistic fuzzy subgroups into structures that allow consistent analysis of their structural properties. That is, under such an environment a topological space allows researchers to create continuity definitions that enable them to examine how fuzzy subgroups transform throughout space.

Although, there are theoretical instances, such as, intuitionistic fuzzy distance measures, class grouping, etc. where topological spaces can be integrated to study the translational outcome of intuitionistic fuzzy subgroups, specific researches are few in number to emphasize and elaborate the context providing more justifiable validation (Khan, 2024). Acknowledging this scarcity, this research aims to study the purpose of topological space connected with invariant translational spaces of intuitionistic fuzzy subgroups and validate the theoretical suitability as applicable in finding the solutions of the following problems of intuitionistic fuzzy subgroups:

- Analyzing homomorphism and isomorphism of intuitionistic fuzzy subgroups under the condition of translation in invariant space and formulate a generalization notion

The research purpose stems from increasing demand for intuitionistic fuzzy subgroups in practical applications that do complex categorization and comparison tasks. Standards for theoretical backing of translational behavior via topological spaces remain insufficient to support reliable real-time solutions development. The research seeks to establish a strong theoretical framework which enhances effective task implementation of these concepts.

The research is organized in sections arranged in sequence intended to provide clarity and understanding in this way: Related works, followed by Problem Statement and motivation. Thereafter, Analytical discussion and Conclusion are provided as final notes.

Let's proceed with the elaboration of these sections one by one.

2. Related Works

2.1 Purpose of Fuzzy Sets, Fuzzy Subgroups, Intuitionistic Fuzzy Subgroups, Intuitionistic Fuzzy Numbers, Topological Space, Homomorphism and Isomorphism of Intuitionistic Fuzzy Subgroups

Multiple research studies jointly show that fuzzy set theory has grown from its beginning while its algebraic and topological extensions became crucial for computational applications and require further development of advanced methods for vague and uncertain systems.

Sostaks (2011) establishes the basis by explaining fuzzy sets function as mappings that transform universes into the $[0, 1]$ interval while explaining their origin from crisp sets. The paper describes how fuzzy alternatives have emerged alongside standard mathematical fields as it discusses topology in particular. Sostaks describes a significant development within fuzzy topology which transformed from applying crisp topologies to fuzzy sets into defining fuzzy topologies through fuzzy families of fuzzy sets because this method demonstrates better fidelity to fuzzy structures and inherent vagueness in the topological system. The paper builds the initial path toward transforming established mathematical constructions into fuzzy versions.

The research documents proceed to analyze algebraic structures of fuzzy sets while exploring fuzzy subgroups alongside their intuitionistic extensions. **Choudhary and Biswas (2020)** and **Rahmawati et al. (2020)** present definitions for fuzzy subgroups together with level subgroups properties and homomorphism evaluations. The studies demonstrate crucial computational benefits in using fuzzy subgroups for managing algebraic structures under conditions of uncertainty.

Moving ahead, according to **Palaniappan and Karjunan (2007)** intuitionistic fuzzy sets (IFSs) exist as an advanced form of fuzzy sets which uses membership and non-membership degrees for better handling of vague elements. The document develops the intuitionistic fuzzy subgroup theory after which it examines homomorphisms in this extended framework. Extending this approach remains essential since it enables researchers to develop models for conditions involving uncertainty about both membership level and hesitation or non-membership degree.

Several research papers demonstrate why topological spaces are needed when dealing with fuzzy and intuitionistic fuzzy structures. **Jana, Patel and Mahanta (2024)** recommends utilizing Pythagorean fuzzy sets (a derivative of IFSs) in topological spaces because they enhance spatial object indeterminacy detection better than traditional fuzzy models. By introducing the proof of topological invariance for Pythagorean fuzzy 9-intersection matrices researchers provide evidence of better computational facilities when analyzing uncertain relationships through topological approaches.

Nayagam et al. (2008) and **Sharma and Sharma (2015)** independently conduct research on fuzzy topological groups and group intuitionistic fuzzy topological spaces. The research presents systematic efforts at adding topological elements to fuzzy algebraic structures which leads to continuous and proximity analysis in ambiguous settings. Such integration aims to utilize topological analysis methods in studying fuzzy algebraic structures thus examining their transformed structural properties under continuous operations.

