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Poisson-Modified Linear-Exponential Distribution

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Abstract:

This proposed distribution is named as Poisson-Modified Linear-exponential distribution (PMLED) which has only one parameter and it is a discrete compound probability distribution. The statistical characteristics required for studying, investigating and analysing the proposed distribution are well explained, defined and derived. When we applied the Chi-square goodness of fit test to some count secondary data related to error proneness and biological sciences, we found that PMLED seems to be more suitable for statistical modelling than Poisson-Lindley distribution (PLD) of Shankaran (1970).

Key-words: Probability distribution, Poisson distribution, Chi-square goodness of fit test, Moments, Compounding, Distribution.

1.0 Introduction:

Just as agricultural scientists who are responsible for generating new but improved quality of plants by using agricultural compounding process of plants of different nature, similarly statistician are also responsible for obtaining many compound distributions by mixing Poisson distribution (PD) with many countable and continuous probability distributions. Here, the concept used are under continuous mixture of Poisson distribution [1]. To give shape of the proposed distribution, Poisson distribution is mixing with Modified Linear-exponential distribution (MLED) of Sah and Sahani [2]. In this process, the parameter of PD works as a continuous variable which follows MLED. The expression (1) is probability density function of MLED

$$h(u,\tau)=\frac{(\pi\tau)^2}{[1+(\pi\tau)^2]}(\pi\tau+u)\,e^{-(\pi\tau u)} \tag{1}$$

Provided that $\tau>0$ and $u>0$.

The references ([3] to [9]) show the previous work on countable and continuous mixtures of Poisson distribution and related to statistical concepts which are very helpful to improve quality of the proposed distribution. The important work of this distribution has been placed under the following headings

1.0 Introduction
2.0 Results
3.0 Applications
4.0 Conclusions

2.0 Results

2.1 Probability Mass Function (pmf) and Probability Generating Function(pgf) and Moment Generating Function (mgf) of PMLED:

The pmf, pgf and mgf of this distribution are denoted by respectively $P(u,\tau), P_u^{(t)}$ and $M_u^{(t)}$,and these are defined and derived as follows and given by the expression (2), (3) and (4) respectively.

$$P(u,\tau)=\frac{(\pi\tau)^2}{[1+(\pi\tau)^2]}\frac{1}{u!}\left[\int_0^\infty\left\{(e^{-\theta}\theta^u)(\pi\tau+\theta)e^{-(\pi\tau\theta)}\right\}d\theta\right]$$

Or, $P(u,\tau)=\frac{(\pi\tau)^2}{[1+(\pi\tau)^2]}\frac{1}{u!}\left[\int_0^\infty\left\{(\pi\tau\theta^u+\theta^{u+1})e^{-(\pi\tau+1)\theta}\right\}d\theta\right]$

Or, $P(u,\tau)=\frac{(\pi\tau)^2}{[1+(\pi\tau)^2]}\left[\frac{\left\{(1+u+\pi\tau+(\pi\tau)^2\right\}}{(1+\pi\tau)^{u+2}}\right]$ (2)

Provided that $\tau>0$ and $u=0,1,2,...$

Graphs of pmf of PMLED:

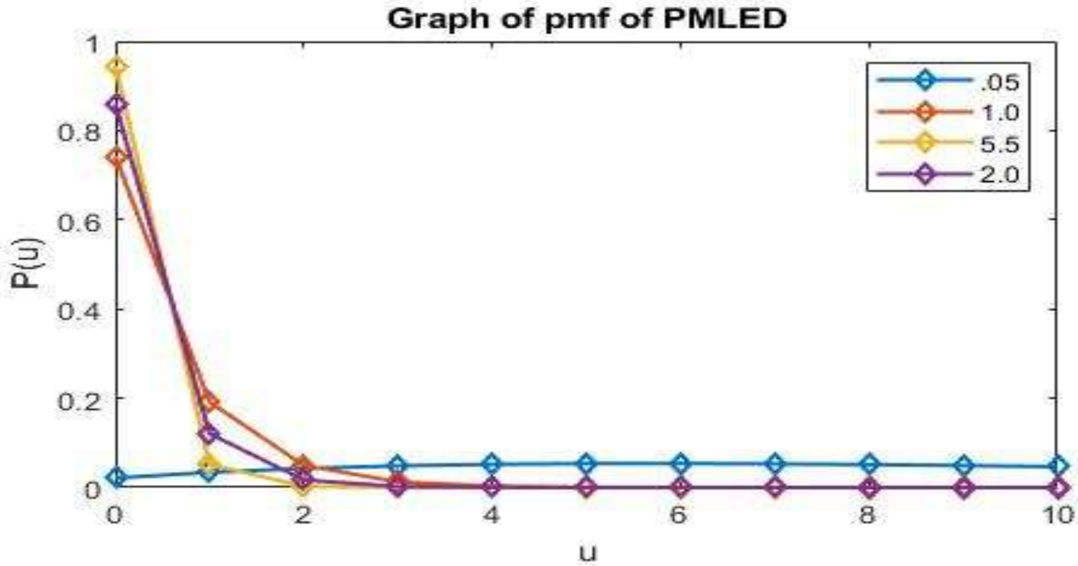

Figure-1: Showing graph of pmf at $\tau=0.05,1.0,5.5,2.0$

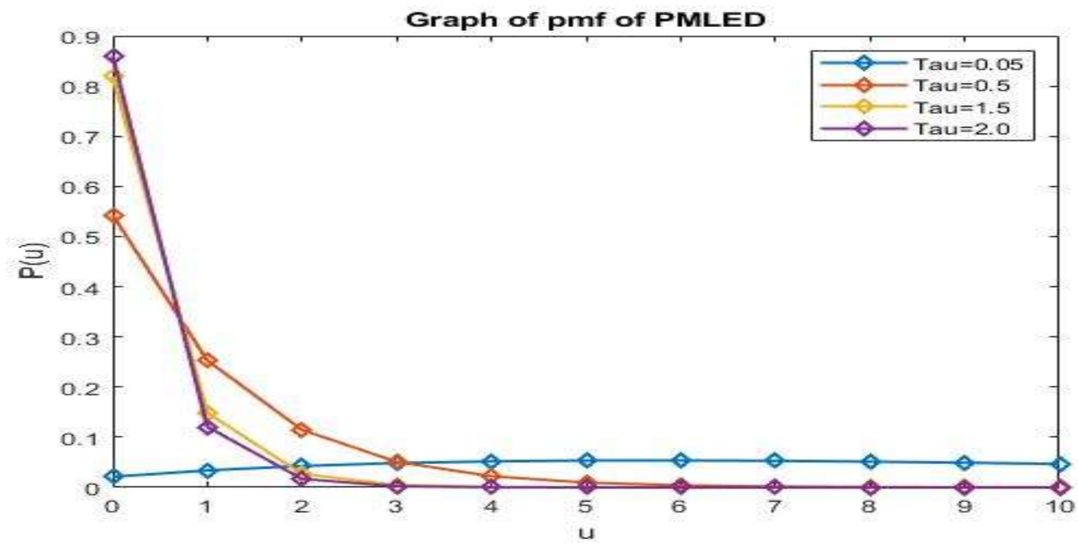

Figure-2: Showing graph of pmf at $\tau=0.05,0.5,1.5,2.0$

Probability Generating Function [$P_u^{(t)}$]:

$$P_u^{(t)}=\frac{(\pi\tau)^2}{[1+(\pi\tau)^2]}\left[\int_0^\infty\left\{(\pi\tau+\theta)e^{-(1+\pi\tau-t)\theta}\right\}d\theta\right]$$

Or, $P_u^{(t)} = \frac{(\pi\tau)^2}{[1+(\pi\tau)^2]}\left[\frac{\pi\tau}{(1+\pi\tau-t)} + \frac{2}{(1+\pi\tau-t)^2}\right]$

Or, $P_u^{(t)} = \frac{(\pi\tau)^2}{[1+(\pi\tau)^2]}\left[\frac{\pi\tau(1+\pi\tau-t)+1}{(1+\pi\tau-t)^2}\right]$ (3)

