# Application of the Hybrid Piecewise Quadratic Triangular Elements (PQTE) method and Taylor Series (TS) method to a Data Set

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## Abstract.

In this article, we proposed a new surface fitting method so-called hybrid PQTE-TS method for fitting surfaces that combines the piecewise quadratic triangular elements (PQTE) method and Tayler series (TS) method to achieve a smooth surface. Then, we investigated the possibility of applying the hybrid PQTE-TS methodology to a data sets taken from Franke [1], to reconstruct a six different surfaces. The accuracy of the proposed methods is evaluated by numerical experimentation and we concluded that the hybrid PQTE-TS method is an effective method for constructing an accurate surfaces and can be employed in the future to reconstruct a real surfaces from a real data such as leaf surface. Such models of leaf simplify an understanding of plant growth and allow the modelling of the plants interaction with their environment. Moreover, Accurate leaves representation is a significant research area because it is a part of plant model development as well as it has many applications, for example, the leaf model can be employed to investigate the droplet path on the leaf surface which is essential for realizing how a droplet of nutrient, pesticide or water will be imbibed within the leaf surface. Therefore, the hybrid PQTE-TS methodology can be applied for this purpose.

**Keywords**. Piecewise quadratic triangular elements method; Tayler series method; Leaf surface model; mathematical modelling.

## 1. Introduction

The application of surface fitting methods to reconstruct the leaf surfaces is the main aim of the research presented in this article. This is a significant research area because a virtual plant model development requires an accurate image of leaves. In this paper, we examine scattered data interpolation techniques, for the purposes of proposing in the future the preferred leaf surface image, based on the piecewise quadratic triangular element (PQTE) method [2] and the Taylor series (TS) method. Finally, a hybrid PQTE-TS method is proposed.

Plant architecture modelling has been investigated widely in the past few decades [6–8]. Accurate Leaf representation is a significant research area since it is a part of plant model development as well as it has many applications, for example, the application of pesticide droplets spreading on the leaf surface. Recentaly, several scientists have been interest in modelling leaf surface such as Oqielat et al. [9] applied Hardy's and gaussion RBF interpolant to model different types of leaves and afterward the author [10-12] included to the RBF a cubic or linear term to extend his proposed model. Furthermore, Oqielat et al. [13-14] suggested a new hybrid technique based finite element method to model leaves surfaces. Finally, oqielat [15] employed the Bernstein polynomials to recreate the leaf surface using 3D data points.

This paper consists of two major sections. In section two, an overview of the PQTE interpolation method is given. Moreover, the Taylor series is discussed here for the purpose of gradients estimate. In Section 3, a numerical experiment is presented to evaluate the accuracy of the Hybrid PQTE-TS method via a set of points taken from (Franke, 1982) for six different test functions. The first, second, and third-order Taylor series are presented in this section for the purpose of estimating the gradient for the PQTE method. The root means square error (RMS) is employed in the analysis of the numerical results, to compute the approximation quality of the hybrid PQTE-TS method.

## 2.1 Piecewise Quadratic triangular elements (PQTE) method

The interpolation technique is stated as:

Given n scattered data point triplets $(x_i, y_i, z_i)$, $i = 1,2, \ldots, n$, find an interpolant $f: D \subset R^2 \to R$ satisfying

$$f(x_i, y_i) = z_i, \quad i = 1,2, \ldots, n, \tag{2.1}$$

D is the function domain. The data points $(x_i, y_i)^T$ are distinct and not all collinear.

The finite element technique (FET) is based on splitting the area where the data point is located into subdomains, and afterward operating a rectangulation or triangulation division to the points to formulate elements on which we can create interpolants in a piecewise way. The most favorably applied technique is triangulation and we will adopt it in this research. In the FET the $z_i$, $i = 1,2, \ldots, n$, is assigned at the vertices of the element and then in each element a polynomial is fitted as well as the derivatives must be approximated. Then by joining the functions on each element we generate the complete surface. For more information, the readers are referred to [2].

Surface reconstruction is frequently tracked by contour maps development to describe the surface. Thus, it is beneficial to have the surface comprised of quadratic pieces, for example, patches of triangles (elements) such that on each element the surface is characterized by a linked quadratic function to guarantee that the entire surface is smooth. Nevertheless, it has been shown by researchers [3] that the PQTE method does not create a smooth surface over the whole domain, unless all triangles are acute, whereas this can be achieved via cubic patches or complete quintic, which is very difficult as it required too many nodes.