Research on intuitionistic fuzzy numbers (IFNs) including their fuzzy set and intuitionistic fuzzy subgroup links was conducted by **Santhi and Kungumaraj (2020)** and **Huang et al. (2023)**. Santhi introduces different approaches for topology generation through IFNs while demonstrating their capacity to unite topological theory and IFN theory. The analysis of IFN comparison difficulties due to hesitation degrees leads Huang to create a probabilistic approach which enhances decision-making processes under intuitionistic fuzzy information.

These research findings demonstrates that efficient techniques must be developed to process and analyze IFNs since decisions under multiple uncertain factors demand reliable approaches in particular contexts. These excerpts do not directly discuss fuzzy subgroup theory but IFN development creates advanced methods for handling the membership and non-membership degrees within intuitionistic fuzzy subgroups.

The research of **Bagchi (2018)** investigates fuzzy set computational properties in topological spaces with monoid embeddings whose focus centers on fuzzy measures and their analytical properties. The research illustrates how fuzzy techniques get incorporated into topological methods to conduct analysis and compute outcomes when systems exhibit natural uncertainty behavior.

2.2 Intuitionistic Fuzzy Subgroups, their Translational Characteristics in Invariant Space, Homomorphism, Isomorphism and Purpose of Topological Space

Multiple scholarly findings indicate about the vitality of expansion of fuzzy subgroups to intuitionistic fuzzy subgroups followed by their property analysis in topological spaces regarding homomorphisms and isomorphisms and translations in invariant spaces. A base theoretical framework sets forth by **Fathi and Salleh (2008)** presents intuitionistic fuzzy groups that emerge from intuitionistic fuzzy spaces for the purpose of extending fuzzy groups and fixing the challenge of determining intuitionistic fuzzy subgroups without a fixed universal set. Intuitionistic

fuzzy subgrouping implements this essential feature by creating an improved method to handle uncertainty which combines membership grade and non-membership analysis.

Al-Husban, Salleh, and Ahmad (2016) takes this research into new directions by introducing complex intuitionistic fuzzy groups which apply membership and non-membership degrees to the complex unit disc to enhance the handling of complex and diverse uncertainties. The pursuit of mathematical models for accurate complexity modeling demonstrates an ongoing need for complex fuzzy structures which **Wu and Gong (2025)** and **Jawad et al. (2025)** have both investigated as part of this research effort.

Topological spaces play a vital role when studying the structure and behavior patterns of complex intuitionistic fuzzy groupings. The development of topology results from an attempt to assess both "shape" and "continuity" features in these algebraic structures. Intuitionistic fuzzy distance measures together with topological data analysis (TDA) help refine uncertain datasets while demonstrating practical topological analysis of intuitionistic fuzzy sets according to **Khan (2024)**. The author **Nayagam et al. (2008)** develops translation invariant spaces in the context of intuitionistic fuzzy theory because such applications require translation invariance in processes like image processing.

Within this context homomorphism and isomorphism function as structural-preserving mappings which relate algebraic properties between intuitionistic fuzzy subgroups. The study of homomorphisms and isomorphisms in intuitionistic fuzzy subgroups and their complex extensions can be comprehended through the work of **Palaniappan and Karjunan (2007)**, **Xiaoping (2001)**, **Massa'deh (2016)**, **Adamu (2020)**, and **Jawad (2025)** who explain structural preservation aspects between different intuitionistic fuzzy algebraic entities.

The purpose of studying these mappings in topological spaces focuses on how spatial transformations affect the preservation of structural properties as indicated by **Sharma and Sharma (2015, 2016)** and **Venkateshwari and Lakshmana Gomathi Nayagam (2011)**. The topological existence of homomorphisms or isomorphisms that link intuitionistic fuzzy subgroups with their translation shifts indicates a spatial transformation preserves their algebraic structure at specific membership levels.