Moment Generating Function [$M_u^{(t)}$]:

$$M_u^{(t)} = \frac{(\pi\tau)^2}{[1+(\pi\tau)^2]}\left[\int_0^\infty \left\{(\pi\tau+\theta)e^{-(1+\pi\tau-e^t)\theta}\right\} d\theta\right]$$

Or, $M_u^{(t)} = \frac{(\pi\tau)^2}{[1+(\pi\tau)^2]}\left[\frac{\pi\tau}{(1+\pi\tau-e^t)} + \frac{1}{(1+\pi\tau-e^t)^2}\right] = \frac{(\pi\tau)^2}{[1+(\pi\tau)^2]}\left[\frac{\pi\tau(1+\pi\tau-e^t)+1}{(1+\pi\tau-e^t)^2}\right]$ (4)

2.2 Moments about the Origin (μ_r') and Moments about the Mean (μ_r) of PMLED:

The general form of the r^{th} moment about the origin and the first four moments about the origin are given by the equations from (5) to (9) respectively.

$$\mu_r' = \frac{(\pi\tau)^2}{[1+(\pi\tau)^2]}\left[\int_0^\infty \left\{\sum_0^\infty \frac{u^r e^{-\theta}\theta^u}{u!}\right\}(\pi\tau+\theta)e^{-\pi\tau\theta} d\theta\right] \quad (5)$$

$$\mu_1' = \frac{(\pi\tau)^2}{[1+(\pi\tau)^2]}\left[\int_0^\infty \theta(\pi\tau+\theta)e^{-\pi\tau\theta} d\theta\right]$$

Or, $\mu_1' = \frac{(\pi\tau)^2}{[1+(\pi\tau)^2]}\left[\pi\tau\int_0^\infty \theta e^{-\pi\tau\theta} d\theta + \int_0^\infty \theta^2 e^{-\pi\tau\theta} d\theta\right] = \frac{(\pi\tau)^2}{[1+(\pi\tau)^2]}\left[\frac{(\pi\tau)\Gamma 2}{(\pi\tau)^2} + \frac{\Gamma 3}{(\pi\tau)^3}\right]$

Or, $\mu_1' = \frac{[(\pi\tau)^2+2]}{(\pi\tau)[1+(\pi\tau)^2]}$ (6)

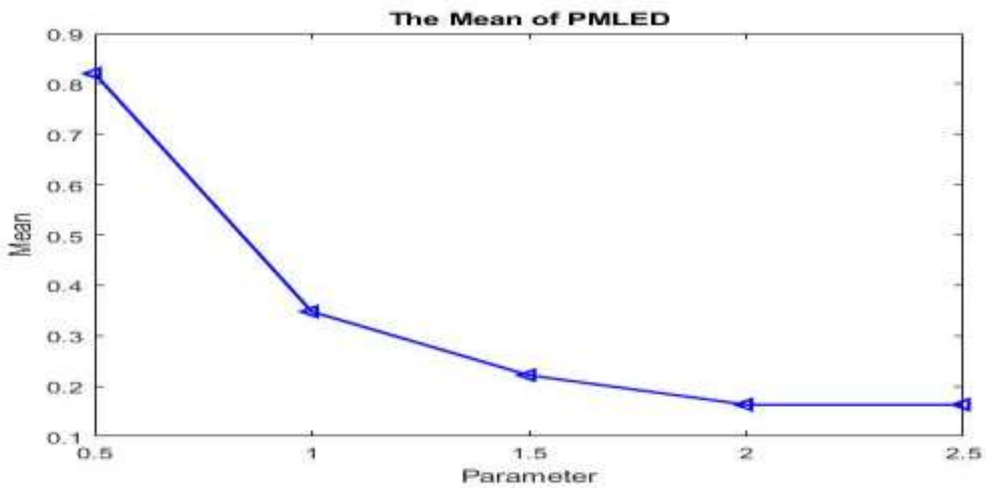

Figure-3: Showing graph of the mean of PMLED

$$\mu_2' = \frac{(\pi\tau)^2}{[1+(\pi\tau)^2]}\left[\int_0^\infty (\theta+\theta^2)(\pi\tau+\theta)e^{-\pi\tau\theta} d\theta\right] = \frac{(\pi\tau)^2}{[1+(\pi\tau)^2]}\left[\int_0^\infty (\pi\tau\theta+(1+\pi\tau)\theta^2+\theta^3)e^{-\pi\tau\theta} d\theta\right]$$

$$\mu_2' = \frac{(\pi\tau)[(\pi\tau)^2+2]+2[(\pi\tau)^2+3]}{(\pi\tau)^2[1+(\pi\tau)^2]} \quad (7)$$

