Computational Analysis of Prey-Predator Model with Fear and Toxic Factors by Fractional Operator

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Abstract

This work suggests a prey-predator model influenced by poisonous compounds and the impacts of fear. We demonstrated the model solution's uniqueness using the Lipschitz condition and its boundedness using the Laplace transform. We chose Yang-Abdel-Cattani derivative to analyze the system. Finally, numerical simulation is used to validate the theoretical results. The results suggest that both population size and presence of poisoned matter exert a noteworthy influence on system's solidity. Additionally, the value of α is shown to impact the stability of the system as well.

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Key Words: Prey-predator model; Yang-Abdel-Cattani fractional derivative; fractional operator; fixed point condition; Homotopy; HPM.

1. Origination

The study of population dynamics has advanced significantly since the eighteenth century, and the intricate inter-specific interactions that exist between many populations have been thoroughly examined. Fractional calculus has high memory qualities, hence it can be used to simulate biological population evolution in biological models more accurately. The challenge of aligning the traditional integer-order differential model with empirical outcomes is addressed by fractional calculus, which offers improved results with fewer parameters. In disciplines such as physics, mechanics, scientific inquiry, and mathematics, fractional-order differential operators serve as valuable tools for numerical analysis [1-10]. We get better findings using fractional calculus than from traditional models in comparison to integer calculus.

Using the Caputo operator as the basis, [11] investigated fractional multi-dimensional Navier-Stokes equation. We have [11] explored behavior of a fractional ordered predator-prey model that incorporates prey shelters. Consequently, the prey-predator system will be examined in this work. Complex interspecific connections between distinct populations have been widely explored with the emergence of population dynamics, and numerous exogenous influences impacting biological populations have also garnered substantial attention. Field studies [12] have underscored the importance of fear effects in prey-predator models. Despite their widespread occurrence in nature and profound impact on populations, fear effects are normally neglected in conservation and wildlife management practices. The quantity of prey animals decreases as a result of the physiological and behavioural changes brought about by predators' indirect effects on prey, which include decreased foraging activity, increased vigilance, and altered habitat utilisation. Prey may experience hormonal alterations as a result of predators' indirect actions, according to certain research [13]. Depending on the species, the fear impact can cause them to move to other locations to reside, change how they forage, have fewer offspring, and so on [14, 15]. Through tests, the author shown in [16] that the fear impact decreased song sparrow offspring by 40 percentage.

Reference [18] examined the fear effect's impact on the biological reproduction rate model, investigated the connection between the fear effect and other physiologically significant factors, and provided additional evidence about the fear effect's influence on prey-predator system. According to results of a group of hunt-vampire systems influenced by infectious illness that Ref. [19] investigated, where prey is vulnerable and will experience fear effect, the bulk of diseased hunt is correlated with the level of panic that predator instills in the prey. A grouping of biological systems impacted by group defence effect as well as the fear effect were examined in Ref. [20], and the findings indicated that an overabundance of fear might cause prey species to go extinct. A group of hunt-vampire systems with prey shelter and panic effects was presented by Ref. [21], and its equilibrium and bifurcation difficulties were examined. The effect of terror on the prey was taken into consideration by the authors in [22], who examined a biological system with two vampires cooperating in their hunt. In [23], a system with several Allee effects brought on by panic results was taken into consideration. As a result of the widespread industrialization trend of today, several factories release hazardous wastes and poisonous gases that are highly contaminated with heavy metals. These pollutants severely damage the ecosystem and ruin the habitats of wild animals [24, 25].

Numerous wild creatures have suffered as a result of harmful substances destroying the environment; some families have gone lost and the quantity of several creature has drastically decreased. According to Ref. [26], harmful compounds can enter organisms directly or go up the food chain through the consumption of prey, whereupon they end up in the stomachs of predators. The authors of [27] examined the issue of fishing between two rival fish species that both emit noxious compounds that are detrimental to one another. Ref. [28] investigated populations of organisms that have long been present in contaminated environments and deduced the prerequisites for population persistence. Authors examined the non-negative result of model along with many other properties like its nature, boundedness, and others and they provided conditions for lost of hunt and vampires as well as the conditions for system's existence. Ref. [29] is about a group of hunt-vampire systems with panic effects and toxic substances, where hunt is directly influenced by poisonous stuff and the vampire is affected by eating hunt.