When complex grouping from topological space is applied to intuitionistic fuzzy subgroups it describes the sophisticated relationship between these dual-degree characterized subgroups as they interact with their topological environment. Through a framework of topology one can define both neighborhoods and continuity to analyze the entire intuitionistic fuzzy subgroup behavior within the space during translation-based operations. The mapping through homomorphisms and isomorphisms creates a link between original and translated groupings so that we can identify how fundamental algebraic and fuzzy components remain intact.

3. Problem Statement and Motivation

When analyzed under the invariant spaces conditions in particular, there is a substantial resource/validation/functional vacuum within the researches on intuitionistic fuzzy subgroups. Although, there are multiple works done that expanded fuzzy sets and subgroups into intuitionistic fuzzy subgroups and used topological spaces to define complex groupings. The analysis of complex intuitionistic fuzzy subgroup structures by topological spaces and the characterization of untranslated intuitionistic fuzzy subgroups by homomorphisms and isomorphisms function separately from each other.

The current body of work fails to unite topological grouping approaches with algebraic mapping methods through homomorphism and isomorphism to build a complete theoretical structure for studying transformations in intuitionistic fuzzy subgroups and numbers. The current inability to link between topological grouping and algebraic mappings (homomorphism and isomorphism) produces obstacles in generalization and scalability as well as decision-making efficiency for intuitionistic fuzzy subgroups and numbers under dynamic spatial or transformational circumstances. A concentrated examination of unifying techniques between these concepts represents a necessary step to achieve improved comprehension and better implementation results of intuitionistic fuzzy algebraic structures throughout computational fields.

4. Theoretical Formulation of the Proposed Notion and its Discussion

4.1 Preliminary Definitions

1. Bijection: The conceptual connection between a bijection function ensures perfect mapping between source elements and target elements while maintaining the original structure across fuzzy systems.

2. Intuitionistic Fuzzy Set (IFS): IFS over a universe X assigns both a membership degree $\mu(x)$ and a non-membership degree $\nu(x)$ to each element $x \in X$ under the restriction $0 \leq \mu(x) + \nu(x) \leq 1$ (Parvathi and Radhika, 2015).

3. Intuitionistic Topological Fuzzy Space (IFTS): A pair (X, τ) such that X is a set and τ is a collection of intuitionistic fuzzy sets on X that fulfill open set axioms like classical topology.

4. Translation Invariance: A group G has a topology τ being translation invariant if for any open set A of τ and for any element x in G , the translated set $A + x$ belongs to τ .

5. Intuitionistic Fuzzy Translation Invariant Space: A topological group (G, τ) is intuitionistic fuzzy translation invariant if the topology τ is closed under translation, maintaining fuzzy openness at any point in the group.

6. Intuitionistic Fuzzy Translation Invariant Space: A topological group (G, τ) is intuitionistic fuzzy translation invariant if the topology τ is translation closed and fuzzy openness at any point in the group is preserved.

7. Homomorphism: A mapping $f : G \rightarrow H$ between groups is a homomorphism if the group operation is preserved. $f(x*y) = f(x) \cdot f(y)$ for all x, y belonging to the group G (Palaniappan and Karjunan, 2007).

8. Intuitionistic Fuzzy Continuous Map: Let f be a function from an Intuitionistic Fuzzy Topological Space (IFTS) to another one. f is intuitionistic fuzzy continuous if for all intuitionistic fuzzy open sets, the pre-image is an intuitionistic fuzzy open set.

9. Intuitionistic Fuzzy Open Map: A mapping f is fuzzy open if the image of any intuitionistic fuzzy open set in the domain is open in the codomain.

10. Onto: A group homomorphism $f : G \rightarrow H$ is surjective if, for every $h \in H$, there is some $g \in G$ such that $f(g) = h$.

11. Isomorphism: An invertible group homomorphism $f : G \rightarrow H$ for which the map inverse is itself a homomorphism, hence defining structural isomorphism of G and H (Santhi and Udhayarani, 2023).

