

Green Supply Chain Decision Making Using Interval-Valued Picture Fuzzy Geometric Aggregation Methods

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Abstract

Green supply chain management (GSCM) involves inclusion of sustainable environmental processes into the conventional supply chain. The demand in designing new practices and formulation of policies also interrogates new procedures for evaluation of efficiency of sustainable supply chains. This study presents a new multi-attribute decision-making (MADM) process that is selection of appropriate green supplier which is based on interval-valued picture fuzzy sets (IVPFS) to deal adequately with imprecise, hesitant, and contradictory evaluations made by experts. We propose three novel aggregation operators based on the Yager t-norm and t-conorm in the context of interval-valued picture fuzzy which are: Interval Valued Picture Fuzzy Yager Weighted Geometric Aggregation (IVPFYWGA), Interval Valued Picture Fuzzy Yager Ordered Weighted Geometric Aggregation (IVPFYOWGA) and are Interval Valued Picture Fuzzy Yager Hybrid Weighted Geometric Aggregation (IVPFYHWGA).

Keywords: Interval-valued logic; Picture Fuzzy set; Yager Aggregation operators; Geometricity; Decision making; Green Supply Chain Management.

1.Introduction

Due to growing environmental consciousness, regulatory pressures and stakeholder expectations, organizations are being pressured to transform their supply chain practices to be more sustainable. Research has suggested that the traditional linear supply chain is inadequate to fulfil environmental and resource efficiency related objectives. Due to global sustainability goals and climate mandates, organizations are embedding green behaviours into their supply chains from sourcing raw materials, managing a products life-cycle, including end of life disposal. Green Supply Chain Management (GSCM), which can be defined as an organized confluence of economic, environmental and social elements, requires organizations to balance economic and operational efficiency with environmental accountability. Economic consideration (in dollars) on impact, for example reduction in carbon footprint, energy consumption, waste production, controlling supplier environmental performance, regulatory compliance, etc., influences supply chain decisions. At the same time, consumer environmental consciousness, market competitiveness, innovation, and corporate social responsibility affects green adoption. GSCM is multi-dimensional, complex, and uncertain. This is characterized by decision making in unclear, incomplete, imprecise, or vague data. As such, a number of decision-making models have emerged to reduce ambiguity in assessing sustainability performance, particularly MCDM methods that incorporates fuzzy and intuitionistic fuzzy logic. These methods allow for expert judgment and stakeholder preferences under uncertainty to be cast, thus providing an even stronger evaluation of green alternatives. As companies continue to migrate toward greener businesses, GSCM becomes an important area of research and practices on environmental and economic sustainability.

Recent research studies have used multi-criteria decision-making (MCDM) approaches such as AHP, TOPSIS, and the extensions of fuzzy logic (e.g., intuitionistic fuzzy sets and neutrosophic fuzzy sets) to analyse and rank the green supply chain alternatives. Such methods provide a mechanism by which not only objective environmental metrics, but also subjective managerial decisions are combined for performance evaluation under complex, uncertain circumstances. For example, Govindan reviewed the literature on green supplier assessment and selection [1]. They noted that the green supply chain literature is dominated with fuzzy based single model approaches while also recommending several areas where future studies were needed. Zimmer analysed the sets of models that assist in sustainable supplier selection, sustainable supplier monitoring, and the sustainable supplier development decision for collaboration with suppliers and they concluded that more integrated decision-making models are needed to simultaneously inform sustainable supplier management decisions [2]. Overall, these articles speak to the need for more adaptive, integrated, and behaviourally aligned decision models that both model uncertainty in data and allow for better responsiveness to changing regulatory pressures and stakeholder expectations.

The binary distinction between "green" and "non-green" supply chains is moving toward a constructed continuum of hybrid models that trade-off ecological responsibility, operational efficiency, and economic feasibility. The traditional crisp-value-based models or even classical fuzzy MCDM approaches usually cannot deal with the high degrees of uncertainty, vagueness and partial ignorance contained in the green supply chain performance assessments. To bridge this methodological gap, this study introduces a decision framework this study presents a decision framework that uses the relatively new concept based on Interval-Valued Picture Fuzzy Sets (IVPFS), which captures uncertainty and the possibility of truth, falsity and neutrality as intervals. The study also provides methodology using Geometric Yager Aggregation Operators to combine expert data from multiple intervals and dimensions, giving a more rigorous and flexible evaluation option for more effective green supply chain evaluations than traditional attributes may allow. The use of interval-based models will provide informative approaches for sustainability strategists, operations managers, and policymakers who want to develop more resilient sustainably-focused supply chains.

2. Theoretical Framework:

Intuitionistic fuzzy sets (IFS) extend classical fuzzy sets by assigning to each element both a membership degree μ and a non-membership degree γ with $\mu + \gamma \leq 1$, thereby making explicit the residual hesitation $\pi = 1 - (\mu + \gamma)$. To acknowledge that experts may themselves be uncertain about exactly how much they agree or disagree, interval-valued IFS replace each crisp μ and γ with intervals $[\mu^l, \mu^u]$ and $[\gamma^l, \gamma^u]$, widening the hesitation into an interval and capturing second-order imprecision. Meanwhile, picture fuzzy sets (PFS) introduce a third neutral degree ξ so that $\mu + \xi + \gamma \leq 1$ enabling a clear distinction between outright acceptance (μ), outright rejection (γ) and true ambivalence (ξ). Finally, interval-valued picture fuzzy sets merge these two enhancements—each of μ , ξ and γ becomes an interval so that one not only models positive, neutral and negative attitudes but also quantifies the expert's uncertainty about each attitude. Together, these nested extensions furnish researchers with progressively richer tools for faithfully representing and aggregating the multifaceted uncertainty inherent in human judgments.

Qiyas and Abdullah investigated the application of Yager aggregation operators in picture fuzzy sets (PFS) to improve emergency program selection decisions, demonstrating its adaptability and accuracy in multi-attribute decision situations [3]. Cuong and Kreinovich proposed Picture Fuzzy Sets, an extension of fuzzy and intuitionistic fuzzy sets that improve computational intelligence efficiency through increased imprecision and uncertainty [4]. Hussain and Liu presented a decision algorithm for picture fuzzy sets utilizing the Aczel Alsina aggregation operators, which can aid in improving decision

accuracy in multi-attribute group decision-making (MAGDM). Subsequently, the authors introduce new picture fuzzy operators to address several uncertainties and diminish vague and redundant information [5].

Khalil presented new operations on interval-valued picture fuzzy sets (IVPFS) and interval-valued picture fuzzy soft sets (IVPFSS) to enrich their theoretical basis and application in uncertain decision-making issues. Also, constructed algebraic operations and introduced a decision-making algorithm with a numerical example to show practical feasibility [6]. Mahmood and Waqas proposed new Frank aggregation operators in interval-valued picture fuzzy sets and incorporates them into the analytic hierarchy process for addressing multi-attribute decision-making issues. Numerical illustrations and comparative analysis prove the effectiveness and flexibility of the proposed approaches in dealing with uncertain and vague information [7]. Liu and Shahzadi presented a family of Yager aggregation operators within the q-rung picture fuzzy set (q-RPFS) environment to solve multi-attribute decision-making (MADM) issues. The operators improve decisional accuracy by efficiently representing different extents of positive, neutral, and negative preferences, with numerical examples demonstrating their advantages over current approaches.[8]

Zhu, S., & Liu presented new distance measures for Picture Fuzzy Sets and Interval-Valued Picture Fuzzy Sets that are adapted from Hellinger distance, motivating the distance measures by uncertainty and imprecision in real-life complicated decision-making problems. The distance measures were examined with various applications in pattern recognition and the field of medical diagnosis.[9] Jana and Pal presented new aggregation operators based on Interval valued Picture Fuzzy using Dombi operations for multi-attribute decision-making. The aggregation operators are designed to address linguistic uncertainties, and the operators are effectively used for the case of assessment about industrial mutual fund investments.[10] Chen proposed a novel multi-criteria decision-making (MCDM) system for the evaluation of design concepts using interval-valued picture fuzzy sets (IVPFS). The study included an entropy weighting method with an improved TOPSIS method for dealing with uncertainty and improving decision-making and selection of the best design alternatives [11].