This article examined a predator-prey model that is with actual biological chain by taking into account the twin effects of surrounding pollution and panic factor nowadays, together with a predator-prey system influenced by poisonous substances. The following are the primary findings. First, the system's solutions' uniqueness and boundedness are examined then its graphical solutions are also obtained.

2. **Pre-requisites**

This section presents the prey-predator system that is impacted by poisonous substances and fear-inducing factors. Yang-Abdel-Cattani [YAC] derivative [30] is employed in model throughout the paper. This section gives some background information on YAC derivative, a recently developed fractional operator and about Laplace transformation. Here is the description of both in more detail ([1]-[3]):

2.1. **Yang-Abdel-Cattani Differential Operator.**

Consider $\phi \in Y(0, \infty)$, then YAC differential operator of exponent α of ϕ with parameters (α, δ, n), $\alpha \geq 0, \delta > 0$ where $n \in I^+$, is expressed as:

$$ {}_0^{YAC}D_t^{\alpha}\phi(t) = \int_0^t R_{\alpha}\left[-\delta(t-\zeta)^{\alpha}\right]\phi^{(n)}(\zeta)d\zeta; \quad t > 0 \tag{2.1} $$

here R is a fractional exponent (in the Rabotnov sense). Since the Yang-Abdel-Cattani derivative comprises a non-singular kernel and produces results more quickly than other derivatives, we employed it in this situation.

2.2. **Yang-Abdel-Cattani Integral Operator.**

YAC integral of $g(t)$ of exponent α is:

$$ {}_{a}^{YAC}I_0^{\alpha}g(t) = \int_0^t \phi_{\alpha}\left[-\delta(t-\tau)^{\alpha}\right]g(\tau)d\tau. \tag{2.2} $$

2.3. **Laplace Transform.** Suppose, Laplace change of $F(t)$ be expressed by $L\{F(t)\}$ and is defined below:

$$L\{F(t)\} = \int_0^\infty e^{-st}F(t)dt, \quad s > 0 \tag{2.3}$$

here e^{-st} is kernal of transform and 's' is transform variable and is a complex number. This transformation has main advantage over alternatives because it breaks down complicated equations into algebraic ones, which are far simplest to solve.

2.4. **Laplace Change of Yang-Abdel-Cattani Differential Operator.** Suppose $\phi \in Y^{1,n}(0, \infty) \cap C^{n-1}([0, \infty))$, $n \in N$ then Laplace change of YAC operator is given as:

$$L\{{}^{YAC}D_t^{\mu,\lambda,n}\phi\}(s) = \frac{1}{s^{\mu+1}\left(1+\lambda s^{-(\mu+1)}\right)} \times [s^n L\{\phi\}(s) - \sum_{r=1}^{n} s^{n-r}\phi^{(r-1)}(0)], \quad s > 0 \tag{2.4}$$

The six sections that make up this article's structure are as follows: The pre-requisites are defined in section 2. Section 3 is having prey-predator system under consideration. The existence and uniqueness of the result is discussed in section 4. We go with numerical and graphical results in segment 5. In addition, we conclude our findings in section 6.

3. Prey-predator Model

Here, we are going to discuss the model which we are going to analyze.
Generally, in the absence of vampire and panic effects, vegan growth is obtained by natural death, density-dependent death and procreation rates:

$$\frac{dx}{dt} = rx - cx^2 - d_1 x \tag{3.1}$$

where 'x' is bulk of the hunt population, 'r' is procreation rate, 'c' is the bulk-dependent death and 'd' is normal death.
It is also evident that the population's rate of procreation will be influenced by the dread effect. The rate of procreation is then multiplied by a fright factor, where "ρ" indicates the prey's level of dread. This allows us to rewrite equation (1) as follows:

$$\frac{dx}{dt} = rg(\rho, y)x - cx^2 - d_1 x \tag{3.2}$$

here 'y' denotes population density of vampires and $g(\rho, y) = \frac{1}{1+\rho y}$ is the fright function with following constraints:

$$g(0, y) = 1, \; g(\rho, 0) = 1$$

$$\lim_{\rho\to\infty} g(\rho, y) = 0, \ \lim_{y\to\infty} g(\rho, y) = 0$$

$$\frac{\partial g(\rho, y)}{\partial \rho} < 0 \ and \ \frac{\partial g(\rho, y)}{\partial y} < 0$$