A smooth can be obtained using the PQTE method if the domain is tessellated by isosceles triangles, where these isosceles triangles are associated with the equilateral standard triangle $(T_0)$, figure can be found in [2]. The PQTE is a seamed element technique, whereby the dotted lines divide the macro-elements (triangle) denoted by (T) into six micro-elements (sub triangle). Subsequently, on each micro-element, an interpolating quadratic function is constructed to allow piecewise quadratic interpolant (continuously differentiable) over the entire area. It produces that quadratic function space has dimension nine and corresponded with the interpolation nine nodes.

The vertices coordinate of any (T) are given by:

$$(x_o, y_o), (x_1, y_o), \left(\frac{1}{2}(x_o + x_1), y_1\right), \tag{2.2}$$

where $x_1 > x_o$ and $y_1 > y_o$, or the form:

$$(\frac{1}{2}(x_o + x_1), y_1),\ \ (\frac{1}{2}(3x_1 - x_o), y_1),\ \ (x_1, y_o). \tag{2.3}$$

If (T) has vertices as the one given in Eq. (2.2), then (T) is mapped onto $(\mathrm{T}_0)$ by means of:

$$\xi = 3(\frac{x-x_o}{x_1-x_o}) - 1, \quad \eta = \sqrt{3}(\frac{y-y_o}{y1-y_o} - \frac{1}{3}). \tag{2.4}$$

If (T) has vertices as the one given in Eq. (2.3), then (T) is mapped onto $(\mathrm{T}_0)$ by means of:

$$\xi = (\frac{x1+x_o-2x}{x1-x_o}) + 1,\ \ \eta = \sqrt{3}(\frac{y_o-y}{y1-y_o} + \frac{2}{3}). \tag{2.5}$$

Then, the given function values $u(P_j), j = 1,2,3$ and first derivatives$\left(\frac{\partial u}{\partial z_j}, \frac{\partial u}{\partial w_j}\right), j = 1,2,3$, on the vertices of (T) (9 nodal values) are to be transferred to $(\mathrm{T}_0)$ with the aid of Eq. (2.4) and Eq. (2.5) to determine the interpolant on the $(\mathrm{T}_0)$ given by $(\xi, \eta)$ . The PQTE interpolant has the following form:

$$\begin{aligned} u(\xi,\eta) = \sum_{j=1}^{3} u(P_j)B_j(\xi,\eta) + \sum_{j=1}^{3} (\frac{\partial u}{\partial zj})(P_j)B_{3+j}(\xi,\eta) \\ + \sum_{j=1}^{3} \left(\frac{\partial u}{\partial w_j}\right)(P_j)B_{6+j}\,(\xi,\eta), \end{aligned} \tag{2.6}$$

where the nine functions $B_j(\xi,\eta)$ are the cardinal basis functions (see [2]).

As one might expect, the cardinal functions are quite difficult to describe. We do it in two stages [following (Ritchie, 1978)]. We first define nine functions $g_1, g_2, \dots, g_9$ (see eq. 2.7) and then define the cardinal functions $\mathrm{B}_1, \mathrm{B}_2, \dots, \mathrm{B}_9$ (see [2]) in terms of them.

It will be convenient to introduce the truncated function denoted by $(+)$ as follows:

$$S_+ = \begin{cases} 0 \ , \ s < 0 \\ s \ , \ s \geq 0 \end{cases}$$

The function $(\mathrm{x} + \mathrm{y})_+$ in the x-, y-coordinate plane then takes the value x+y when $\mathrm{x} + \mathrm{y} \geq 0$, i.e. to the right of the line y = -x, and takes the identically zero value when $\mathrm{x} + \mathrm{y} < 0$, or to the left of the line y = -x. We will need functions defined by expressions like$\xi_+^2$, or $(\xi - \eta)_+^2$. Here, it is to be understood that the operation of truncation, denoted by the subscript +, is performed first and is followed by the operation of exponentiation. Now define,

$$\begin{aligned} &\mathrm{g}_1(\xi,\eta) = 1, && \mathrm{g}_2(\xi,\eta) = \xi, && \mathrm{g}_3(\xi,\eta) = \eta, \\ &g_4(\xi,\eta) = \xi_+^2, && g_5(\xi,\eta) = (-\xi)_+^2, && g_6(\xi,\eta) = \left(\eta - \frac{\sqrt{3}}{3}\xi\right)_+^2, \\ &g_7(\xi,\eta) = \left(\frac{\sqrt{3}}{3}\xi - \eta\right)_+^2, && g_8(\xi,\eta) = \left(\eta + \frac{\sqrt{3}}{3}\xi\right)_+^2, && g_9(\xi,\eta) = \left(-\eta - \frac{\sqrt{3}}{3}\xi\right)_+^2. \end{aligned} \tag{2.7}$$

An interpolant on the standard triangle is determined by the (transformed) nodal values and cardinal function as given in Eq. (2.6).

