10.48047/jocaaa.2023.33.07.32

Some Characteristics of Double Vertex Graph

Lathadevi Puli[1], Roopa Prabhu[2]
[1]Associate Professor
Government First Grade College
Yelahanka, Bangalore India

[2]Department of Science

Government Polytechnic,

Channapatna, India

Abstract

The *double vertex graph* $U_2(G)$ of a graph G of order $n \geq 2$ is the graph whose vertex set consists of all $\binom{n}{2}$ unordered pairs of vertices of G and where two vertices $\{a, b\}$ and $\{c, d\}$ are adjacent if and only if $|\{a, b\} \cap \{c, d\}| = 1$ and if $a = c$, then b and d are adjacent in G. In this paper, we discuss some properties of $U_2(G)$ related to degree and distances. We also have obtained graphs G whose double vertex graphs contain G as a subgraph.

1. Introduction

All graphs G considered here are simple, finite and connected of order $n \geq 2$ and size m. Let $V(G)$ and $E(G)$ represent the vertex set and edge set of G, respectively. Chartrand introduced the term graph-valued function for any kind of rule or procedure which yeilds a unique graph from a given graph. In literature, there are many graph-valued functions studied,the earliest being line graph of a graph. The *double vertex graph* $U_2(G)$, is one such graph-valued function, introduced by Alavi.et.al[2].

Keywords: Double vertex graphs, Degree,Distance
2010 AMS Classification: *05C07, 05C12, 05C38*
*Corresponding author:*roopa_prabhu@rediffmail.com

10.48047/jocaaa.2023.33.07.32

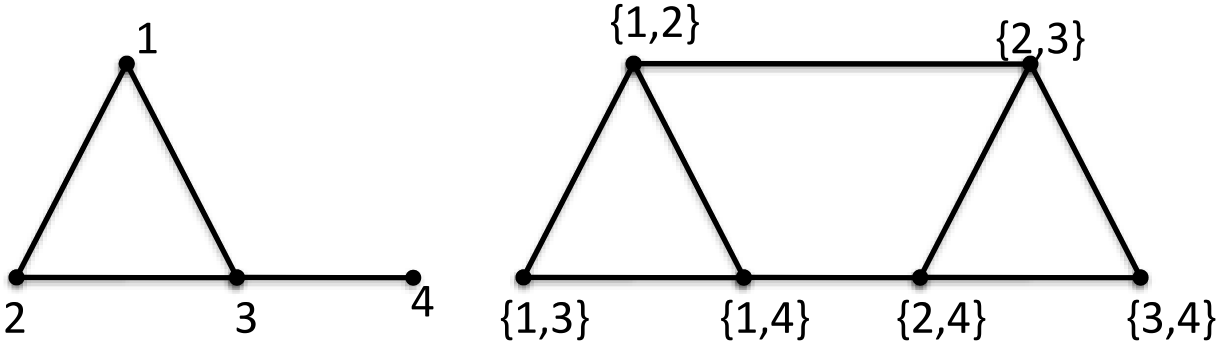

Figure 1: The graph G and its double vertex graph

For a graph G of order $n \geq 2$, $U_2(G)$ is the graph whose vertex set consists of all $\binom{n}{2}$ unordered pairs from $V(G)$ and where two vertices $\{a, b\}$ and $\{c, d\}$ are adjacent if and only if $|\{a, b\} \cap \{c, d\}| = 1$ and if $a = c$, then b and d are adjacent in G. The properties such as connectivity, planarity and traversability of double vertex graphs are well studied in [2,3,4].

In this paper, we characterize double vertex graphs that have pendant vertex, full degree vertex and isolated vertices. The double vertex graph that contains G as subgraph are investigated. Further, tight bounds for eccentricity of a vertex of $U_2(G)$ are obtained.
For undefined terminologies and notations refer to the text by G.Chartrand[1].

2. Preliminaries

A vertex $\{x, y\}$ of $U_2(G)$ is called a line pair or a non line pair according as x and y are adjacent or non adjacent in G. Then the vertex set of $U_2(G)$ can be partitioned into sets U and W where U is the set of line pairs and W the set of nonline pairs.