12. Homeomorphism: A bijective, open, and continuous map between two IFTSs that provides a topological isomorphism.

4.2 Theoretical Formulation of Notion

Homomorphism and Isomorphism Cases Under the Condition of Translation of Intuitionistic Fuzzy in Invariant Space

In this section, through theoretical explanations, it is affirmed that topological space and homomorphism in the intuitionistic fuzzy subgroup setting acquires special significance when it is viewed in the constraint of translation invariance. Such a setting implies that the meaning of translation invariance ensures that the topological properties of intuitionistic fuzzy sets are preserved under group operation, thereby assuring structure consistency at different locations in space (Lee and Lee, 2000). Such preservation is not only to the entire group but also to its subgroups, and this indicates the stability and scalability of the structure.

The following theorems justify the above reasoning (Nayagam, 2008):

Theorem 1a: *Let (G, τ) be an intuitionistic fuzzy translation invariant space. Then, (G, τ) is translation invariant.* (Nayagam, 2008)

Proof: An intuitionistic fuzzy translation invariant space requires that for any x in G and any open set A in τ , the translated set $A + x$ is also an element of τ . This immediately shows that the topology τ is invariant under translations, which satisfies the translation invariance criterion.

Theorem 2a: *Let H be a subgroup of G . If (G, τ) is an intuitionistic fuzzy translation invariant space, then the subspace H , with the subspace topology induced from τ , is also intuitionistic fuzzy translation invariant. (Nayagam, 2008)*

Proof: Since H is a subgroup of G and τ is translation invariant, the translation of any open set in H by an element of H is still in H and is open under the subspace topology. Hence, the subspace H retains the translation invariance property from G .

With the incorporation of homomorphisms, i.e., open, intuitionistic fuzzy continuous, and surjective, it is possible to transfer faithfully the invariant group structure of the source group into the target group. This type of connecting structure brings out the compatibility and stability of intuitionistic fuzzy topological spaces with algebraic modifications and becomes possible to make fuzzy or imprecise systems are more efficiently analyzable and modelable.

Theorem 3a: *Let $f: G \rightarrow G'$ be a surjective intuitionistic fuzzy continuous and intuitionistic fuzzy open homomorphism. Then (G, μ) is intuitionistic fuzzy translation invariant if and only if (G', μ') is intuitionistic fuzzy translation invariant. (Nayagam, 2008)*

Proof: Let $qy \in G'$ be an intuitionistic fuzzy singleton of G' , and let $B \subseteq G'$. To prove that (G', μ') is intuitionistic fuzzy translation invariant, it is enough to prove that: $qy \in B$

Since f is surjective, $f^{-1}(y) \neq \emptyset$; choose $x \in f^{-1}(y)$.

Define an intuitionistic fuzzy singleton $px \in G$ by:

$$\begin{aligned} px^+(z) &= qy^+(f(z-x)) \\ px^-(z) &= qy^-(f(z-x)) \end{aligned}$$

Since f is intuitionistic fuzzy continuous, $A \subseteq f^{-1}(B)$, and hence,

$px \in f^{-1}(B)$ by hypothesis.

Since f is intuitionistic fuzzy open, $f(px) \in B$.

Now we claim that $f(px) = qy$, which completes the proof.

We compute:

$$qy^+(t) = \sup_{z \in G, f(z)=t} px^+(z) = \sup_{z \in G, f(z)=t} qy^+(f(z-x)) = qy^+(t-f(x)) = qy^+(0) = qy^+(t)$$

Similarly,

$$qy^-(t) = \inf_{z \in G, f(z)=t} px^-(z) = qy^-(t)$$

Hence, $f(px) = qy$.

Therefore, $qy \in B$, proving the theorem.

In case of isomorphism, generalization on invariance translation is done in limited researches. Thus, in this work, a generalized case of isomorphism under the condition of intuitionistic invariant translation invariant space is given. Before coming to its formulation, let's understand the base proposition together with the working definitions and theories (Lee and Lee, 2000).