So, a prey-predator system with panic effects under study is given below:

$$\begin{aligned} \frac{dx}{dt} &= \frac{rx}{1+\rho y} - cx^2 - \frac{a_1xy}{1+bx} - d_1x \\ \frac{dy}{dt} &= \frac{a_2xy}{1+bx} - d_2y \end{aligned} \tag{3.3}$$

where 'a_1' is the plundering quantity, 'a_2' is food turning quantity, 'b' is half-saturation constant, 'd_1' is normal death of hunt and 'd_2' is normal death of vampire. Now we insert the effects of poisoned stuff on biological inhabitants, supposing that hunt is directly influenced by external poisoned matters and vampire are indirectly influenced by poisoned matters by preying on hunt:

$$\begin{aligned} \frac{dx}{dt} &= \frac{rx}{1+\rho y} - cx^2 - \frac{a_1xy}{1+bx} - d_1x - e_1x^2 \\ \frac{dy}{dt} &= \frac{a_2xy}{1+bx} - d_2y - e_2y \end{aligned} \tag{3.4}$$

where 'e_1x^2' shows effect of poisoned substances on hunt and 'e_2y' denotes the effect of poisoned substances on vampires. In this article, we propose following system:

$$\begin{aligned} D^\alpha x &= \frac{rx}{1+\rho y} - cx^2 - \frac{a_1xy}{1+bx} - d_1x - e_1x^2 \\ D^\alpha y &= \frac{a_2xy}{1+bx} - d_2y - e_2y \end{aligned} \tag{3.5}$$

where $\alpha \in (0, 1]$

4. Existence and Uniqueness of Solution

We are having fractional prey-predator model as:

$$\begin{aligned} {}^{YAC}D^\alpha x &= \frac{rx}{1+\rho y} - cx^2 - \frac{a_1xy}{1+bx} - d_1x - e_1x^2 \\ {}^{YAC}D^\alpha y &= \frac{a_2xy}{1+bx} - d_2y - e_2y \end{aligned} \tag{4.1}$$

We inquire existence and uniqueness of solutions in the region $B \times [t_0, T]$, here

$$B = \{(x, y) \in R^2 : \max\{|x|, |y|\} \leq \psi, \min\{|x|, |y|\} \geq \psi_0\}$$

where $T < \infty$.

Theorem For every $X_0 = (x_0, y_0) \in B$, $\exists$ a unique solution of the system (4.1) with initial condition X_0 , which is defined for all $t \geq 0$.

Proof We represent $X = (x, y)$ and $\bar{X} = (\bar{x}, \bar{y})$. Now, assume a mapping $M(X) = \{M_1(X), M_2(X)\}$, where

$$M_1(X) = \frac{rx}{1+\rho y} - cx^2 - \frac{a_1xy}{1+bx} - d_1x - e_1x^2$$

,

$$M_2(X) = \frac{a_2xy}{1+bx} - d_2y - e_2y$$

For any $X, \overline{X} \in B$, we have

$$\left\| M(X) - M(\overline{X}) \right\| = \left| M_1(X) - M_1(\overline{X}) \right| + \left| M_2(X) - M_2(\overline{X}) \right|$$

Now,

$$M_1(X) - M_1(\overline{X}) = \frac{rx}{1+\rho y} - cx^2 - \frac{a_1xy}{1+bx} - d_1x - e_1x^2 - \frac{r\overline{x}}{1+\rho\overline{y}} + c\overline{x}^2 + \frac{a_1\overline{x}\overline{y}}{1+b\overline{y}} + d_1\overline{x} + e_1\overline{x}^2$$

$$\left| M_1(X) - M_1(\overline{X}) \right| \le \left(\frac{r}{\rho} + d_1 + \frac{a}{b} + (c_1 + e_1)(x + \overline{x}) \right) |x - \overline{x}|$$