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$$\mu_3' = \frac{(\pi\tau)^2}{[1+(\pi\tau)^2]}\left[\int_0^\infty (\theta+3\theta^2+\theta^3)(\pi\tau+\theta)e^{-\pi\tau\theta}d\theta\right]$$

Or, $\mu_3' = \frac{(\pi\tau)^2}{[1+(\pi\tau)^2]}\left[\pi\tau\int_0^\infty (\theta+3\theta^2+\theta^3)e^{-\pi\tau\theta}d\theta + \int_0^\infty (\theta^2+3\theta^3+\theta^4)e^{-\pi\tau\theta}d\theta\right]$

Or, $\mu_3' = \frac{(\pi\tau)^2[(\pi\tau)^2+6(\pi\tau)+6]+[2(\pi\tau)^2+18(\pi\tau)+24]}{(\pi\tau)^3[1+(\pi\tau)^2]}$ (8)

$$\mu_4' = \frac{(\pi\tau)^2}{[1+(\pi\tau)^2]}\left[\int_0^\infty (\theta+7\theta^2+6\theta^3+\theta^4)(\pi\tau+\theta)e^{-\pi\tau\theta}d\theta\right]$$

Or, $\mu_4' = \frac{(\pi\tau)^2}{[1+(\pi\tau)^2]}\left[\pi\tau\int_0^\infty (\theta+7\theta^2+6\theta^3+\theta^4)e^{-\pi\tau\theta}d\theta + \int_0^\infty (\theta^2+7\theta^3+6\theta^4+\theta^5)e^{-\pi\tau\theta}d\theta\right]$

Or, $\mu_4' = \frac{(\pi\tau)^2[(\pi\tau)^3+14(\pi\tau)^2+36(\pi\tau)+24]+[2(\pi\tau)^3+42(\pi\tau)^2+144(\pi\tau)+120]}{(\pi\tau)^4[1+(\pi\tau)^2]}$ (9)

And the first four moment about the mean have been obtained and expressed by the equations from (10) to (13) given as

$\mu_1 = 0$ (10)

$$\mu_2 = \frac{(\pi\tau)[(\pi\tau)^2+2]+2[(\pi\tau)^2+3]}{(\pi\tau)^2[1+(\pi\tau)^2]} - \left[\frac{[(\pi\tau)^2+2]}{(\pi\tau)[1+(\pi\tau)^2]}\right]^2$$

Or, $\mu_2 = \frac{[2+2(\pi\tau)+4(\pi\tau)^2+3(\pi\tau)^3+(\pi\tau)^4+(\pi\tau)^5]}{[(\pi\tau)^2+2(\pi\tau)^3+(\pi\tau)^5]}$ (11)