Jalil explored the dynamics of customer trust in e-commerce, emphasizing the essential role of GRSCM in making online purchasing more empirically viable [12]. Letunovska addressed the impact of long-term viability on reverse logistics and also suggests that the firms can collaborate with environmental preservation authorities to manufacture goods that are ecologically friendly [13]. Ahmad presented the study which focuses on examining the influence of GRSCM methods on the sustainable achievement of the fabric and automobile sectors [14].Bhatia evaluated the research methodology of 216 empirical papers in GRSCM and proposed future research paths for empirical study in GRSCM [15].Fahimnia presented a pragmatic supply chain strategy model that may be used to explore cost-environmental trade-offs while also integrating other features of practical supply chains[16].Lee analysed 133 data sets from Malaysian manufacturing enterprises to identify a link between GRSCM procedures and technological progression [17]. Mathiyazhagan observed 26 hurdles to GRSCM deployment and aims to determine the most dominant one. [18]. Iddik discovered that non-bureaucratic culture, familial social structure, and structured culture had a favourable impact on the execution of GRSCM activities and also demonstrated how ecological preservation and culture are interconnected [19]. Abdallah assessed the impact of GRSCM on circular economy performance by collecting data from 278 firms using an elementary stochastic approach, revealing that GRSCM had an immediate beneficial effect on circular economy performance [20]. Xing's presentation on product-service innovation and GRSCM sheds light on how it contributes to ecological resilience [21]. Saini developed a theoretical model that comprises an effective theory-building method for the essential aspects of global logistics management practices

[22].Tiep Le studied how business social accountability activities impact diverse stakeholders, including GRSCM, in fostering equitable patterns of consumption and production [23].Roh investigated the effect of ethical managerial creativity and rights of intellectual property on GRSCM and environmental efficacy, emphasizing the key role of supply chain administration [24].The study aims to understand the basis for poor supply chain quality at a steel product manufacturing setup and finds key issues such as inventory stock-out, deviations in process, and unreliable ERP systems. The study proposes better inventory management techniques, supplier evaluation, and ERP system customization as potential solutions aimed to improve supply chain efficiency and reduce costs [25].

2.1 Preliminaries

In this section we recall some definitions of interval valued intuitionistic fuzzy set and Interval Valued Picture Fuzzy Set. We also discussed the fundamental properties of Yager t-norm and t-conorm.

Definition 1[26]:

Let $D[0,1]$ be the set of all closed subintervals of the interval $[0,1]$.Let $X(\neq \emptyset)$ be a given set. An interval valued intuitionistic fuzzy set (IVIFS) in X is an expression given by $A = \{x, \mu_A(x), \gamma_A(x) \mid x \in X\}$ where $\mu_A(x) : X \rightarrow D[0,1]$, $\gamma_A(x) : X \rightarrow D[0,1]$ with the condition $0 < \sup_x \mu_A(x) + \sup_x \gamma_A(x) < 1$.

We denote $\langle \mu_A(x), \gamma_A(x) \rangle = \langle [a, b], [c, d] \rangle$ where $[a, b] \in D[0,1], [c, d] \in D[0,1]$.

Definition 2 [4]:

A picture fuzzy set A on a universe $X(\neq \emptyset)$ is an object in the form of

$A = \{x, \mu_A(x), \eta_A(x), \gamma_A(x) \mid x \in X\}$ where $\mu_A(x) \in [0,1]$ is called degree of positive membership of x in A , $\eta_A(x) \in [0,1]$ is called degree of neutral membership of x in A and $\gamma_A(x) \in [0,1]$ is called degree of negative membership of x in A , and where μ_A, η_A, γ_A satisfy the following condition

$$\forall x \in X \mu_A(x) + \eta_A(x) + \gamma_A(x) \leq 1$$

Now, $1 - (\mu_A(x) + \eta_A(x) + \gamma_A(x))$ could be called the degree of refusal membership of x in A .

Definition 3 [27]:

An IVPFS B on the universe of discourse X is defined as follows:

$$B = \left(r, [\mu_B^l(r), \mu_B^u(r)], [(\xi_B^l(r), \xi_B^u(r)), (\vartheta_B^l(r), \vartheta_B^u(r))] \mid r \in X \right)$$

Where $[\mu_B^l(r), \mu_B^u(r)], [(\xi_B^l(r), \xi_B^u(r)), (\vartheta_B^l(r), \vartheta_B^u(r))]$ are closed sub intervals of the $[0,1]$ denoting the positive membership degree , neutral membership degree and negative membership degree provided that $\mu_B^u(r) + \xi_B^u(r) + \vartheta_B^u(r) \leq 1$

And, $\delta_B(r) = [(1 - \mu_B^u(r) + \xi_B^u(r) + \vartheta_B^u(r)), (1 - \mu_B^l(r) + \xi_B^l(r) + \vartheta_B^l(r))]$ is called the degree of refusal membership degree of r in X .

Definition 4[28]:

Let us consider three IVPFVs $\beta = (r, [\mu_{\beta}^l(r), \mu_{\beta}^u(r)], [(\xi_{\beta}^l(r), \xi_{\beta}^u(r)), (\vartheta_{\beta}^l(r), \vartheta_{\beta}^u(r))]|r \in X)$,

$$\beta_1 = (r, [\mu_{\beta_1}^l(r), \mu_{\beta_1}^u(r)], [(\xi_{\beta_1}^l(r), \xi_{\beta_1}^u(r)), (\vartheta_{\beta_1}^l(r), \vartheta_{\beta_1}^u(r))]|r \in X),$$

$\beta_2 = (r, [\mu_{\beta_2}^l(r), \mu_{\beta_2}^u(r)], [(\xi_{\beta_2}^l(r), \xi_{\beta_2}^u(r)), (\vartheta_{\beta_2}^l(r), \vartheta_{\beta_2}^u(r))]|r \in X)$, then

1. $\beta_1 \subseteq \beta_2$ iff $\mu_{\beta_1}^l \leq \mu_{\beta_2}^l, \mu_{\beta_1}^u \leq \mu_{\beta_2}^u, \xi_{\beta_1}^l(r) \geq \xi_{\beta_2}^l, \xi_{\beta_1}^u \geq \xi_{\beta_2}^u, \vartheta_{\beta_1}^l \geq \vartheta_{\beta_2}^l$ and $\vartheta_{\beta_1}^u \geq \vartheta_{\beta_2}^u$
2. $\beta_1 = \beta_2$ iff $\beta_1 \subseteq \beta_2$ and $\beta_2 \subseteq \beta_1$
3. $\beta_1 \cup \beta_2 = ([\max\{\mu_{\beta_1}^l(r), \mu_{\beta_2}^l(r)\}, \max\{\mu_{\beta_1}^u(r), \mu_{\beta_2}^u(r)\}], [\min\{\xi_{\beta_1}^l(r), \xi_{\beta_2}^l(r)\}, \min\{\xi_{\beta_1}^u(r), \xi_{\beta_2}^u(r)\}], [\min\{\vartheta_{\beta_1}^l(r), \vartheta_{\beta_2}^l(r)\}, \min\{\vartheta_{\beta_1}^u(r), \vartheta_{\beta_2}^u(r)\}])$
4. $\beta_1 \cap \beta_2 = ([\min\{\mu_{\beta_1}^l, \mu_{\beta_2}^l\}, \min\{\mu_{\beta_1}^u, \mu_{\beta_2}^u\}], [\max\{\xi_{\beta_1}^l, \xi_{\beta_2}^l\}, \max\{\xi_{\beta_1}^u, \xi_{\beta_2}^u\}], [\max\{\vartheta_{\beta_1}^l, \vartheta_{\beta_2}^l\}, \max\{\vartheta_{\beta_1}^u, \vartheta_{\beta_2}^u\}])$
5. $\beta^c = ([\mu_{\beta}^l(r), \mu_{\beta}^u(r)], [(\xi_{\beta}^l(r), \xi_{\beta}^u(r)), (\vartheta_{\beta}^l(r), \vartheta_{\beta}^u(r))]|r \in X)$