Similarly,

$$M_2(X) - M_2(\overline{\overline{X}}) = \frac{a_2xy}{1+bx} - d_2y - e_2y - \frac{a_2\overline{x}\overline{y}}{1+b\overline{\overline{x}}} + d_2\overline{y} + e_2\overline{y}$$

$$\left| M_2(X) - M_2(\overline{X}) \right| = \left(\frac{a_2}{b} + d_2 + e_2 \right) |y - \overline{y}|$$

or,

$$\left\| M(X) - M(\overline{X}) \right\| = \left| M_1(X) - M_1(\overline{X}) \right| + \left| M_2(X) - M_2(\overline{X}) \right|$$

$$= \left(\frac{r}{\rho} + d_1 + \frac{a}{b} + (c_1 + e_1)(x + \overline{x}) \right) |x - \overline{x}| + \left(\frac{a_2}{b} + d_2 + e_2 \right) |y - \overline{y}|$$

$$= \left(\frac{r}{\rho} + d_1 + \frac{a}{b} + (c_1 + e_1)\psi \right) |x - \overline{x}| + \left(\frac{a_2}{b} + d_2 + e_2 \right) |y - \overline{y}|$$

$$\left\| M(X) - M(\overline{X}) \right\| = L \left\| X - \overline{X} \right\|$$

where

$$L = \max \left(\frac{r}{\rho} + d_1 + \frac{a}{b} + (c_1 + e_1)\psi \, , \, \frac{a_2}{b} + d_2 + e_2 \right)$$

Hence $M(X)$ fulfil Lipschitz condition shows existence and uniqueness of results of the system.

5. Numerical Analysis

Consider the given prey-predator model:

$$D^{\alpha}x = \frac{rx}{1+\rho y} - cx^2 - \frac{a_1xy}{1+bx} - d_1x - e_1x^2$$
$$D^{\alpha}y = \frac{a_2xy}{1+bx} - d_2y - e_2y \quad (5.1)$$

Now replace the derivative by Yang-Abdel-Cattani fractional derivative, we get

$$^{YAC}D^{\alpha}x = \frac{rx}{1+\rho y} - cx^2 - \frac{a_1xy}{1+bx} - d_1x - e_1x^2$$
$$^{YAC}D^{\alpha}y = \frac{a_2xy}{1+bx} - d_2y - e_2y \quad (5.2)$$

Here, taking the L.T. both sides in first equation of above model, we obtain

$$L\left\{^{YAC}D^{\alpha}x\right\} = L\left[\frac{rx}{1+\rho y} - cx^2 - \frac{a_1xy}{1+bx} - d_1x - e_1x^2\right] \quad (5.3)$$

or,

$$\frac{1}{(1+s^{\alpha+1})}.\left[sL\{x(t)\} - x(0)\right] = L\left[\frac{rx}{1+\rho y} - cx^2 - \frac{a_1xy}{1+bx} - d_1x - e_1x^2\right] \quad (5.4)$$

or,

$$sL\{x(t)\} - x(0) = \left(1+s^{\alpha+1}\right).L\left[\frac{rx}{1+\rho y} - cx^2 - \frac{a_1xy}{1+bx} - d_1x - e_1x^2\right] \quad (5.5)$$

or,

$$sL\{x(t)\} = x(0) + \left(1+s^{\alpha+1}\right).L\left[\frac{rx}{1+\rho y} - cx^2 - \frac{a_1xy}{1+bx} - d_1x - e_1x^2\right] \quad (5.6)$$

or,

$$L\{x(t)\} = \frac{x(0)}{s} + \left(\frac{1}{s} + s^{\alpha}\right).L\left[\frac{rx}{1+\rho y} - cx^2 - \frac{a_1xy}{1+bx} - d_1x - e_1x^2\right] \quad (5.7)$$

Now taking the inverse Laplace transform both sides,

$$x(t) = x(0) + L^{-1}\left[\left(\frac{1}{s} + s^{\alpha}\right).L\left[\frac{rx}{1+\rho y} - cx^2 - \frac{a_1xy}{1+bx} - d_1x - e_1x^2\right]\right] \quad (5.8)$$