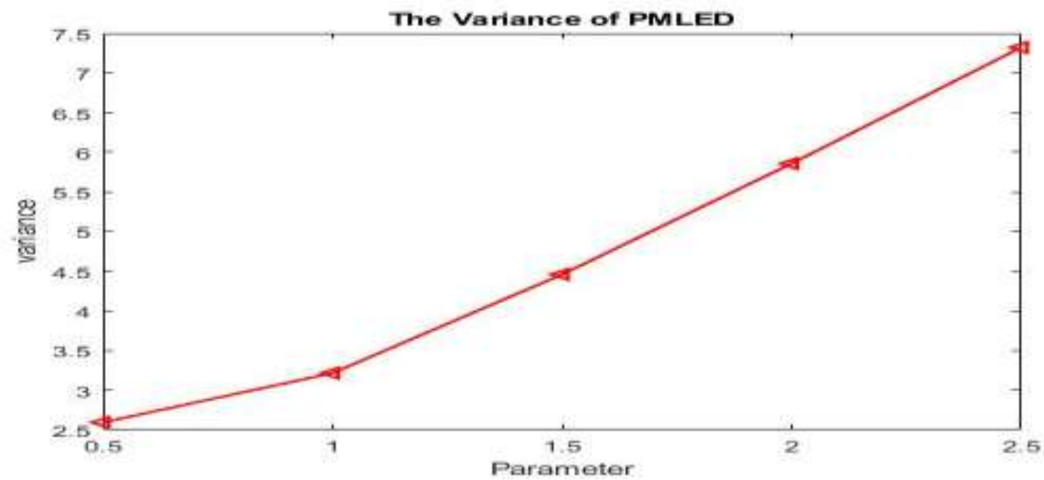

Figure-4: Showing graph of the variance of PMLED

$$\mu_3 = \frac{(\pi\tau)^2[(\pi\tau)^2+6(\pi\tau)+6]+[2(\pi\tau)^2+18(\pi\tau)+24]}{(\pi\tau)^3[1+(\pi\tau)^2]}$$
$$-3\left[\frac{(\pi\tau)[(\pi\tau)^2+2]+2[(\pi\tau)^2+3]}{(\pi\tau)^2[1+(\pi\tau)^2]}\right]\left[\frac{[(\pi\tau)^2+2]}{(\pi\tau)[1+(\pi\tau)^2]}\right]+\left[\frac{[(\pi\tau)^2+2]}{(\pi\tau)[1+(\pi\tau)^2]}\right]^3$$

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$$\text{Or, } \mu_3 = \frac{[4+6(\pi\tau)+14(\pi\tau)^2+18(\pi\tau)^3+17(\pi\tau)^4+15(\pi\tau)^5+6(\pi\tau)^6+3(\pi\tau)^7+(\pi\tau)^8]}{\left[(\pi\tau)\left\{1+(\pi\tau)^2\right\}\right]^3} \quad (12)$$

$$\mu_4 = \frac{(\pi\tau)^2[(\pi\tau)^3+14(\pi\tau)^2+36(\pi\tau)+24]+[2(\pi\tau)^3+42(\pi\tau)^2+144(\pi\tau)+120]}{(\pi\tau)^4[1+(\pi\tau)^2]}$$

$$-4\left[\frac{(\pi\tau)^2[(\pi\tau)^2+6(\pi\tau)+6]+[2(\pi\tau)^2+18(\pi\tau)+24]}{(\pi\tau)^3[1+(\pi\tau)^2]}\right]\left[\frac{[(\pi\tau)^2+2]}{(\pi\tau)[1+(\pi\tau)^2]}\right]$$

$$+6\left[\frac{(\pi\tau)[(\pi\tau)^2+2]+2[(\pi\tau)^2+3]}{(\pi\tau)^2[1+(\pi\tau)^2]}\right]\left[\frac{[(\pi\tau)^2+2]}{(\pi\tau)[1+(\pi\tau)^2]}\right]^2-3\left[\frac{[(\pi\tau)^2+2]}{(\pi\tau)[1+(\pi\tau)^2]}\right]^4$$

$$\text{Or, } \mu_4 = \frac{\begin{array}{l}[24+48(\pi\tau)+122(\pi\tau)^2+182(\pi\tau)^3+224(\pi\tau)^4+247(\pi\tau)^5+188(\pi\tau)^6+135(\pi\tau)^7\\ +69(\pi\tau)^8+23(\pi\tau)^9+10(\pi\tau)^{10}+(\pi\tau)^{11}]\end{array}}{\left[(\pi\tau)\left\{1+(\pi\tau)^2\right\}\right]^4} \quad (13)$$

2.3 Features of PMLED: Condition for over-dispersion, co-efficient of skewness and kurtosis have been obtained and given by the equations (14) to (16) respectively.