Theorem 2.1. $U_2(G)$ is connected if and only if G is connected.

Theorem 2.2. For any vertex $\{x, y\}$ of $U_2(G)$,

$$\deg\{x, y\} = \begin{cases} \deg x + \deg y - 2 & \text{if } xy \in E(G) \\ \deg x + \deg y & \text{otherwise} \end{cases}$$

10.48047/jocaaa.2023.33.07.32

3.Results

We begin with the following result:

Theorem3.1.$U_2(G)$*is totally disconnected if and only if* G *is totally disconnected.*

Proof: Suppose G is totally disconnected then every two vertices are non adjacent. Therefore every vertex in $U_2(G)$ is a non line pair and for a non line pair $\{x, y\}$ of $U_2(G)$, $deg\{x, y\} = 0$.

Conversely, if $U_2(G)$ is totally disconnected then G must be same as well since an edge e of G gives rise to $n - 2$ edges in $U_2(G)$. ■

Theorem 3.2. $U_2(G)$ *has a full degree vertex if and only if* $G \cong K_3, P_3$.

Proof: Suppose $U_2(G)$ has a full degree vertex. Then $U_2(G)$ is connected and by Theorem 2.1,G is connected. Let $n \geq 4$ and u, v, w, x be any four distinct vertices of G. Then the vertices $\{u, v\}$ and $\{w, x\}$ of $U_2(G)$ are non adjacent. Therefore, if G has four or more vertices then $U_2(G)$ has no full degree vertex.

However, if $n = 3$ the possible connected graphs are K_3 and P_3 and $U_2(K_3) \cong K_3$, $U_2(P_3) \cong P_3$. Both these graphs have full degree vertex.

If $n = 2, G \cong K_2$ is the only connected graph and $U_2(K_2) \cong K_1$.

Converse is obvious. ■

Lemma 3.3. *If* $H(\neq G)$*is any vertex induced subgraph of a graph* G *then for each* $w \in V(G) - V(H)$ *there corresponds a subgraph of* $U_2(G)$ *isomorphic to* H.

Proof: Consider a vertex induced subgraph $H(\neq G)$ of G and for each $w \in V(G) - V(H)$, $H_w = \{\{v, w\}: v \in V(H)\}$ is a subset of $V(U_2(G))$. Define the funtion $\phi: V(H) \longrightarrow V(H_w)$ given by $\phi(v) = \{v, w\}$. Clearly ϕ is a bijection and any two vertices $\{x, w\}$ and $\{y, w\}$ of H_w are adjacent in $U_2(G)$ if and only if x and y are adjacent in G(therefore in H). Thus $< H_w >$ of $U_2(G)$ is isomorphic to H.

10.48047/jocaaa.2023.33.07.32

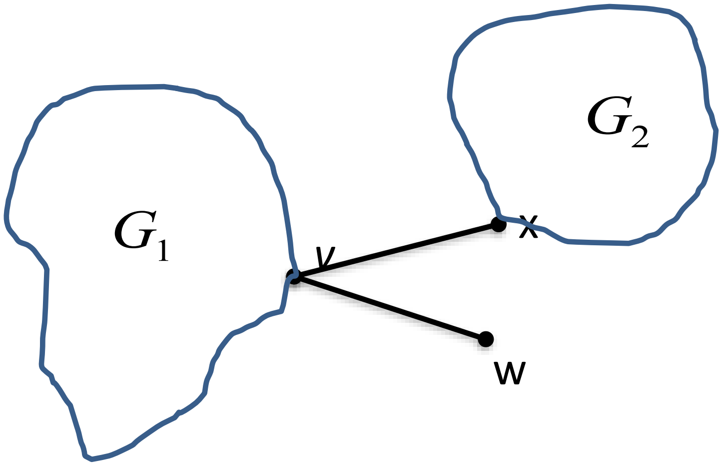

Figure 2: Illustration of Theorem 3.4

In particular, for a vertex $v \in V(G)$, the induced subgraph $G - v$ of G is isomorphic to $< H_v >$ of $U_2(G)$. ■

Theorem 3.4. *$U_2(G)$ has a subgraph isomorphic to G if one of the following condition holds:*

i) *G has a vertex of degree two*

ii) *G has a cut vertex incident with atleast two bridges one of which is a pendant edge*

Proof: Let v be a vertex of degree two and vx and vy be the edges incident at v. Then by Lemma 3.3, $< H_v >$ is isomorphic to $G - v$ and the vertex $\{x, y\}$ (not in H_v) is adjacent to both $\{v, x\}$ and $\{v, y\}$ of H_v and the subgraph induced by $H_v \cup \{\{x, y\}\}$ is isomorphic to G.