Definition 5 [29]: Given any two real values p and q , Yager t-norms and t-conorms are given as follows:

$$T(p, q) = 1 - \min(1, ((1 - p)^{\gamma} + (1 - q)^{\gamma})^{1/\eta})$$

$$T'(p, q) = \min(1, (p^{\gamma} + q^{\gamma})^{1/\gamma}), \gamma \in (0, \infty)$$

Definition 6 [28]:

Suppose $\beta = ([(\mu_{\beta}^l(r), \mu_{\beta}^u(r)), [(\xi_{\beta}^l(r), \xi_{\beta}^u(r)), (\vartheta_{\beta}^l(r), \vartheta_{\beta}^u(r))]|r \in X)$ be the Interval valued picture fuzzy value. The Score Function $S(\beta) = \frac{\mu_{\beta}^l(1 - \xi_{\beta}^l - \vartheta_{\beta}^l) + \mu_{\beta}^u(1 - \xi_{\beta}^u - \vartheta_{\beta}^u)}{3}$ where $S(\beta) \in [0, 1]$

3.Interval Valued Picture Fuzzy Yager Geometric Aggregation Operators

We now define Yager weighted geometric operators and ordered weighted geometric operators under interval valued picture fuzzy environment.

Definition 7:

Let us consider three IVPFVs $\beta = (r, [\mu_{\beta}^l(r), \mu_{\beta}^u(r)], [(\xi_{\beta}^l(r), \xi_{\beta}^u(r)), (\vartheta_{\beta}^l(r), \vartheta_{\beta}^u(r))]|r \in X)$, $\beta_1 = (r, [\mu_{\beta_1}^l(r), \mu_{\beta_1}^u(r)], [(\xi_{\beta_1}^l(r), \xi_{\beta_1}^u(r)), (\vartheta_{\beta_1}^l(r), \vartheta_{\beta_1}^u(r))]|r \in X)$, $\beta_2 = (r, [\mu_{\beta_2}^l(r), \mu_{\beta_2}^u(r)], [(\xi_{\beta_2}^l(r), \xi_{\beta_2}^u(r)), (\vartheta_{\beta_2}^l(r), \vartheta_{\beta_2}^u(r))]|r \in X)$, $\eta > 0$ and $\lambda > 0$. The Yager t-norm and t-conorm functions for IVPFNs have been defined as

$$1. \beta_1 \oplus \beta_2 = 1 - \min(1, (1 - \mu_{\beta_1}^l)^{\eta})^{\frac{1}{\eta}} + 1 - \min(1, (1 - \mu_{\beta_2}^l)^{\eta})^{\frac{1}{\eta}},$$

$$\begin{aligned}
 & 1 - \min\left(1, (1 - \mu_{\beta_1}^u)^\eta\right)^{\frac{1}{\eta}} + 1 - \min\left(1, (1 - \mu_{\beta_2}^u)^\eta\right)^{\frac{1}{\eta}}, \min\left(1, ((\xi_{\beta_1}^l)^\eta)^\eta\right)^{\frac{1}{\eta}} + \\
 & \min\left(1, ((\xi_{\beta_2}^l)^\eta)^\eta\right)^{\frac{1}{\eta}}, \min\left(1, ((\xi_{\beta_1}^u)^\eta)^\eta\right)^{\frac{1}{\eta}} + \min\left(1, ((\xi_{\beta_2}^u)^\eta)^\eta\right)^{\frac{1}{\eta}}, \min\left(1, ((\vartheta_{\beta_1}^l)^\eta)^\eta\right)^{\frac{1}{\eta}} + \\
 & \min\left(1, (\alpha_2(\vartheta_{\beta_2}^l)^\eta)^\eta\right)^{\frac{1}{\eta}}, \min\left(1, ((\vartheta_{\beta_1}^u)^\eta)^\eta\right)^{\frac{1}{\eta}} + \min\left(1, ((\vartheta_{\beta_2}^u)^\eta)^\eta\right)^{\frac{1}{\eta}} \\
 2. \quad & \beta_1 \otimes \beta_2 = \min\left(1, (\mu_{\beta_1}^l)^\eta\right)^{\frac{1}{\eta}} + \min\left(1, (\mu_{\beta_2}^l)^\eta\right)^{\frac{1}{\eta}}, \\
 & \min\left(1, (\mu_{\beta_1}^u)^\eta\right)^{\frac{1}{\eta}} + \min\left(1, (\mu_{\beta_2}^u)^\eta\right)^{\frac{1}{\eta}}, 1 - \min\left(1, ((1 - \xi_{\beta_1}^l)^\eta)^\eta\right)^{\frac{1}{\eta}} + 1 - \min\left(1, ((1 - \xi_{\beta_2}^l)^\eta)^\eta\right)^{\frac{1}{\eta}}, 1 - \\
 & \min\left(1, ((1 - \xi_{\beta_1}^u)^\eta)^\eta\right)^{\frac{1}{\eta}} + \min\left(1, ((1 - \xi_{\beta_2}^u)^\eta)^\eta\right)^{\frac{1}{\eta}}, 1 - \min\left(1, ((1 - \vartheta_{\beta_1}^l)^\eta)^\eta\right)^{\frac{1}{\eta}} + \\
 & 1 - \min\left(1, (\alpha_2(1 - \vartheta_{\beta_2}^l)^\eta)^\eta\right)^{\frac{1}{\eta}}, 1 - \min\left(1, ((1 - \vartheta_{\beta_1}^u)^\eta)^\eta\right)^{\frac{1}{\eta}} + 1 - \min\left(1, ((1 - \vartheta_{\beta_2}^u)^\eta)^\eta\right)^{\frac{1}{\eta}} \\
 3. \quad & \lambda\beta = ([1 - \min\left(1, (\lambda(1 - \mu_{\beta_1}^l)^\eta)^\eta\right)^{\frac{1}{\eta}}, 1 - \min\left(1, (\lambda(1 - \mu_{\beta_2}^l)^\eta)^\eta\right)^{\frac{1}{\eta}}], [\min\left(1, (\lambda(1 - \xi_{\beta_1}^l)^\eta)^\eta\right)^{\frac{1}{\eta}}, \\
 & \min\left(1, (\lambda(1 - \xi_{\beta_2}^l)^\eta)^\eta\right)^{\frac{1}{\eta}}], [\min\left(1, (\lambda(1 - \vartheta_{\beta_1}^l)^\eta)^\eta\right)^{\frac{1}{\eta}}, \min\left(1, (\lambda(1 - \vartheta_{\beta_2}^l)^\eta)^\eta\right)^{\frac{1}{\eta}}]
 \end{aligned}$$

Definition 8:

Suppose $\beta_j = ([(u_{\beta_j}^l(r), u_{\beta_j}^u(r)], [(\xi_{\beta_j}^l(r), \xi_{\beta_j}^u(r)], (\vartheta_{\beta_j}^l(r), \vartheta_{\beta_j}^u(r)])$ be the set of IVPFNs. The IVPFYWGA is a function $\mathbb{J}_n \rightarrow \mathbb{J}$ such that

$$IVPFYWGA_\alpha(\beta_1, \beta_2, \dots, \beta_n) = \otimes_{j=1}^n (\alpha_j \beta_j)$$

where $\alpha_j = (\alpha_1, \alpha_2, \dots, \alpha_n)^T$ is the weight vector with $\alpha_j > 0$ and $\sum_{i=1}^n \alpha_j = 1$.