In many applications, the derivative at the vertices is not existing. The vertex gradient approximates are usually produced from neighboring data. Breslin [5] introduced an approach to estimate the directional derivatives necessary for the Clough-Tocher method (CTM) information based on difference quotients derived using a Taylor expansion and then applied direct least-squares fit the difference quotients. Loch [4] estimated the gradients at the vertices for use in the CTM following the approach introduced in [5] and then estimated directional derivatives at each edge midpoint by averaging the gradients (on the same edge) at the two vertices. Oqielat et [16-18] proposed a new technique based on using radial basis function (RBF) to find the CT tiangle required derivatives. Moreover, Belward [16] and Turner [17] assessed the accuracy of the least squares methods.

In this paper, we took the same gradient estimation idea and used the set of neighbors closest to the triangle vertex to create the estimated derivative using a truncated TS. This process allows an overdetermined arrangment to be produced. Then, the required gradients are obtained by solving this system in the least squares sense. The neighboring data were employed to approximate the $1^{st}, 2^{nd}$ and $3^{rd}$ derivatives in the TS approximation.

To employ our method, the PQTE-element is transferred to a standard triangle $T_0$ which is split into six sub-triangles and has the vertices located at

$$P_1 = \left(0, \frac{2\sqrt{3}}{3}\right), P_2 = \left(-1, \frac{-\sqrt{3}}{3}\right), P_3 = \left(1, \frac{-\sqrt{3}}{3}\right),$$

such that the interpolation on each sub-element corresponds to the interpolation on the sub- element on a standard triangle and can always be done on a $T_0$ and then transformed to any given triangle $T$. The mapping of an arbitrary triangle $T$ having vertices $(\mathrm{x}_0, \mathrm{y}_0), (\mathrm{x}_1, \mathrm{y}_1)$, and $(\mathrm{x}_2, \mathrm{y}_2)$ onto a $T_0$, can be found in [2].

## 2.2. Taylor Series Method

As stated earlier, the PQTE requires derivative estimate at the triangle vertices. Now, the process of estimating the gradients using the Taylor series is summarized in the following paragraphs.

Let $Z = f(x, y)$ be the interest surface. Our aim here is to estimate the derivative of $f$ at a point by calculating the difference quotients from near scattered points. Let $(x_i, y_i)$, $i$=1, 2…m, be the neighbors of the target that needs the gradient estimate. By using TS given by:

$$f(X_j + h_i) \cong f(X_j) + \sum_{k=1}^{\mathrm{m}} \frac{1}{k!} (h_i \nabla)^k f(X_j), \qquad i = 1,2, \dots, m, \ j = 1,2,3, \tag{2.8}$$

where $X_j = (x_j, y_j), j = 1,2,3$ represents the vertices of the triangle and $h = (h_x, h_y)$ represents the distance from the vertex to the neighbors points .

In this research, we used a truncated Taylor series expansion (first, second, and third order) to produce estimate derivatives from a set of nearby neighbors closest to the triangle vertices. This method allows for the creation of an overdetermined system (see Eq. 2.9). Then, by solving this system (using least-squares approach), the required gradients (see Eq. 2.10) are obtained.