Similarly, we have other expression as:

$$y(t) = y(0) + L^{-1}\left[\left(\frac{1}{s} + s^{\alpha}\right).L\left[\frac{a_2xy}{1+bx} - d_2y - e_2y\right]\right] \quad (5.9)$$

or, by iterative technique, we have

$$x_{n+1}(t) = x(0) + L^{-1}\left[\left(\frac{1}{s} + s^{\alpha}\right).L\left[\frac{rx_n}{1+\rho y_n} - cx_n^2 - \frac{a_1x_ny_n}{1+bx_n}\right] - d_1x_n - e_1x_n^2\right],$$
$$y_{n+1}(t) = y(0) + L^{-1}\left(\frac{1}{s} + s^{\alpha}\right).L\frac{a_2x_ny_n}{1+bx_n} - d_2y_n - e_2y_n \quad (5.10)$$

From equation (5.10), we find the various iterations of x(t) and y(t) and we can find the final solution by

$$x(t) = \lim_{n\to\infty} x_n(t) \tag{5.11}$$

and

$$y(t) = \lim_{n\to\infty} y_n(t) \tag{5.12}$$

Now, to get the graphical results of the model, we have used the following data [41]:

Parameter	Value
r	0.3
a_1	0.5
a_2	0.3
b	5
c	0.2
d_1	0.01
d_2	0.12
e_1	0.01
e_2	0.1
ρ	1

TABLE 1. Table with initial value and parameters

By using equations (5.11), (5.12) and the above parametric values, we have the following graphical representations:

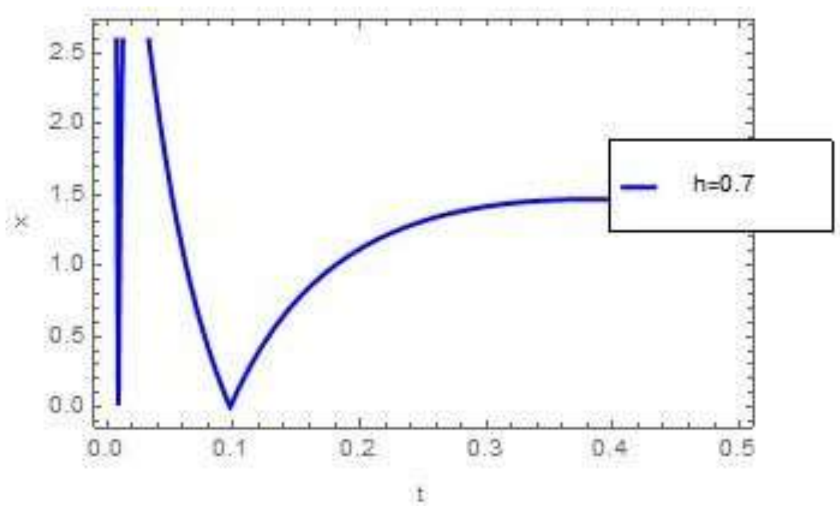

FIGURE 1. Growth rate of prey population for $\alpha = 0.7$

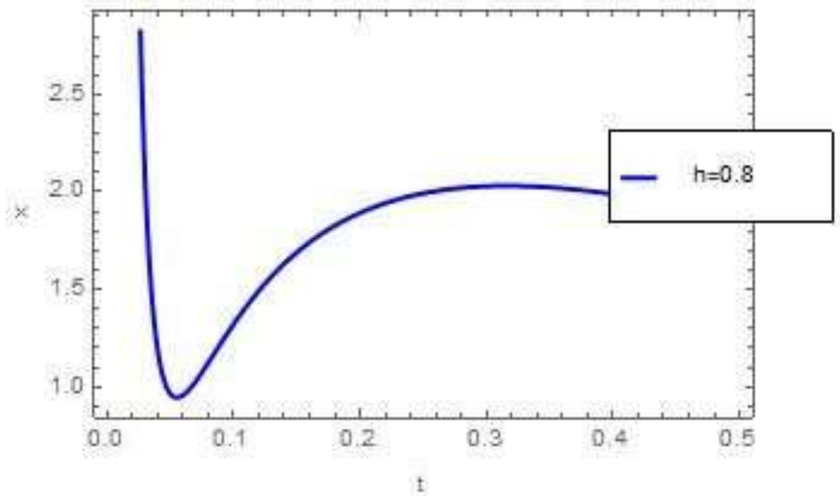

FIGURE 2. Growth rate of prey population for $\alpha = 0.8$

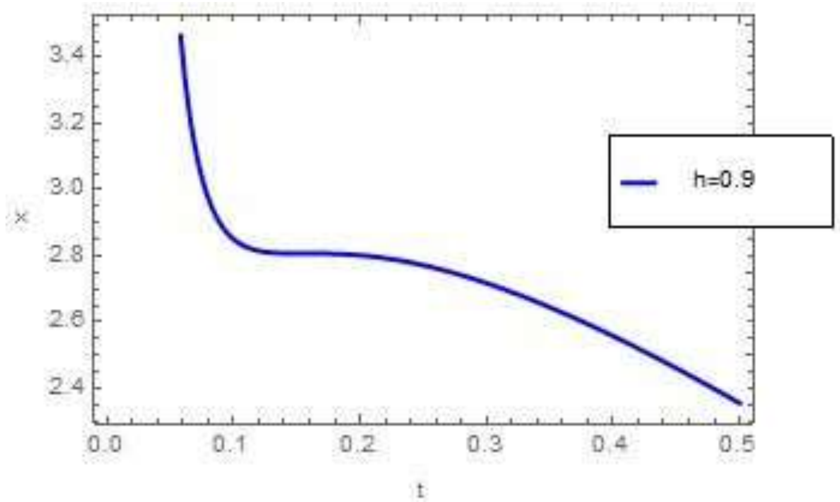

FIGURE 3. Growth rate of prey population for $\alpha = 0.9$

6. **Conclusion**

This work presents a model of prey-predator fear responses to harmful compounds. It analyses dynamic behaviours of system, including uniqueness, boundedness and stability. Theoretical study and simulation can confirm that the system is stable and has a fast rate of convergence. Furthermore, bifurcations for parameters b and e_2 are produced by the system. It is evident that System (5) gradually moves from stable to an unstable state as