Theorem 1:

Suppose $\beta_j = ([(u_{\beta_j}^l(r), u_{\beta_j}^u(r)], [(\xi_{\beta_j}^l(r), \xi_{\beta_j}^u(r)], (\vartheta_{\beta_j}^l(r), \vartheta_{\beta_j}^u(r)])$ be the set of IVPFNs. Then the aggregated value of IVPFYWGA is given by

$$IVPFYWGA_\alpha(\beta_1, \beta_2, \dots, \beta_n) =$$

$$\left(\begin{aligned}
 & \min\left(1, \left(\sum_{j=1}^n (\alpha_j (\mu_{\beta_j}^l)^\eta)^\eta\right)^{\frac{1}{\eta}}\right), \min\left(1, \left(\sum_{j=1}^n (\alpha_j (\mu_{\beta_j}^u)^\eta)^\eta\right)^{\frac{1}{\eta}}\right) \\
 & 1 - \min\left(1, \left(\sum_{j=1}^n (\alpha_j (1 - \xi_{\beta_j}^l)^\eta)^\eta\right)^{\frac{1}{\eta}}\right), 1 - \min\left(1, \left(\sum_{j=1}^n (\alpha_j (1 - \xi_{\beta_j}^u)^\eta)^\eta\right)^{\frac{1}{\eta}}\right) \\
 & 1 - \min\left(1, \left(\sum_{j=1}^n (\alpha_j (1 - \vartheta_{\beta_j}^l)^\eta)^\eta\right)^{\frac{1}{\eta}}\right), 1 - \min\left(1, \left(\sum_{j=1}^n (\alpha_j (1 - \vartheta_{\beta_j}^u)^\eta)^\eta\right)^{\frac{1}{\eta}}\right)
 \end{aligned} \right)$$

Proof:

Suppose $\beta_j = ([(\mu_{\beta_j}^l(r), \mu_{\beta_j}^u(r)), [(\xi_{\beta_j}^l(r), \xi_{\beta_j}^u(r)), (\vartheta_{\beta_j}^l(r), \vartheta_{\beta_j}^u(r))])$ be the set of IVPFNs. We prove the above theorem by using mathematical induction. For $n = 2$, we have

$$\alpha_1\beta_1 = \left(\begin{array}{c} \min(1, (\alpha_1(\mu_{\beta_1}^l)^\eta)^{\frac{1}{\eta}}), \min(1, (\alpha_1(\mu_{\beta_1}^u)^\eta)^{\frac{1}{\eta}}) \\ 1 - \min(1, (\alpha_1(1 - \xi_{\beta_1}^l)^\eta)^{\frac{1}{\eta}}), 1 - \min(1, (\alpha_1(1 - \xi_{\beta_1}^u)^\eta)^{\frac{1}{\eta}}) \\ 1 - \min\left(1, (\alpha_1(\vartheta_{\beta_1}^l)^\eta)^{\frac{1}{\eta}}\right), 1 - \min(1, (\alpha_1(1 - \vartheta_{\beta_1}^u)^\eta)^{\frac{1}{\eta}}) \end{array} \right)$$

$$\alpha_2\beta_2 = \left(\begin{array}{c} \min(1, (\alpha_2(\mu_{\beta_2}^l)^\eta)^{\frac{1}{\eta}}), \min(1, (\alpha_2(\mu_{\beta_2}^u)^\eta)^{\frac{1}{\eta}}) \\ 1 - \min(1, (\alpha_2(1 - \xi_{\beta_2}^l)^\eta)^{\frac{1}{\eta}}), 1 - \min(1, (\alpha_2(1 - \xi_{\beta_2}^u)^\eta)^{\frac{1}{\eta}}) \\ 1 - \min(1, ((\alpha_2(\vartheta_{\beta_2}^l)^\eta)^{\frac{1}{\eta}}), 1 - \min(1, ((\alpha_2(1 - \vartheta_{\beta_2}^u)^\eta)^{\frac{1}{\eta}}) \end{array} \right)$$

Now from the definition 7 and 8

$$\alpha_1\beta_1 + \alpha_2\beta_2 =$$

$$\min(1, (\alpha_1(\mu_{\beta_1}^l)^\eta)^{\frac{1}{\eta}}), \min(1, (\alpha_1(\mu_{\beta_1}^u)^\eta)^{\frac{1}{\eta}})$$

$$1 - \min(1, (\alpha_1(1 - \xi_{\beta_1}^l)^\eta)^{\frac{1}{\eta}}), 1 - \min(1, (\alpha_1(1 - \xi_{\beta_1}^u)^\eta)^{\frac{1}{\eta}})$$

$$1 - \min(1, ((\alpha_1(\vartheta_{\beta_1}^l)^\eta)^{\frac{1}{\eta}}), 1 - \min(1, ((\alpha_1(1 - \vartheta_{\beta_1}^u)^\eta)^{\frac{1}{\eta}})$$

$$\otimes$$

$$\min(1, (\alpha_2(\mu_{\beta_2}^l)^\eta)^{\frac{1}{\eta}}), \min(1, (\alpha_2(\mu_{\beta_2}^u)^\eta)^{\frac{1}{\eta}})$$

$$1 - \min(1, (\alpha_2(1 - \xi_{\beta_2}^l)^\eta)^{\frac{1}{\eta}}), \min(1, (\alpha_2(1 - \xi_{\beta_2}^u)^\eta)^{\frac{1}{\eta}})$$

$$1 - \min(1, ((\alpha_2(1 - \vartheta_{\beta_2}^l)^\eta)^{\frac{1}{\eta}}), 1 - \min(1, ((\alpha_2(1 - \vartheta_{\beta_2}^u)^\eta)^{\frac{1}{\eta}})$$

$$\min(1, (\alpha_1(\mu_{\beta_1}^l)^\eta)^{\frac{1}{\eta}}) + \min(1, (\alpha_2(\mu_{\beta_2}^l)^\eta)^{\frac{1}{\eta}}),$$

$$\min(1, (\alpha_1(\mu_{\beta_1}^u)^\eta)^{\frac{1}{\eta}}) + \min(1, (\alpha_2(\mu_{\beta_2}^u)^\eta)^{\frac{1}{\eta}})$$

$$1 - \min(1, (\alpha_1(1 - \xi_{\beta_1}^l)^\eta)^{\frac{1}{\eta}}) + 1 - \min(1, (\alpha_2(1 - \xi_{\beta_2}^l)^\eta)^{\frac{1}{\eta}}),$$

$$1 - \min(1, (\alpha_1(1 - \xi_{\beta_1}^u)^\eta)^{\frac{1}{\eta}}) + 1 - \min(1, (\alpha_2(1 - \xi_{\beta_2}^u)^\eta)^{\frac{1}{\eta}})$$

$$1 - \min(1, ((\alpha_1(\vartheta_{\beta_1}^l)^\eta)^{\frac{1}{\eta}}) + 1 - \min(1, ((\alpha_2(1 - \vartheta_{\beta_2}^l)^\eta)^{\frac{1}{\eta}})$$

$$1 - \min(1, ((\alpha_1(\vartheta_{\beta_1}^u)^\eta)^{\frac{1}{\eta}}) + 1 - \min(1, ((\alpha_2(1 - \vartheta_{\beta_2}^u)^\eta)^{\frac{1}{\eta}})$$

$$\left(\begin{array}{l} \min(1, (\sum_{j=1}^2 (\alpha_j (\mu_{\beta_j}^l)^\eta)^{\frac{1}{\eta}})), \min((1, (\sum_{j=1}^2 (\alpha_j (1 - \mu_{\beta_j}^u)^\eta)^{\frac{1}{\eta}})) \\ 1 - \min(1, (\sum_{j=1}^2 (\alpha_j (1 - \xi_{\beta_j}^l)^\eta)^{\frac{1}{\eta}})), 1 - \min((1, (\sum_{j=1}^2 (\alpha_j (1 - \xi_{\beta_j}^u)^\eta)^{\frac{1}{\eta}})) \\ 1 - \min(1, (\sum_{j=1}^2 (\alpha_j (1 - \vartheta_{\beta_j}^l)^\eta)^{\frac{1}{\eta}})), 1 - \min((1, (\sum_{j=1}^2 (\alpha_j (1 - \vartheta_{\beta_j}^u)^\eta)^{\frac{1}{\eta}})) \end{array} \right)$$