$$Au = q, \tag{2.9}$$

where $q, u$ and $A$ are given as follow:

$$\mathrm{A} = \begin{bmatrix} h_{x1} & h_{y1} & \frac{1}{2}\, h_{x1}^2 & h_{x1}h_{y1} & \frac{1}{2}\, h_{y1}^2 & \frac{1}{6} h_{x1}^3 & \frac{1}{2} h_{x1}^2\, h_{y1} & \frac{1}{2}\, h_{x1}h_{y1}^2 & \frac{1}{6} h_{y1}^3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ h_{x20} & h_{y20} & \frac{1}{2}\, h_{x20}^2 & h_{x20}h_{y20} & \frac{1}{2}\, h_{y20}^2 & \frac{1}{6} h_{x20}^3 & \frac{1}{2} h_{x20}^2\, h_{y20} & \frac{1}{2}\, h_{x20}h_{y20}^2 & \frac{1}{6} h_{y20}^3 \end{bmatrix},$$

$$u = \left[ f_x \quad f_y \quad \frac{d^2 f}{dx^2} \quad f_x f_y \quad \frac{d^2 f}{dy^2} \quad \frac{d^3 f}{dx^3} \quad \frac{d^2 f}{dx^2}\frac{df}{dy} \quad \frac{d^2 f}{dy^2}\frac{df}{dx} \quad \frac{d^3 f}{dy^3} \right]^T, \tag{2.10}$$

$$q = [\Delta f_1 \quad \Delta f_2 \quad \Delta f_3 \quad \dots \quad \Delta f_{19} \quad \Delta f_{20}]^T.$$

## 3. Numerical Experimentation for the hybrid PQTE-TS method

The results of our numerical experiments using the hybrid PQTE-TS method given in section 2 is presented here. A data set was taken from [1] to test the accuracy of our method. The data contains two sets and 6 test functions defined on$[0,1]^2$, (see Figure 1). The first set consists of 100 points scattered uniformly over $[0,1]^2$ and used to triangulate the surface for the PQTE method using MATLAB command called Delaunay. While the other set consists of 30 points which we have replaced it with 900 points. The reason behind taking 900 point instead of 30 point was to have a closer points to the triangles vertices so we can have a better derivative estimation at those vertices. The 900 points are employed to evaluate the accuracy of the hybrid PQTE-TS technique (see Table 3.1) by calculating the RMS (root mean square error)

$$RMS = \sqrt{\frac{\sum_{i=1}^{900}\left(f(a_i,b_i) - Q(a_i,b_i)\right)^2}{900}},$$

where $f(a_i, b_i)$, $Q(a_i, b_i)$ represent respectively the exact function value and the approximate value for the set of data using the hybrid PQTE-TS technique. Following are the 6 test functions (see Figure 2):

$$f_1(x,y) = 0.75e^{-\frac{(9x-2)^2}{4}-\frac{(9y-2)^2}{4}} + 0.75e^{-\frac{(9x+1)^2}{49}-\frac{9y}{10}-\frac{1}{10}} + 0.5e^{-\frac{(9x-7)^2}{4}-\frac{(9y-3)^2}{4}} - 0.2e^{-(9x-4)^2-(9y-7)^2},$$

$$f_2(x,y) = \frac{\tanh\big(9(y-x)\big)+1}{9},$$

$$f_3(x,y) = \frac{1.25+\cos(5.4y)}{6+6*(3x-1)^2},$$

$$f_4(x,y) = \left(\frac{1}{3}\right)e^{\left(\left(\frac{-81}{16}\right)*(x-0.5)^2+2+\left(\frac{-81}{16}\right)*(y-0.5)^2\right)},$$