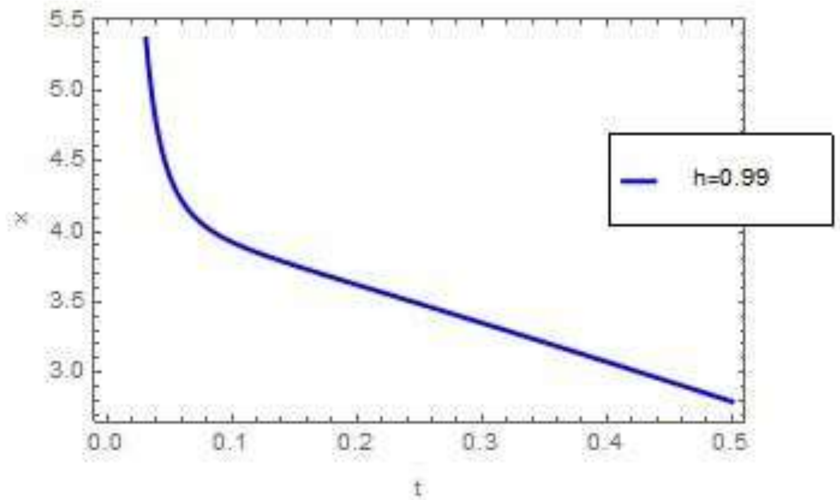

FIGURE 4. Growth rate of prey population for $\alpha = 0.99$

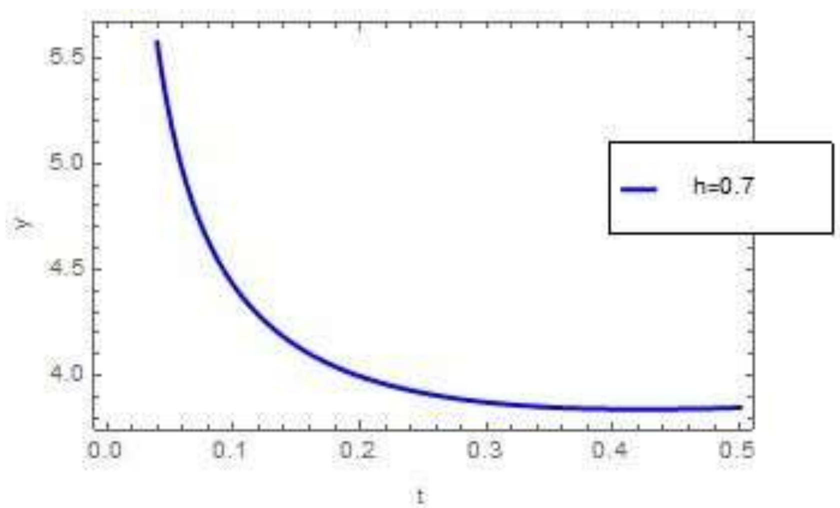

FIGURE 5. Growth rate of predator population for $\alpha = 0.7$

parameter b increases, and that System (5) gradually moves from an unstable to stable state as parameter e_2 increases. In the future, we can think about how environmental pollution-related metabolic problems affect the prey-predator model and whether or not prey would be affected by fear.

Availability of statistics and materials: Availability of statistics is cited in paper.
Statement of disagreement: There are no conflicts of interest to disclose about the article presented here.

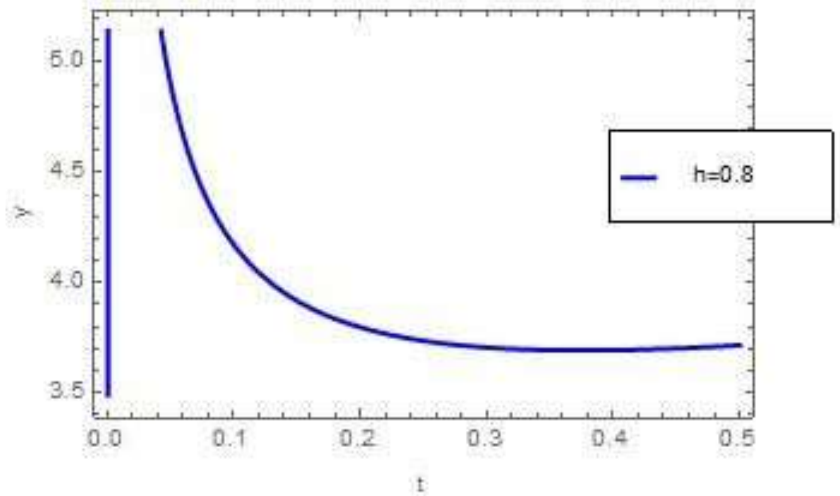

FIGURE 6. Growth rate of predator population for $\alpha = 0.8$

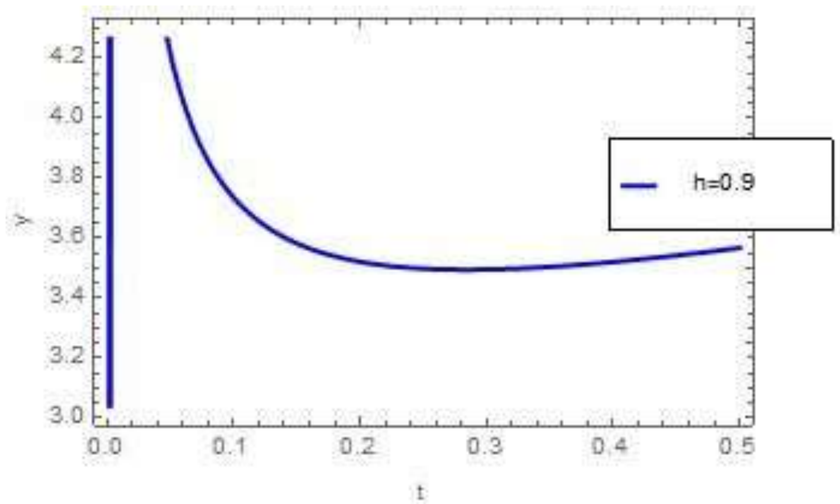

FIGURE 7. Growth rate of predator population for $\alpha = 0.9$

Author's Contribution: The study was directed by R. S. Dubey, who also constructed the study map, analysed the findings, and organised the necessary research materials. S. Kumar drew the figures/graphs, and structured the final paper while M. N. Mishra prepared the article and carried out all the mathematical computations. The draft was read and corrected by all authors.