Therefore, it is true for $n = 2$

Suppose it is also true for $n = k$, we have

$$IVPFYWGA_\alpha(\beta_1, \beta_2, \dots, \beta_k) = \otimes_{j=1}^k (\alpha_j \beta_j)$$

$$\left(\begin{array}{l} \min(1, (\sum_{j=1}^k (\alpha_j (\mu_{\beta_j}^l)^\eta)^{\frac{1}{\eta}})), \min((1, (\sum_{j=1}^k (\alpha_j (\mu_{\beta_j}^u)^\eta)^{\frac{1}{\eta}})) \\ 1 - \min(1, (\sum_{j=1}^k (\alpha_j (1 - \xi_{\beta_j}^l)^\eta)^{\frac{1}{\eta}})), 1 - \min((1, (\sum_{j=1}^k (\alpha_j (1 - \xi_{\beta_j}^u)^\eta)^{\frac{1}{\eta}})) \\ 1 - \min(1, (\sum_{j=1}^k (\alpha_j (1 - \vartheta_{\beta_j}^l)^\eta)^{\frac{1}{\eta}})), 1 - \min((1, (\sum_{j=1}^k (\alpha_j (1 - \vartheta_{\beta_j}^u)^\eta)^{\frac{1}{\eta}})) \end{array} \right)$$

Now we will prove that for $n = k + 1$

$$IVPFYWGA_\alpha(\beta_1, \beta_2, \dots, \beta_k, \beta_{k+1}) = \otimes_{j=1}^k (\alpha_j \beta_j) \otimes (\alpha_{k+1} \beta_{k+1})$$

$$\min(1, (\sum_{j=1}^k (\alpha_j (\mu_{\beta_j}^l)^\eta)^{\frac{1}{\eta}})), \min((1, (\sum_{j=1}^k (\alpha_j (\mu_{\beta_j}^u)^\eta)^{\frac{1}{\eta}}))$$

$$1 - \min(1, (\sum_{j=1}^k (\alpha_j (1 - \xi_{\beta_j}^l)^\eta)^\eta)^{\frac{1}{\eta}}), 1 - \min(1, (\sum_{j=1}^k (\alpha_j (1 - \xi_{\beta_j}^u)^\eta)^\eta)^{\frac{1}{\eta}})$$

$$1 - \min(1, (\sum_{j=1}^k (\alpha_j (1 - \vartheta_{\beta_j}^l)^\eta)^\eta)^{\frac{1}{\eta}}), 1 - \min(1, (\sum_{j=1}^k (\alpha_j (1 - \vartheta_{\beta_j}^u)^\eta)^\eta)^{\frac{1}{\eta}})$$

⊗

$$\min(1, (1, (\alpha_{k+1} (\mu_{\beta_{k+1}}^l)^\eta)^\eta)^{\frac{1}{\eta}}), \min(1, (1, (\alpha_{k+1} (\mu_{\beta_{k+1}}^u)^\eta)^\eta)^{\frac{1}{\eta}})$$

$$1 - \min(1, (\alpha_{k+1} (1 - \xi_{\beta_{k+1}}^l)^\eta)^\eta)^{\frac{1}{\eta}}, 1 - \min(1, (\alpha_{k+1} (1 - \xi_{\beta_{k+1}}^u)^\eta)^\eta)^{\frac{1}{\eta}}$$

$$1 - \min(1, (\alpha_{k+1} (1 - \vartheta_{\beta_{k+1}}^l)^\eta)^\eta)^{\frac{1}{\eta}}, 1 - \min(1, (\alpha_{k+1} (1 - \vartheta_{\beta_{k+1}}^u)^\eta)^\eta)^{\frac{1}{\eta}}$$

$$\min(1, (\sum_{j=1}^{k+1} (\alpha_j (\mu_{\beta_j}^l)^\eta)^\eta)^{\frac{1}{\eta}}), \min(1, (\sum_{j=1}^{k+1} (\alpha_j (1 - \mu_{\beta_j}^u)^\eta)^\eta)^{\frac{1}{\eta}})$$

$$1 - \min(1, (\sum_{j=1}^{k+1} (\alpha_j (\xi_{\beta_j}^l)^\eta)^\eta)^{\frac{1}{\eta}}), 1 - \min(1, (\sum_{j=1}^{k+1} (\alpha_j (1 - \xi_{\beta_j}^u)^\eta)^\eta)^{\frac{1}{\eta}})$$

$$1 - \min(1, (\sum_{j=1}^{k+1} (\alpha_j (1 - \vartheta_{\beta_j}^l)^\eta)^\eta)^{\frac{1}{\eta}}), 1 - \min(1, (\sum_{j=1}^{k+1} (\alpha_j (1 - \vartheta_{\beta_j}^u)^\eta)^\eta)^{\frac{1}{\eta}})$$

Hence Proved.

Theorem 2:

If all IVPFNs are similar (i.e.) $\beta_i = \beta$ then $IVPFYWGA(\beta_1, \beta_2, \dots, \beta_n) = \beta$

Proof:

Suppose $\beta_j = ([(\mu_{\beta_j}^l(r), \mu_{\beta_j}^u(r)), [(\xi_{\beta_j}^l(r), \xi_{\beta_j}^u(r)), (\vartheta_{\beta_j}^l(r), \vartheta_{\beta_j}^u(r))])$

As all IVPFNs are similar, $\beta_j = ([(\mu_{\beta}^l(r), \mu_{\beta}^u(r)), [(\xi_{\beta}^l(r), \xi_{\beta}^u(r)), (\vartheta_{\beta}^l(r), \vartheta_{\beta}^u(r))])$

Then the Then aggregated value of IVPFYWGA is given by

$$\begin{aligned}
 &IVPFYWGA_{\alpha}(\beta_1, \beta_2, \dots, \beta_n) = \\
 &\min(1, (\sum_{j=1}^n (\alpha_j (\mu_{\beta_j}^l))^{\eta})^{\frac{1}{\eta}}), \min((1, (\sum_{j=1}^n (\alpha_j (\mu_{\beta_j}^u))^{\eta})^{\frac{1}{\eta}}) \\
 &1 - \min(1, (\sum_{j=1}^n (\alpha_j (1 - \xi_{\beta_j}^l))^{\eta})^{\frac{1}{\eta}}), 1 - \min((1, (\sum_{j=1}^n (\alpha_j (1 - \xi_{\beta_j}^u))^{\eta})^{\frac{1}{\eta}}) \\
 &1 - \min(1, (\sum_{j=1}^n (\alpha_j (1 - \vartheta_{\beta_j}^l))^{\eta})^{\frac{1}{\eta}}), 1 - \min((1, (\sum_{j=1}^n (\alpha_j (1 - \vartheta_{\beta_j}^u))^{\eta})^{\frac{1}{\eta}}) \\
 &= \min(1, (\sum_{j=1}^n (\alpha_j (\mu_{\beta}^l))^{\eta})^{\frac{1}{\eta}}), 1 - \min((1, (\sum_{j=1}^n (\alpha_j (\mu_{\beta}^u))^{\eta})^{\frac{1}{\eta}}) \\
 &1 - \min(1, (\sum_{j=1}^n (\alpha_j (1 - \xi_{\beta}^l))^{\eta})^{\frac{1}{\eta}}), 1 - \min((1, (\sum_{j=1}^n (\alpha_j (1 - \xi_{\beta}^u))^{\eta})^{\frac{1}{\eta}}) \\
 &1 - \min(1, (\sum_{j=1}^n (\alpha_j (1 - \vartheta_{\beta}^l))^{\eta})^{\frac{1}{\eta}}), 1 - \min((1, (\sum_{j=1}^n (\alpha_j (1 - \vartheta_{\beta}^u))^{\eta})^{\frac{1}{\eta}}) \\
 &= \beta
 \end{aligned}$$