$$f_5(x,y) = \frac{e^{-20.25\left((x-0.5)^2+(y-0.5)^2\right)}}{3},$$

$$f_6(x,y) = \sqrt{\frac{64}{81} - (x-0.5)^2 - (y-0.5)^2} - 0.5.$$

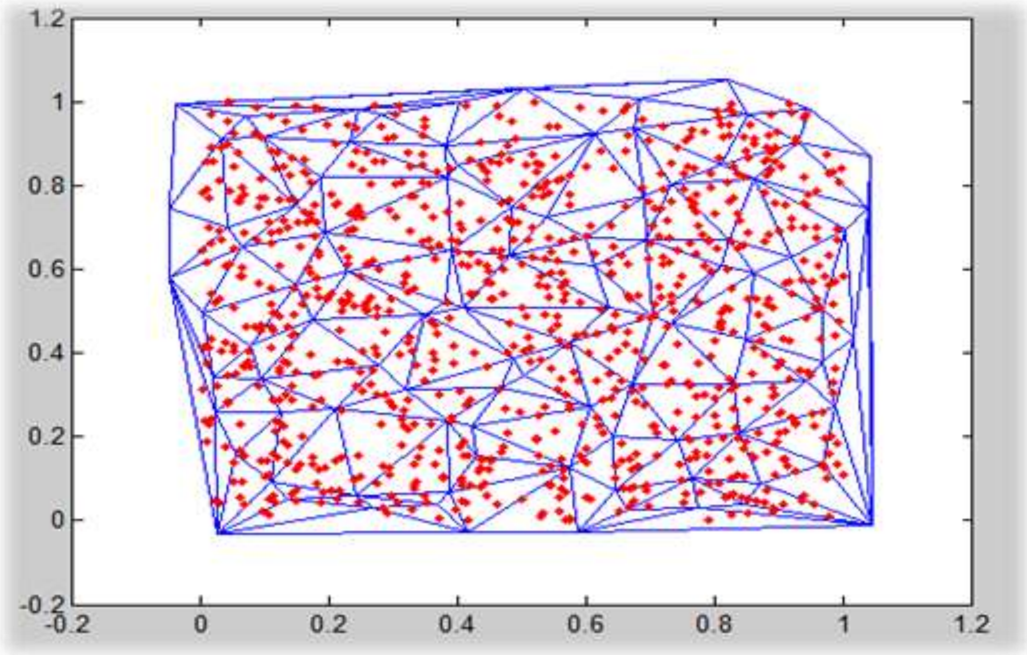

**Figure 1: The Triangulation of the 100 points, the red dots represent the 900 points.**

**Table 3.1: RMS errors of the six functions by employing the first, second, and third-order TS to approximate the derivative at the PQTE triangle vertices.**

| Function | PQTE method & Exact Gradient | PQTE method &$1^{st}$ Taylor | PQTE method & $2^{nd}$ Taylor | PQTE method & $3^{rd}$Taylor |
|---|---|---|---|---|
| $f_1$ | 0.0713 | 0.0764 | 0.0741 | 0.0779 |
| $f_2$ | 0.0247 | 0.0247 | 0.0247 | 0.0259 |
| $f_3$ | 0.0258 | 0.0264 | 0.0259 | 0.0267 |
| $f_4$ | 0.0251 | 0.0258 | 0.0255 | 0.0261 |
| $f_5$ | 0.0279 | 0.0276 | 0.0271 | 0.0288 |
| $f_6$ | 0.0265 | 0.0265 | 0.0267 | 0.0265 |