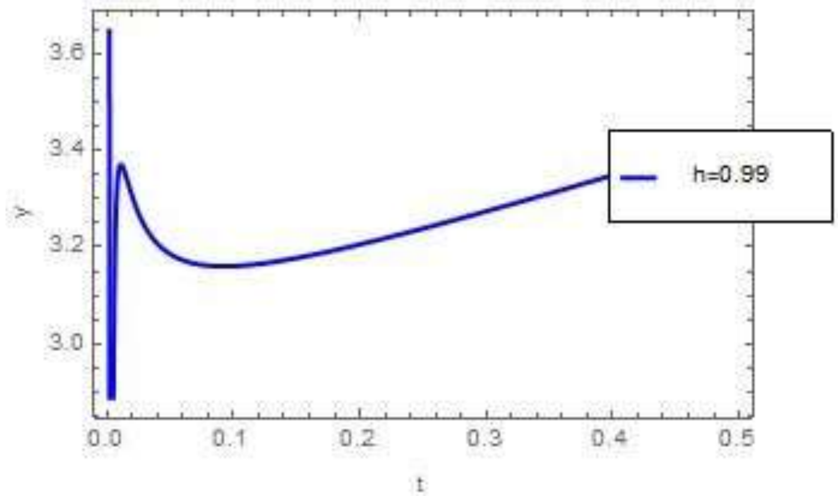

FIGURE 8. Growth rate of predator population for $\alpha = 0.99$

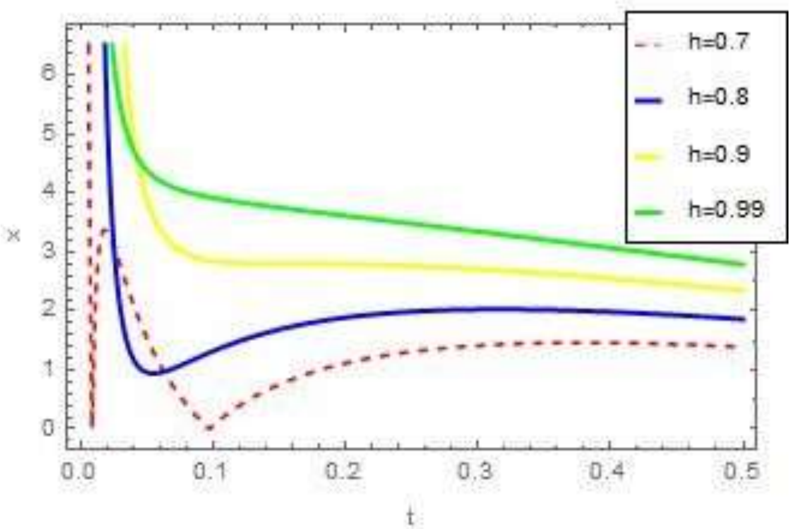

FIGURE 9. Comparison of growth in prey population for various values of h

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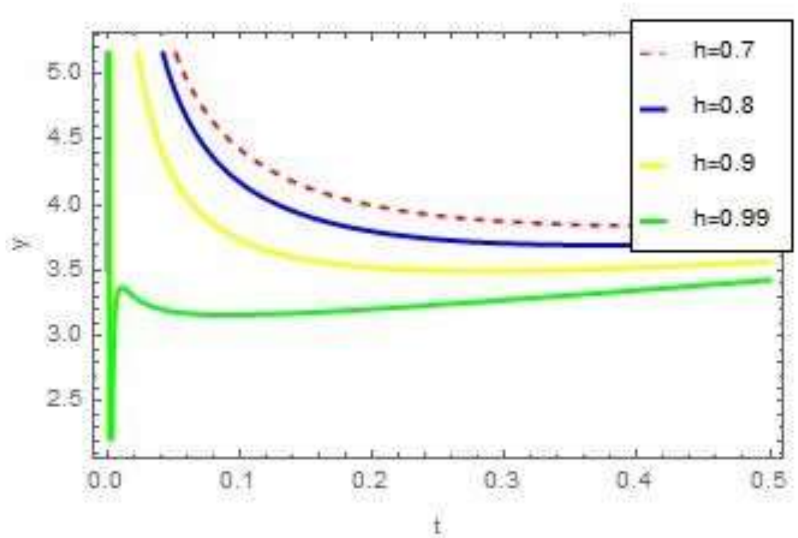

FIGURE 10. Comparison of growth in predator population for various values of h

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