Theorem 3 (Boundedness)

Suppose $\beta_j = ([(\mu_{\beta_j}^l(r), \mu_{\beta_j}^u(r)), [(\xi_{\beta_j}^l(r), \xi_{\beta_j}^u(r)), (\vartheta_{\beta_j}^l(r), \vartheta_{\beta_j}^u(r))])$ be a collection of IVPFNs. Let $\beta^- = \min(\beta_1, \beta_2, \dots, \beta_n)$, $\beta^+ = \max(\beta_1, \beta_2, \dots, \beta_n)$ PFNs, then $\beta^- \leq IVPFYWGA(\beta_1, \beta_2, \dots, \beta_n) \leq \beta^+$

Proof:

Suppose that $\beta^- = \min(\beta_1, \beta_2, \dots, \beta_n) = \langle [\mu_{\beta}^{l-}, \mu_{\beta}^{u-}], [(\xi_{\beta}^{l-}, \xi_{\beta}^{u-}), (\vartheta_{\beta}^{l-}, \vartheta_{\beta}^{u-})] \rangle$ and

$$\beta^+ = \max(\beta_1, \beta_2, \dots, \beta_n) = \langle [(u_{\beta}^{l+}, u_{\beta}^{u+}), [\xi_{\beta}^{l+}, \xi_{\beta}^{u+}], [\vartheta_{\beta}^{l+}, \vartheta_{\beta}^{u+}]] \rangle$$

The inequalities for membership values are

Since $[(\mu_{\beta}^{l-}, \mu_{\beta}^{u-})] \leq [(\mu_{\beta_j}^l, \mu_{\beta_j}^u)] \leq (\mu_{\beta}^{l+}, \mu_{\beta}^{u+}), \alpha_i \in [0, 1]$; we have

$$\begin{aligned} & \min(1, (\sum_{j=1}^n (\alpha_j (\mu_{\beta_j}^{l-})^\eta)^{\frac{1}{\eta}})), \min(1, (\sum_{j=1}^n (\alpha_j (\mu_{\beta_j}^{u-})^\eta)^{\frac{1}{\eta}})) \\ & \leq \min(1, (\sum_{j=1}^n (\alpha_j (\mu_{\beta_j}^l)^\eta)^{\frac{1}{\eta}})), \min(1, (\sum_{j=1}^n (\alpha_j (\mu_{\beta_j}^u)^\eta)^{\frac{1}{\eta}})) \\ & \leq \min(1, (\sum_{j=1}^n (\alpha_j (\mu_{\beta_j}^{l+})^\eta)^{\frac{1}{\eta}})), \min(1, (\sum_{j=1}^n (\alpha_j (\mu_{\beta_j}^{u+})^\eta)^{\frac{1}{\eta}})) \end{aligned}$$

The inequalities for neutral membership

$$\begin{aligned} & 1 - \min(1, (\sum_{j=1}^n (\alpha_j (1 - \xi_{\beta_j}^{l-})^\eta)^{\frac{1}{\eta}})), 1 - \min(1, (\sum_{j=1}^n (\alpha_j (1 - \xi_{\beta_j}^{u-})^\eta)^{\frac{1}{\eta}})) \\ & \leq 1 - \min(1, (\sum_{j=1}^n (\alpha_j (1 - \xi_{\beta_j}^l)^\eta)^{\frac{1}{\eta}})), 1 - \min(1, (\sum_{j=1}^n (\alpha_j (1 - \xi_{\beta_j}^u)^\eta)^{\frac{1}{\eta}})) \\ & \leq 1 - \min(1, (\sum_{j=1}^n (\alpha_j (1 - \xi_{\beta_j}^{l+})^\eta)^{\frac{1}{\eta}})), 1 - \min(1, (\sum_{j=1}^n (\alpha_j (1 - \xi_{\beta_j}^{u+})^\eta)^{\frac{1}{\eta}})) \end{aligned}$$

The inequalities for non-membership

$$\begin{aligned} & 1 - \min(1, (\sum_{j=1}^n (\alpha_j (1 - \vartheta_{\beta_j}^{l-})^\eta)^{\frac{1}{\eta}})), 1 - \min(1, (\sum_{j=1}^n (\alpha_j (1 - \vartheta_{\beta_j}^{u-})^\eta)^{\frac{1}{\eta}})) \\ & \leq 1 - \min(1, (\sum_{j=1}^n (\alpha_j (1 - \vartheta_{\beta_j}^l)^\eta)^{\frac{1}{\eta}})), 1 - \min(1, (\sum_{j=1}^n (\alpha_j (1 - \vartheta_{\beta_j}^u)^\eta)^{\frac{1}{\eta}})) \\ & \leq 1 - \min(1, (\sum_{j=1}^n (\alpha_j (1 - \vartheta_{\beta_j}^{l+})^\eta)^{\frac{1}{\eta}})), 1 - \min(1, (\sum_{j=1}^n (\alpha_j (1 - \vartheta_{\beta_j}^{u+})^\eta)^{\frac{1}{\eta}})) \end{aligned}$$

Hence $\beta^- \leq IFPFYWGA(\beta_1, \beta_2, \dots, \beta_n) \leq \beta^+$

Theorem 4(Monotonicity)

Suppose $\beta_j = [(\mu_{\beta_j}^l(r), \mu_{\beta_j}^u(r)), [(\xi_{\beta_j}^l(r), \xi_{\beta_j}^u(r)), (\vartheta_{\beta_j}^l(r), \vartheta_{\beta_j}^u(r))]]$ be a collection of IVPFNs.

Assume that $\beta'_j = (\beta'_1, \beta'_2, \dots, \beta'_n)$ and $\beta_j = (\beta_1, \beta_2, \dots, \beta_n)$ are two sets of IVPFNs. If $\mu_{\beta'_j}^l(r) \leq \mu_{\beta_j}^l(r)$, $\mu_{\beta'_j}^u(r) \leq \mu_{\beta_j}^u(r)$, $\xi_{\beta'_j}^l(r) \leq \xi_{\beta_j}^l(r)$, $\xi_{\beta'_j}^u(r) \leq \xi_{\beta_j}^u(r)$, $\vartheta_{\beta'_j}^l(r) \leq \vartheta_{\beta_j}^l(r)$ and $\vartheta_{\beta'_j}^u(r) \leq \vartheta_{\beta_j}^u(r) \quad \forall j$ then,

$$IVPFYWGA(\beta'_1, \beta'_2, \dots, \beta'_n) \leq IVPFYWGA(\beta_1, \beta_2, \dots, \beta_n)$$

Proof:

Let $IVPFYWGA(\beta'_1, \beta'_2, \dots, \beta'_n) = (G', K', S')$ and

$$IVPFYWGA(\beta_1, \beta_2, \dots, \beta_n) = (G, K, S)$$

First, we will show that $G' \leq G$. As $\mu_{\beta'_j}^l(r) \leq \mu_{\beta_j}^l(r)$, $\mu_{\beta'_j}^u(r) \leq \mu_{\beta_j}^u(r)$ Moreover,

$$\sum_{j=1}^n (\alpha_j (\mu_{\beta'_j}^l)^{\eta})^{\frac{1}{\eta}} \leq \sum_{j=1}^n (\alpha_j (\mu_{\beta_j}^l)^{\eta})^{\frac{1}{\eta}}$$

Thus,

$$\min(1, (\sum_{j=1}^n (\alpha_j (\mu_{\beta'_j}^l)^{\eta})^{\frac{1}{\eta}})) \leq \min(1, (\sum_{j=1}^n (\alpha_j (\mu_{\beta_j}^l)^{\eta})^{\frac{1}{\eta}}))$$

And,

$$\sum_{j=1}^n (\alpha_j (\mu_{\beta'_j}^u)^{\eta})^{\frac{1}{\eta}} \leq \sum_{j=1}^n (\alpha_j (\mu_{\beta_j}^u)^{\eta})^{\frac{1}{\eta}}$$

Thus,

$$\min(1, (\sum_{j=1}^n (\alpha_j (\mu_{\beta'_j}^u)^{\eta})^{\frac{1}{\eta}})) \leq \min(1, (\sum_{j=1}^n (\alpha_j (\mu_{\beta_j}^u)^{\eta})^{\frac{1}{\eta}}))$$

Hence $G' \leq G$.