Condition for Over-dispersion:

$(Variance - Mean) > 0$

$$\text{Or, } \left[\frac{[2+2(\pi\tau)+4(\pi\tau)^2+3(\pi\tau)^3+(\pi\tau)^4+(\pi\tau)^5]}{[(\pi\tau)^2+2(\pi\tau)^3+(\pi\tau)^5]}-\left\{\frac{[(\pi\tau)^2+2]}{(\pi\tau)[1+(\pi\tau)^2]}\right\}\right] > 0$$

$$\text{Or, } [2(\pi\tau+2(\pi\tau)^3+2(\pi\tau)^4+3(\pi\tau)^5+3(\pi\tau)^6+(\pi\tau)^7+(\pi\tau)^8] > 0 \quad (14)$$

Which is true because $\tau > 0$.

$$\gamma_1 = \frac{[4+6(\pi\tau)+14(\pi\tau)^2+18(\pi\tau)^3+17(\pi\tau)^4+15(\pi\tau)^5+6(\pi\tau)^6+3(\pi\tau)^7+(\pi\tau)^8]}{[2+2(\pi\tau)+4(\pi\tau)^2+3(\pi\tau)^3+(\pi\tau)^4+(\pi\tau)^5]^{3/2}} \quad (15)$$

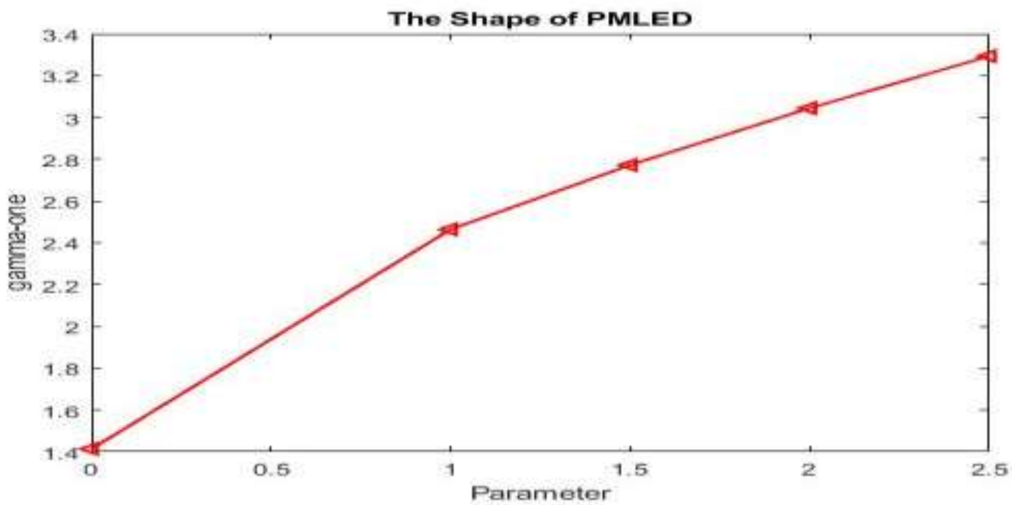

Figure-5: Showing graph of shape of PMLED

$$\beta_2 = \frac{\begin{array}{l}[24+48(\pi\tau)+122(\pi\tau)^2+182(\pi\tau)^3+224(\pi\tau)^4+247(\pi\tau)^5+188(\pi\tau)^6+135(\pi\tau)^7\\ +69(\pi\tau)^8+23(\pi\tau)^9+10(\pi\tau)^{10}+(\pi\tau)^{11}]\end{array}}{[2+2(\pi\tau)+4(\pi\tau)^2+3(\pi\tau)^3+(\pi\tau)^4+(\pi\tau)^5]^2} \quad (16)$$