Base Proposition: *Let A be an intuitionistic fuzzy subgroup (IFSG) of a group H , and let $f : G \rightarrow H$ be an isomorphism from a group G onto H . Then the composition $A \circ f$ is an intuitionistic fuzzy subgroup of G .*

– used in the work of Palaniappan and Karjunan (2007)

Base Definition 1: Let (X, T) and (Y, U) be two **intuitionistic** fuzzy topological spaces (IFTS). A function $f : X \rightarrow Y$ is said to be (Lee and Lee, 2000):

1. Continuous if for every intuitionistic fuzzy open set (IFOS) B in Y , the pre-image $f^{-1}(B)$ is an IFOS in X ,
or equivalently, for every intuitionistic fuzzy closed set (IFCS) B in Y , $f^{-1}(B)$ is an IFCS in X .
2. Open if for every IFOS A in X , the image $f(A)$ is an IFOS in Y .
3. Closed if for every IFCS A in X , the image $f(A)$ is an IFCS in Y .
4. A homeomorphism if f is bijective, continuous, and open (Lee and Lee, 2000).

Base Theorem 1b: Let $f : (X, T) \rightarrow (Y, U)$ be a function. Then f is continuous if and only if for every intuitionistic fuzzy point (IFP) $x(\mu, \nu) \in X$ and every **intuitionistic** fuzzy neighborhood (IFN) B of $f(x(\mu, \nu))$, there exists an IFN A of $x(\mu, \nu)$ in X such that $x(\mu, \nu) \in A$ and $f(A) \subseteq B$.

Proof: Suppose f is continuous. Let $x(\alpha, \beta) \in X$ and let B be an IFN of $f(x(\alpha, \beta))$. Then there exists an IFOS $C \subseteq B$ in Y such that $f(x(\alpha, \beta)) \in C$. Since f is continuous, $f^{-1}(C)$ is an IFOS in X , and we have: $x(\alpha, \beta) \in f^{-1}(f(x(\alpha, \beta))) \subseteq f^{-1}(C) \subseteq f^{-1}(B)$

Let $A = f^{-1}(C)$. Then A is an IFN of $x(\alpha, \beta)$, and clearly: $x(\alpha, \beta) \in A$ and $f(A) \subseteq B$.

Conversely, let B be an IFOS in Y . If $f^{-1}(B) = \emptyset$, the claim is trivial. Suppose $x(\alpha, \beta) \in f^{-1}(B)$. Then B is an IFN of $f(x(\alpha, \beta))$. By hypothesis, there exists an IFN $A_{x(\alpha, \beta)}$ of $x(\alpha, \beta)$, such that:

$$x(\alpha, \beta) \in A_{x(\alpha, \beta)} \text{ and } f(A_{x(\alpha, \beta)}) \subseteq B.$$

Since $A_{x(\alpha, \beta)}$ is an IFN, there exists an IFOS $C_{x(\alpha, \beta)} \subseteq A_{x(\alpha, \beta)}$. Then:

$$x(\alpha, \beta) \in C_{x(\alpha, \beta)} \subseteq f^{-1}(f(A_{x(\alpha, \beta)})) \subseteq f^{-1}(B).$$

Hence: $f^{-1}(B) = \bigcup_{x(\alpha,\beta) \in f^{-1}(B)} C_{x(\alpha,\beta)} \in \mathcal{F}^{-1}$

which implies $f^{-1}(B)$ is a union of IFOSs and thus itself an IFOS in X . Therefore, f is continuous.

Base Theorem 2b: Let $f : (X, T) \rightarrow (Y, U)$ be a function. Then the following statements are equivalent:

1. f is continuous.
2. For every intuitionistic fuzzy set (IFS) $A \subseteq X$, $f(\text{cl}(A)) \subseteq \text{cl}(f(A))$.
3. For every IFS $B \subseteq Y$, $\text{cl}(f^{-1}(B)) \subseteq f^{-1}(\text{cl}(B))$.
4. For every IFS $B \subseteq Y$,

Proof: Equivalence among (1), (3), and (4) follows from [3].