Similarly, we can prove that $K' \geq K$ and $S' \geq S$. Hence

$$IVPFYWGA(\beta'_1, \beta'_2, \dots, \beta'_n) \leq IVPFYWGA(\beta_1, \beta_2, \dots, \beta_n)$$

Definition 9:

Assume that $\beta_j = ([(\mu_{\beta_j}^l(r), \mu_{\beta_j}^u(r)), [(\xi_{\beta_j}^l(r), \xi_{\beta_j}^u(r)), (\vartheta_{\beta_j}^l(r), \vartheta_{\beta_j}^u(r))])$ be the set of IVPFNs. The IVPFYOWGA is a function $\mathbb{J}_n \rightarrow \mathbb{J}$ such that

$$IVPFYOWGA_\alpha(\beta_1, \beta_2, \dots, \beta_n) = \otimes_{j=1}^n (\alpha_j \beta_{\rho(j)})$$

where $\alpha_j = (\alpha_1, \alpha_2, \dots, \alpha_n)^T$ is the weight vector with $\alpha_j > 0$ and $\sum_{i=1}^n \alpha_j = 1$; where $\rho(1), \rho(2), \dots, \rho(n)$ is the permutation of $j = 1, 2, \dots, n$

Theorem 5:

Assume that $\beta_j = ([(\mu_{\beta_j}^l(r), \mu_{\beta_j}^u(r)), [(\xi_{\beta_j}^l(r), \xi_{\beta_j}^u(r)), (\vartheta_{\beta_j}^l(r), \vartheta_{\beta_j}^u(r))])$ be the set of IVPFNs. The IVPFYOWGA is a function $\mathbb{J}_n \rightarrow \mathbb{J}$ such that

$$IVPFYOWGA_\alpha(\beta_1, \beta_2, \dots, \beta_n) =$$

$$\left(\begin{aligned} & \min(1, (\sum_{j=1}^n (\alpha_j (\mu_{\beta_{\rho(j)}}^l))^{\frac{1}{\eta}}))^{\frac{1}{\eta}}, \min(1, (\sum_{j=1}^n (\alpha_j (\mu_{\beta_{\rho(j)}}^u))^{\frac{1}{\eta}}))^{\frac{1}{\eta}} \\ & 1 - \min(1, (\sum_{j=1}^n (\alpha_j (1 - \xi_{\beta_{\rho(j)}}^l))^{\frac{1}{\eta}}))^{\frac{1}{\eta}}, 1 - \min(1, (\sum_{j=1}^n (\alpha_j (1 - \xi_{\beta_{\rho(j)}}^u))^{\frac{1}{\eta}}))^{\frac{1}{\eta}} \\ & 1 - \min(1, (\sum_{j=1}^n (\alpha_j (1 - \vartheta_{\beta_{\rho(j)}}^l))^{\frac{1}{\eta}}))^{\frac{1}{\eta}}, 1 - \min(1, (\sum_{j=1}^n (\alpha_j (1 - \vartheta_{\beta_{\rho(j)}}^u))^{\frac{1}{\eta}}))^{\frac{1}{\eta}} \end{aligned} \right)$$

Proof:

We can demonstrate the result by utilizing analogous explanations as used in theorem 1.

Definition 10:

Assume that $\beta_j = ([(\mu_{\beta_j}^l(r), \mu_{\beta_j}^u(r)), [(\xi_{\beta_j}^l(r), \xi_{\beta_j}^u(r)), (\vartheta_{\beta_j}^l(r), \vartheta_{\beta_j}^u(r))])$ be the set of IVPFNs. The IVPFYHWGA is a function $\mathbb{J}_n \rightarrow \mathbb{J}$ such that

$$IVPFYHWGA_\alpha(\beta_1, \beta_2, \dots, \beta_n) = \otimes_{j=1}^n (\alpha_j \hat{\beta}_{\rho(j)})$$

where $\alpha_j = (\alpha_1, \alpha_2, \dots, \alpha_n)^T$ is the weight vector with $\alpha_j > 0$ and $\sum_{i=1}^n \alpha_j = 1$; where $\rho(1), \rho(2), \dots, \rho(n)$ is the permutation of $j = 1, 2, \dots, n$

Theorem 6:

Suppose $\beta_j = ([(\mu_{\beta_j}^l(r), \mu_{\beta_j}^u(r)), [(\xi_{\beta_j}^l(r), \xi_{\beta_j}^u(r)), (\vartheta_{\beta_j}^l(r), \vartheta_{\beta_j}^u(r))])$ be the set of IVPFNs. Then the aggregated value of IVPFYHWGA is given by

$$IVPFYHWGA_{\alpha}(\beta_1, \beta_2, \dots, \beta_n) =$$

$$\left(\begin{aligned} & \min(1, (\sum_{j=1}^n (\alpha_j (\mu_{\beta_{\rho(j)}}^l))^{\frac{1}{\eta}}))^{\frac{1}{\eta}}, \min(1, (\sum_{j=1}^n (\alpha_j (\mu_{\beta_{\rho(j)}}^u))^{\frac{1}{\eta}}))^{\frac{1}{\eta}} \\ & 1 - \min(1, (\sum_{j=1}^n (\alpha_j (1 - \xi_{\beta_{\rho(j)}}^l))^{\frac{1}{\eta}}))^{\frac{1}{\eta}}, 1 - \min(1, (\sum_{j=1}^n (\alpha_j (1 - \xi_{\beta_{\rho(j)}}^u))^{\frac{1}{\eta}}))^{\frac{1}{\eta}} \\ & 1 - \min(1, (\sum_{j=1}^n (\alpha_j (1 - \vartheta_{\beta_{\rho(j)}}^l))^{\frac{1}{\eta}}))^{\frac{1}{\eta}}, 1 - \min(1, (\sum_{j=1}^n (\alpha_j (1 - \vartheta_{\beta_{\rho(j)}}^u))^{\frac{1}{\eta}}))^{\frac{1}{\eta}} \end{aligned} \right)$$

Where $\alpha_i = (\alpha_1, \alpha_2, \dots, \alpha_n)^T$ is the weight vector of I_i with $\alpha_i > 0$ and $\sum_{i=1}^n \alpha_i = 1$; Where $I_{\rho(i)}$ is i^{th} largest weighted Intuitionistic fuzzy values $I_i (I_i = n\alpha_i I_i, i = 1, 2, \dots, n)$ and n is the balancing co-efficient.

Proof: We can demonstrate the result by utilizing analogous explanations as used in theorem 1.

4. An Interval Valued Picture Fuzzy Decision Framework for Optimal Green Supplier Selection

To address the selection of appropriate green supplier under uncertainty, the IVPFYWGA operators are incorporated in a framework of MADM problem using interval-valued picture fuzzy data. The step-by-step explanation of the proposed method is given as follows:

Step 1: The set of alternatives is called $\mathcal{L} = \{\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_m\}$ which are to be evaluated according to a list of criteria $K = \{K_1, K_2, \dots, K_n\}$. In order to make a meaningful and fair assessment, we need to assign a weight to each criterion which we can do with a weight vector $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)^T$ with $\alpha_j > 0$ and $\sum_{j=1}^n \alpha_j = 1$, The weights reflect the comparative significance of each criterion in the decision-making process. The performance or evaluation of each alternative with respect to the criteria is set up as a decision matrix with each entry to be represented by interval-valued picture fuzzy values (IVPFVs), which capture the imprecise, uncertain, and indeterminate nature of the information available.