Now, assume G has no vertex of degree two. Let v be a cut vertex incident with bridges vx and vw where vw be a pendant edge. Deletion of vw and vx disconnects G giving rise to components,say, G_1, G_2 and G_3. Let G_1 be the component containing v, G_2 containing x and G_3 is a trivial component containing isolated vertex w. Now in $U_2(G)$ the subgraph G_{1x} induced by the vertices $\{\{x, v_i\}/v_i \in V(G_1)\}$ is isomorphic to G_1 and the subgraph G_{2w} induced by the vertices $\{\{w, x_i\}/x_i \in V(G_2)\}$ is isomorphic to G_2 in G. Further the vertices $\{v, x\} \in G_{1x}$ and $\{w, x\} \in G_{2w}$ are adjacent in $U_2(G)$ since vw is an edge in G. Therefore the subgraph $G_{1x} \cup G_{2w}$ of $U_2(G)$ is isomorphic to $G - w$. Since $deg x \geq 3$ there exists atleast one vertex x_k adjacent to x in G such that vertex $\{v, x\}$ of $G_{1x} \cup G_{2w}$ is adjacent to the vertex $\{v, x_k\}$ in $U_2(G)$. Therefore the subgraph formed by $G_{1x} \cup G_{2w} \cup \{\{v, x_k\}\}$ is isomorphic to G. ■

10.48047/jocaaa.2023.33.07.32

Corollary 3.5. *If $G \cong C_n$ or P_n or $K_4 - e$ then $U_2(G)$ has exactly $n-2$ edge disjoint subgraphs each isomorphic to G.*

Corollary 3.6. *For a path P_n, $U_2(P_n)$ has k vertex disjoint paths each isomorphic to P_n, $k = \left\lfloor \frac{n-1}{2} \right\rfloor$.* However, for $G \cong K_n, K_{1,n}$, $U_2(G)$ has no subgraph isomorphic to G. ■

Theorem 3.7. *$U_2(G)$ is claw free if and only if G has no*

- *i) induced subgraph isomorphic to C_4 or P_4,or*
- *ii) subgraph isomorphic to $K_{1,2} \cup K_2$,or*
- *iii) proper subgraph isomorphic to $K_{1,3}$.*

Proof: We prove the equivalent statement that $U_2(G)$ has an induced subgraph H' isomorphic to $K_{1,3}$ if and only if G has an induced subgraph H isomorphic to C_4 or P_4 or a subgraph isomorphic to $K_{1,2} \cup K_2$ and a proper subgraph isomorphic to $K_{1,3}$.

For the claw H', let $\{u, v\}$ be the central vertex. Then three cases arise due to nature of the end vertices of H'.

Case 1 Let $\{u, w\}, \{u, p\}, \{v, w\}$ be the end vertices of the claw H'. Then $\{u, w\}, \{u, p\}$ and $\{v, w\}$ induce a null graph in $U_2(G)$. So in G, v and w; v and p; u and w are adjacent whereas w and p; u and v are not adjacent. However, u and p can be either adjacent or not adjacent. If u and p are non adjacent then the vertex set $\{u, w, v, p\}$ induce a P_4 in G on the other hand $\{u, w, v, p\}$ induce a subgraph $H \cong P_4$ in G on the other hand u, w, v, p induce a C_4 if u and p are adjacent.

Case 2 If $\{u, w\}, \{u, p\}, \{v, z\}$ are the end vertices of the claw H' then $\{u, w\}, \{u, p\}, \{v, z\}$ induce a null graph in $U_2(G)$. So in G,v and w; v and p; u and z are adjacent whereas w and p are non adjacent. Then the vertex set $\{v, w, p, u, z\}$ induces a $K_2 \cup K_{1,2}$ in G.