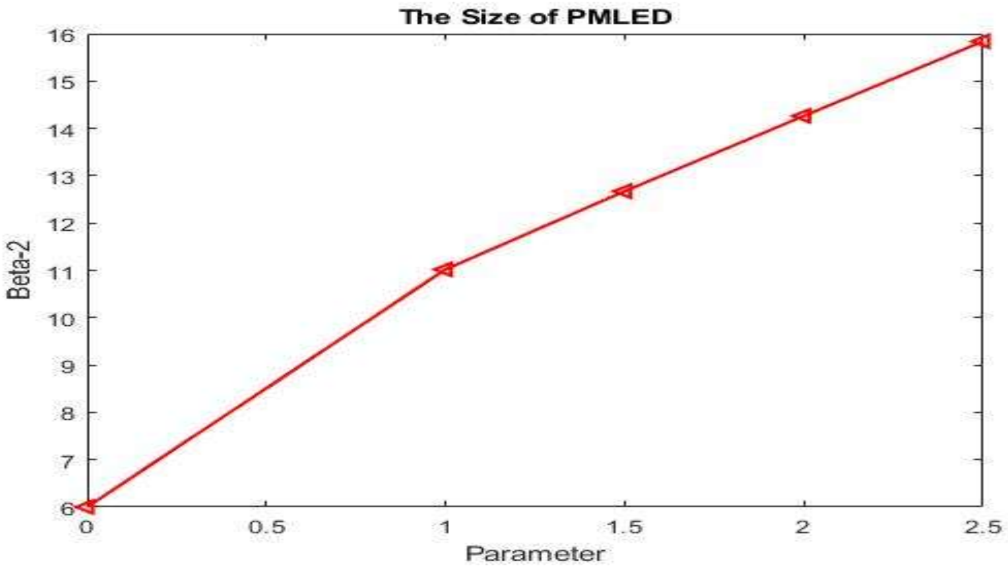

Figure-6: Showing graph of size of PMLED

From (15) and (16), we get $(\sqrt{2}) < \gamma_1 < \infty$ and $(6) < \beta_2 < \infty$ which shows this distribution is positively skewed and Leptokurtic in nature.

2.4 Estimation of Parameter: There are several methods for estimating population parameter but we estimate the parameter by using the method of moments (MoM) only, which can be obtained the expression of the mean of this distribution and given by the equation (17) as follows

$$f(\tau) = \bar{u}(\pi\tau)^3 - (\pi\tau)^2 + \bar{u}(\pi\tau) - 2 = 0 \tag{17}$$

We replace the population mean by the sample mean ($\bar{u}$).

3.0 Applications of PMLED:

This distribution is considered more suitable for statistical modelling of count data related to error proneness, accident proneness, ecology and biological sciences. We have calculated the theoretical frequency by using PLD as well as PMLED on some of these types of data and applied the $\chi^2 - goodness\ of\ fit\ test$ to the following examples.

Example-1: Distribution of mistakes in copying group of random digits reported by Kemp and Kemp [10].

Table-1

Number of Error Per Page	0	1	2	3	4
Number of Pages	35	11	8	4	2

Example-2: Distribution of Pyrausta Nablilalis in in 1937 reported by Beall [11].

Table-2

Number of insects per leaf	0	1	2	3	4	5
Number of leaves	33	12	6	3	1	1

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Example-3: Distribution of mammalian cytogenic dosimetry lesions in Rabbit lymphoblast included by [NSC-45383] [12].

Table-3

Class/Exposure ($\mu g / kg$)	0	1	2	3	4	5
Observed Frequency	33	12	6	3	1	1

Table numbered from (4) to (6) contains theoretical frequencies due to PLD and PMLED. They also contain $\hat{\tau}$,degrees of freedom, calculated values of chi-square at given level of significance and P-values of calculated value of chi-square at given level of significance of examples 1,2 and 3 respectively, which makes comparison easy and simple.

Table-4

Tabulation of the theoretical frequencies, estimated values of the parameters, degrees of freedom, calculated value of chi-square and P-values due to PLD and PMLED of example (1).

U	O	Theoretical frequencies due to	
		PLD	MPLED
0	35	33.0	33.3
1	11	15.3	15.0
2	8	6.8	6.7
3	4	2.9	2.9
4	2	2.0	2.1
T0tal	60	60.0	60.0
$\bar{u} = 0.7833333$	$\mu_2' = 1.85$	-	-
$\hat{\tau}$	-	1.7434	0.51779
$d.f.$	-	2	3
χ^2	-	1.78	1.61
$P-value$	-	0.41	0.657

Table-5

Tabulation of the theoretical frequencies, estimated values of the parameters, degrees of freedom, calculated value of chi-square and P-values due to PLD and PMLED of example (2).