To show that (1) \Rightarrow (2):

Assume f is continuous and $A \subseteq X$ is an IFS. Then, $\text{cl}(f(A))$ is an IFCS in Y , so $f^{-1}(\text{cl}(f(A)))$ is an IFCS in X . Hence,

$$\text{cl}(A) \subseteq \text{cl}(f^{-1}(\text{cl}(f(A)))) \subseteq f^{-1}(\text{cl}(f(A))).$$

Applying f to both sides,

$$f(\text{cl}(A)) \subseteq f(f^{-1}(\text{cl}(f(A)))) \subseteq \text{cl}(f(A)).$$

To show that (2) \Rightarrow (3):

Let $B \subseteq Y$ be any IFS. Then

$$f(\text{cl}(f^{-1}(B))) \subseteq \text{cl}(f(f^{-1}(B))) \subseteq \text{cl}(B).$$

Applying f^{-1} to both sides,

$$\text{cl}(f^{-1}(B)) \subseteq f^{-1}(\text{cl}(B)).$$

Note: The condition $\text{int}(f(A)) \subseteq f(\text{int}(A))$ for each IFS $A \subseteq X$ is **not** equivalent to continuity. This is demonstrated in the original text using counterexamples.

Base Definition 2: Let $\{(G_i, \tau_i)\}_{i \in I}$ be an indexed family of intuitionistic fuzzy topological spaces. The product intuitionistic fuzzy topology τ on $\prod_{i \in I} G_i$ is defined as the smallest intuitionistic fuzzy topology such that all canonical projection maps $\pi_i : \prod_{j \in I} G_j \rightarrow G_i$, given by $\pi_i((x_j)_{j \in I}) = x_i$, are intuitionistic fuzzy continuous for each $i \in I$ (Gandhi, 2011).

Lemma: Let $\{(X_i, \tau_i)\}_{i \in I}$ be an indexed family of intuitionistic fuzzy topological spaces, and let $(\prod_{i \in I} X_i, \tau)$ denote their product intuitionistic fuzzy topological space. Then for any finite sub-collection $J \subset I$, the product space $(\prod_{j \in J} X_j, \tau)$ is **intuitionistic fuzzy homeomorphic** to a subspace of $(\prod_{i \in I} X_i, \tau)$, where τ is the product intuitionistic fuzzy topology on $(\prod_{i \in I} X_i, \tau)$, provided that each $X_i \times X_i$ is nonempty (Gandhi, 2011).

Proof: Fix $x_i \in X_i$ for each $i \in I$, and define the map $f : \prod_{j \in J} X_j \rightarrow \prod_{i \in I} X_i$ by

$$f((x_j)_{j \in J}) = (y_i)_{i \in I},$$

where $y_i = \begin{cases} x_j, & \text{if } i \in J \\ x_i, & \text{if } i \notin J \end{cases}$

Then f is well-defined, injective, and continuous with respect to the intuitionistic fuzzy topologies. The image of f is a subspace of $\prod_{i \in I} X_i$, and f is a homeomorphism onto its image, completing the proof.

Base Theorem 3b: Let $\{(G_i, \tau_i)\}_{i \in I}$ be an indexed family of intuitionistic fuzzy topological spaces. The product space $(\prod_{i \in I} G_i, \tau)$ is intuitionistic fuzzy translation invariant if and only if each (G_i, τ_i) is intuitionistic fuzzy translation invariant (Gandhi, 2011).

Proof: Assume the product space $(\prod_{i \in I} G_i, \tau)$ is intuitionistic fuzzy translation invariant. Let $J \subset I$ be a finite index set, and let e_i be the identity element of each G_i . Define the map $f : \prod_{j \in J} G_j \rightarrow \prod_{i \in I} G_i$ by

$$f((x_j)_{j \in J}) = (y_i)_{i \in I}, \text{ where}$$

$$y_i = \begin{cases} x_j, & \text{if } i \in J \\ e_i, & \text{if } i \notin J \end{cases}$$

By the Lemma, the image of f is a subspace of the product space. Since the product space is translation invariant, this subspace is also translation invariant. Hence, each (G_j, τ_j) is translation invariant.

Conversely, assume each (G_i, τ_i) is intuitionistic fuzzy translation invariant. Let $p = (p_i) \in \prod_{i \in I} G_i$ and consider a basic open set of the form $\pi_j^{-1}(A_j)$, where $A_j \in \tau_j$.

Define the fuzzy singleton π_i in G_i as

$$\mu_{\pi_i}(z) = \begin{cases} \mu(x_i), & z = x_i \\ 0, & \text{otherwise,} \end{cases} \quad \nu_{\pi_i}(z) = \begin{cases} \nu(x_i), & z = x_i \\ 1, & \text{otherwise} \end{cases}$$

By the hypothesis, since each (G_i, τ_i) is translation invariant, we conclude that the product space is also translation invariant.