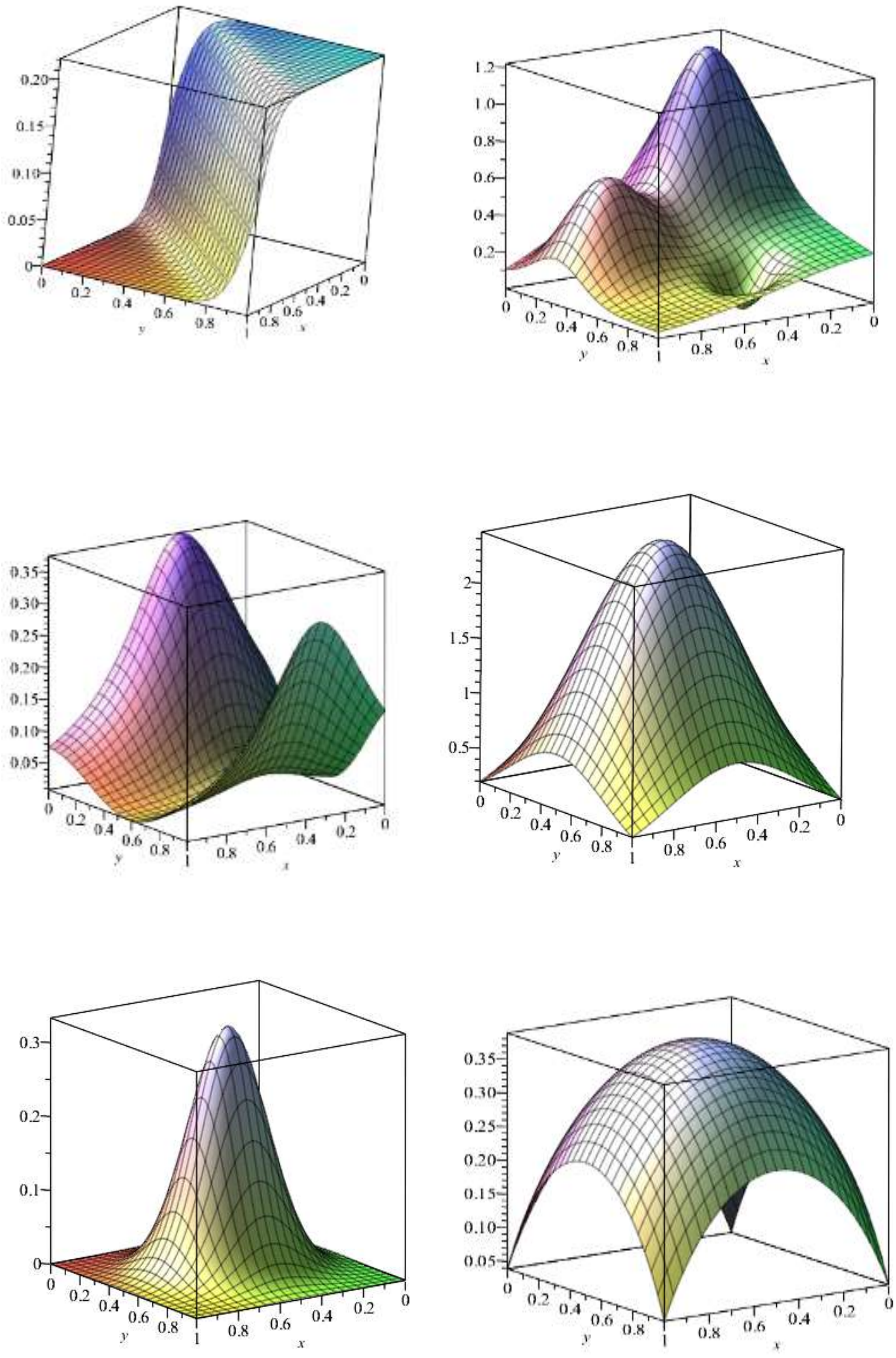

**Figure 2: Six test function taken from (Franke, 1982).**

Table 3.1 display the error (RMS) acquired using hybrid PQTE-TS technique using the exact function gradient and TS technique (the $1^{st}$, $2^{\mathrm{nd}}$ and $3^{\mathrm{rd}}$ order) given in Eq. (2.8) to find the derivatives at the PQTE triangle vertices. The approach applied here is based on the set of (7, 12, and 20 points) nearest neighbors closest to each vertex for derivative estimates, see Eq. (2.9). The 7 closest points are used to estimate the first derivatives at each vertex while 12 points are used to find the second derivative and 20 points are used to estimate the third derivative. Note that the second column in Table 3.1, shows the exact gradients, using the six test functions, which construct the best error (RMS) using the hybrid PQTE-TS method. Therefore, these gradients should be considered as the benchmark for our comparison between the $1^{st}$, $2^{\mathrm{nd}}$ and $3^{\mathrm{rd}}$ order TS to estimate the required gradients for the PQTE method.

We observe from Table 3.1 that the errors obtained using the $3^{\mathrm{rd}}$ order TS to estimate the derivative did not offer accurate RMS more than that constructed using the $1^{st}$ and $2^{\mathrm{nd}}$ order TS which is perhaps related to the function behavior and the selection of the points as well as to the. Moreover, $2^{\mathrm{nd}}$ order TS offered a few enhancements in RMS over the $1^{st}$ order TS. Note, though, that when rising the order of the TS, the computational process for the least-squares solution also increases. Therefore, we must question whether this extra calculation effort that gave small improvements is needed. Moreover, the $1^{st}$, $2^{\mathrm{nd}}$ and $3^{\mathrm{rd}}$ order Taylor offers gradients are approximately very close to the one produced using the exact gradients thus the hybrid PQTE-TS method works well, and we can justify its use.

### Conclusion

A hybrid piecewise quadratic triangular elements Taylor series (PQTE-TS) method is proposed in this article, which is a finite element method, to reconstruct a smooth surfaces. Then we looked into the possibility of using the PQTE-TS methodology to reconstruct surfaces using two data sets. Gradient estimation at the vertices of the triangular elements is required by the PQTE method. Thus, we employed Taylor series method to estimate these gradients from nearby data information.

Our numerical experiments results for the surface fitting PQTE-TS method were presented to test the accuracy of OQTE-TS method, we used two sets of data and six test functions (Franke, 1982). We found that the accuracy of the PQTE-TS method is acceptable, therefore we can justify its use.

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