Step 2: Use the IVPFYWGA Operator for each alternative across all criteria. The operator can be used to combine uncertain, unclear and ambiguous data together into a single representative value for each alternative with respect to each group of criteria together.

Step 3: Calculate the overall evaluation scores for each alternative by aggregating the assessment for each criterion, which gives an overall score for each alternative with respect to the relative importance assigned to the criteria.

Step 4: Determine which alternative has the highest overall aggregated score. The alternative with the highest overall score will be selected as the preferred alternative, as it has the maximum alignment with the objectives and weightings, and thus represents the preferred option.

This study examines the performance of four leading companies denoted as $\mathcal{L} = \{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4\}$ that recognized and incorporate Green Supply Chain Management practices. This assessment is conducted in five critical criteria important to green operations: K_1 represents green design, K_2 represents green manufacturing K_3 represents green logistics K_4 represents green procurement K_5 represents green inventory. Overall, all criteria are important to improving environmental performance, meeting regulations, and increasing resource efficiency. To highlight the discrepancies in importance associated with each criterion for decision-making processes, we used a predetermined weight vector which is $\alpha = (0.18, 0.21, 0.25, 0.16, 0.13)^T$ Such that $\sum_{i=1}^n \alpha_i \leq 1$. These weights ensure that the evaluation process reflects the importance of each green initiative. The selected criteria, taken all together, provide an exhaustive overall framework for evaluating the company, considering both sustainability importance and effectiveness, allowing us to fully compare and understand each firm and its respective green supply chain performance. Table 1 shows the decision matrix of alternative according to the different attributes.

Table 1. Interval Valued Picture Fuzzy Decision Matrix

\mathcal{L}_i/K_i	K_1	K_2	K_3	K_4	K_5
\mathcal{L}_1	[0.48,0.52] [0.17,0.19] [0.24,0.28]	[0.43,0.47] [0.18,0.20] [0.30,0.32]	[0.46,0.51] [0.10,0.13] [0.20,0.25]	[0.41,0.46] [0.22,0.27] [0.20,0.24]	[0.47,0.52] [0.11,0.14] [0.23,0.27]
\mathcal{L}_2	[0.44,0.48] [0.16,0.20] [0.27,0.30]	[0.40,0.45] [0.18,0.21] [0.26,0.28]	[0.45,0.49] [0.16,0.20] [0.23,0.27]	[0.43,0.46] [0.17,0.19] [0.28,0.31]	[0.43,0.47] [0.12,0.18] [0.24,0.28]
\mathcal{L}_3	[0.24,0.28] [0.23,0.27] [0.47,0.51]	[0.21,0.23] [0.25,0.27] [0.43,0.46]	[0.20,0.25] [0.18,0.21] [0.48,0.52]	[0.23,0.27] [0.20,0.24] [0.42,0.44]	[0.14,0.18] [0.22,0.26] [0.50,0.53]
\mathcal{L}_4	[0.41,0.47] [0.13,0.18] [0.31,0.34]	[0.37,0.40] [0.18,0.22] [0.34,0.37]	[0.38,0.41] [0.12,0.16] [0.34,0.37]	[0.37,0.42] [0.15,0.19] [0.27,0.31]	[0.40,0.44] [0.22,0.25] [0.27,0.30]

Applying IVPFYWGA operator to find the evaluation performance values ξ_i for $\eta = 3$ are

$$\xi_1 = [0.44,0.48], [0.17,0.20], [0.25,0.29]$$

$$\xi_2 = [0.42, 0.46], [0.18, 0.22], [0.27, 0.30]$$

$$\xi_3 = [0.21, 0.24], [0.23, 0.26], [0.47, 0.50]$$

$$\xi_4 = [0.38, 0.42], [0.17, 0.21], [0.32, 0.36]$$

The score metrics of \mathcal{L}_i , for $\eta = 3$ obtained by IVPFYWGA which are given as,

$$S(\mathcal{L}_1) = 0.16719$$

$$S(\mathcal{L}_2) = 0.15054$$

$$S(\mathcal{L}_3) = 0.03928$$

$$S(\mathcal{L}_4) = 0.12202$$

The ranking order obtained from the computed score values is $S(\mathcal{L}_1) > S(\mathcal{L}_2) > S(\mathcal{L}_4) > S(\mathcal{L}_3)$. This ranking is established through the application of the proposed algorithm, Interval-Valued Picture Fuzzy Yager Weighted Geometric aggregation (IVPFYWGA) operators. The results clearly indicate that alternative \mathcal{L}_1 stands out above the viable options for green supply chain performance, being the better, most favourable and strategically to the firms assessed.

Figure 1 illustrates the score function representations of the Companies that implements GSCM, computed using IVPFYWGA for $\eta = 3$.

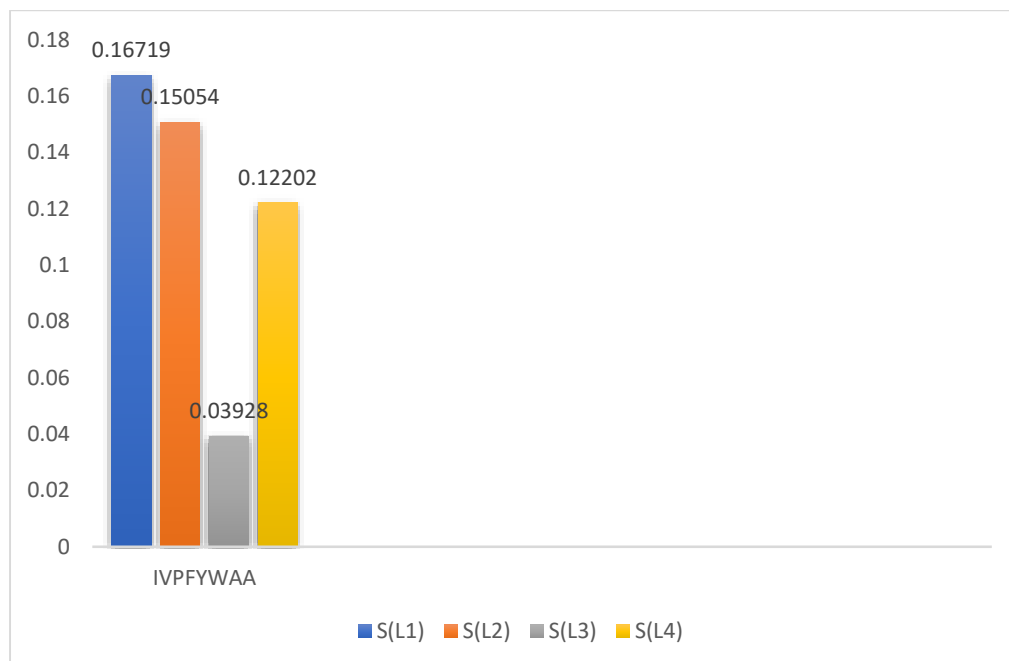


Fig 1 Depiction of score functions of alternatives using IVPFYWGA for $\eta = 3$

5. Conclusion:

In conclusion, Yager aggregation operators are useful for synthesizing information from sources in situations where uncertainty and imprecision are critical factors. These operators provide an approach to data aggregation that can represent uncertainty levels and the preferences of decision makers. Our study developed three combining processes that associate IVPFSs with finite families taking benefit from the Yager t-norms and conorms. The operators include IVPFYWGA, IVPFYOWGA, and IVPFYHWGA. We also examined the properties of the operators we proposed and discussed an MADM problem. In particular, we indicated by applying the IVPFYWGA method to represent the information associated with each attribute. We provided useful results using scoring functions so that we could relate our estimate of the value of each alternative consistently.

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