Base Definition 3: Let (G, τ) be an intuitionistic fuzzy topological space. The operator is defined as (Gandhi, 2011): $T_p : G \rightarrow GT$ by $T_p(q)(x) = \min \{ \mu_p(x), \mu_q(x) \}$, $\nu T_p(q)(x) = \max \{ \nu_p(x), \nu_q(x) \}$

where p, q are intuitionistic fuzzy singletons supported at elements $x, y \in G$ respectively. The resulting singleton pq has support at xy with the above membership and non-membership functions. This operator is called the **intuitionistic fuzzy translation invariant operator**.

In order to confirm and generalize the Base Proposition stated above, in the intuitionistic fuzzy topological context, it must be reformulated with the proved definitions and theorems, in particular with regard to continuity, homeomorphism, and translation invariance in intuitionistic fuzzy topological spaces (IFTS).

So, let's proceed with step by step reformulation as:

Step 1: Continuity Preservation

Proposition: Let $f : (X, \tau_X) \rightarrow (Y, \tau_Y)$ be a continuous function between two IFTSs. If $A \subset Y$ is an intuitionistic fuzzy open set (IFOS) or an intuitionistic fuzzy closed set (IFCS), then $f^{-1}(A) \subset X$ is also an IFOS or IFCS.

Validation: This coincides directly with Base Definition 1 and is justified by Base Theorem 1b, which states that continuity ensures the preimage of all IFOS or IFCS is similarly open or closed. Thus, the structure is preserved under sustained mappings, in the same manner that an IFSG is an IFSG maintained by isomorphism through composition.

Step 2: Topological Subgroup Structure Preservation

Proposition: Let $A \subset H$ be an intuitionistic fuzzy set with a topology τ_H making it an intuitionistic fuzzy topological group, and let $f : G \rightarrow H$ be a bijective, continuous isomorphism that has a continuous inverse. If $A \circ f$ is an IFSG in G , then $(A \circ f, \tau_G)$ forms an intuitionistic fuzzy topological group, where $\tau_G = f^{-1}(\tau_H)$.

Validation: This is a consequence of:

- **Base Definition 1:** The function f is a homeomorphism if it ensures continuity as well as openness.
- **Base Theorem 2b:** Closure operations maintain continuity.
- **Condition:** Since f is bijective and preserves the group operation, and since fuzzy membership is preserved through pre-images (composition), $A \circ f$ preserves the subgroup and topological structure. The result is an IFTS-preserving structure, similar to the preservation of the algebraic IFSG structure in the fundamental premise.

Step 3: Translation Invariance Preservation

Proposition: Let $\{(G_i, \tau_i)\}_{i \in I}$ be a family of intuitionistic fuzzy topological groups. The product group $\prod_{i \in I} G_i$ is an intuitionistic fuzzy topological group and translation invariant if and only if every G_i is translation invariant.

Validation: This is specifically confirmed in Base Theorem 3b. The lemma ensures that finite products (subspaces) gain the topological structure through embedding into the universal product space. The translation invariance operator T_p is preserved from component-based operations.

5. Conclusions

The theoretical extension of homomorphism and isomorphism through invariant translation within intuitionistic fuzzy topological spaces demonstrates the importance of topological relevancy together with value preservation between different algebraic structures which include groups, semigroups and sub groupoids. The constant framework maintains consistent system modeling in cases where imprecise data and uncertainty rules the conditions. The practical applicability of this concept stems from its role in designing secure networks and robotics path planning for fuzzy environments because both areas require translation-invariant subgroup structures for distributed system maintenance of integrity and consistent topological inference support during robotic movements. Such properties become preserved through the implementation of intuitionistic fuzzy homomorphisms and isomorphisms which creates new prospects for scalable interpretable models operating in imprecise domains. The formulation requires future development to scale up from groupoids of first order to higher-order structures and to investigate categorical invariants within intuitionistic fuzzy logic together with producing algorithmic solutions for machine learning, control systems and spatial reasoning under uncertainty conditions.

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