U	O	Theoretical frequencies due to	
		PLD	MPLED
0	33	31.5	31.7
1	12	14.2	14.0
2	6	6.1	6.0
3	3	2.5	2.5
4	1	1.0	1.1
5	1	0.7	0.7
T0tal	56	56.0	56.0
$\bar{u} = 0.75$	$\mu_2' = 1.8571$	-	-
$\hat{\tau}$	-	1.8081	0.53528
$d.f.$	-	2	2
χ^2	-	0.53	0.387
$P-value$	-	0.767	0.824

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Table-6

Tabulation of the theoretical frequencies, estimated values of the parameters, degrees of freedom, calculated value of chi-square and P-values due to PLD and PMLED of example (3).

U	O	Theoretical frequencies due to	
		PLD	MPLED
0	200	191.8	192.5
1	57	70.3	69.4
2	30	24.9	27.7
3	7	8.6	8.7
4	4	2.9	3.0
5	0	1.0	1.1
6	2	0.5	0.6
T0tal	300	300.0	300.0
$\bar{u} = 0.553333$	$\mu_2' = 1.253333$	-	-
$\hat{\tau}$	-	2.35333	0.67892
$d.f.$	-	3	3
χ^2	-	3.91	3.657
$P-value$	-	0.2713	0.3010

4.0 Conclusion:

Table-7

Tabulation of $d.f.$, χ^2 and $P-value$ of PLD and PMLED

Table No.	PLD			PMLED		
	$d.f.$	χ^2	$P-value$	$d.f.$	χ^2	$P-value$
4	2	1.78	0.41	3	1.61	0.657
5	2	0.53	0.767	2	0.387	0.824
6	3	3.91	0.2713	3	3.657	0.3010

- The mean of PMLED is inversely proportion to its estimated value of the parameter.
- The variance of PMLED is directly proportional to its estimated value of the parameter.
- PMLED is always over-dispersed.
- It is positively skewed and leptokurtic in nature.

The table number (7) shows that PMLED is a better alternative of PLD for above mentioned applications under similar statistical condition.

Conflict of Interest:

We are working tirelessly in the field of countable and continuous mixtures of Poisson and generalised Poisson distributions and it is not our intension to hurt anyone's feeling.

Acknowledgment:

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References:

[1] Sah, B.K. (2013). Generalisations of Some Countable and Continuous Mixtures of Poisson Distribution and their Applications. *Doctoral thesis*, Patna University, Patna, India.

[2] Sah, B.K. and Sahani, S.K. (2022). Modified Linear-Exponential Distribution. NeuroQuantology, 20(5), 3520-3528. D0I 10.14704/nq.2022.20.2. NQ22650

[3] Shankaran, M. (1970). The Discrete Poisson-Lindley Distribution. *Biometrics,*26,145-149.

https://doi.org/10.2307/2529053

[4] Sah, B.K. (2022). New Linear-Exponential Distribution. Applied Science Periodical, 26 (2), 1-16.

[5] Lehmann, E.L. and Romano, J.P. (2005). Testing of Statistical Hypothesis. *New York: Sringer.*

[6] Mishra, A. (1979). Generalization of Some Discrete Distributions. *Doctoral Thesis, Patna University*, Patna, India.

[7] Shoukri, M.M. (1980). Estimation Problem of Some Discrete Distribution. *Doctoral Thesis, University of Calgary,* Calgary, Canada.

[8] Sah, B.K. and Mishra, A. (2020). On Generalised Exponential-Lindley Mixture of Generalised Poisson Distribution. *Nepalese Journal of Statistics*, 4, 33-42. https://doi.org/10.3126/njs.v4i0.33449

[9] Sah, B.K. and Mishra, A. (2019). A Generalised Exponential-Lindley Mixture of Poisson Distribution. *Nepalese Journal of Statistics*, 3, 11-20. https://doi.org/10.3126/njs.v3i0.25575

[10] Kemp, C.D. and Kemp, A.W.: Some properties of the Hermite distribution. *Biometrika,* 52.

[11] Beall, G. (1940). The Fit and Significance of Contagious Distribution When Applied to Observations on Larval Insects, Ecology, 21, 460-474.

[12] Catcheside, D.G.; Lea, D.E.and Thoday, J.M. (1946). The production of chromosome structural changes in Tradescantia microspores in relation to dosage, intensity and temperature. *Journal of Genetics,* 47(2) ,137–149.https://doi.org/10.1007/bf02986783