10.48047/jocaaa.2023.33.07.32

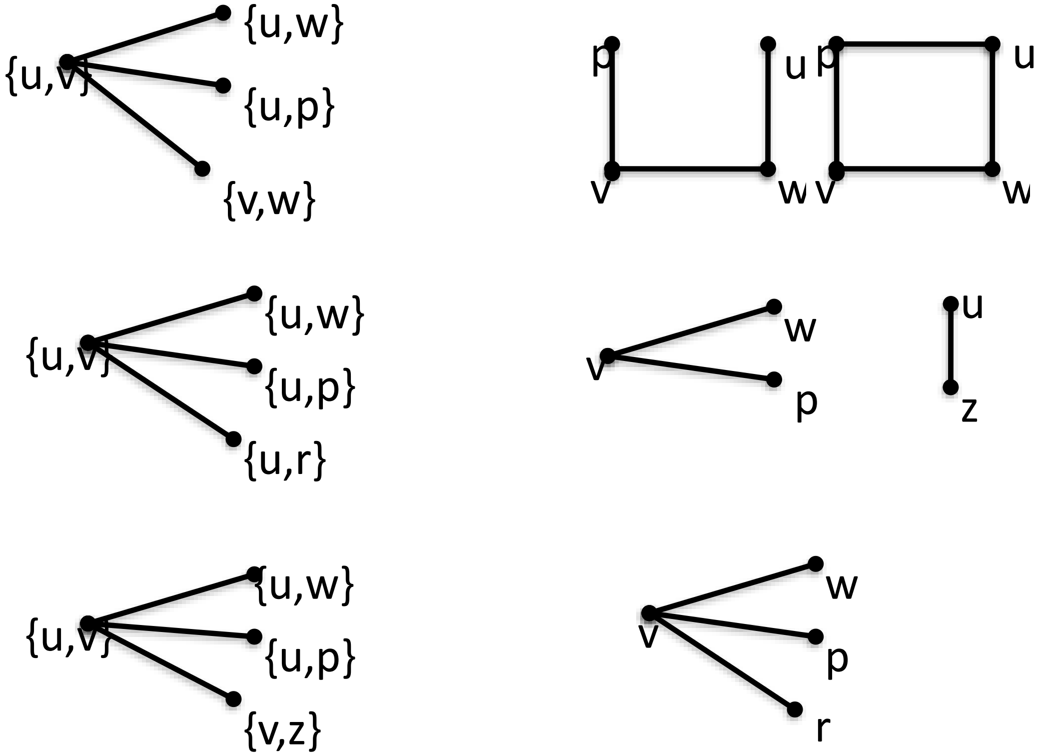

Figure 3: H' and H of Theorem 3.7

Case 3 If $\{u,w\},\{u,p\},\{u,r\}$ are the end vertices of the claw H' then $\{u,w\},\{u,p\}$,$\{u,r\}$ induce a null graph in $U_2(G)$. So in G, v and w; v and p; v and r are adjacent whereas w and p; p and r; w and r are non adjacent. Then the vertex set $\{u,v,w,p,r\}$ induce a proper subgraph $K_{1,3}$in G.

Conversly, if G has an induced C_4: x,y,z,w,x. Then in $U_2(G)$ the subgraph induced by the vertices $\{w,z\},\{y,z\},\{x,y\},\{x,z\}$ is isomorphic to $K_{1,3}$ with $\{x,z\}$ as central vertex.

Similarly, if G has an induced P_4: x,y,z,w then in $U_2(G)$ the subgraph induced by the vertices $\{x,y\},\{w,x\},\{z,y\},\{x,z\}$ is isomorphic to $K_{1,3}$ with $\{x,z\}$ as central vertex.

Further, if G has a subgraph $K_2 \cup K_{1,2}$ where u,w induces a K_2 and z,v,p induces a $K_{1,2}$. Then in $U_2(G)$, the vertices $\{u,z\},\{u,v\},\{u,p\},\{w,v\}$ induce a claw with $\{u,v\}$ as central vertex. Lastly,

10.48047/jocaaa.2023.33.07.32

if G has a proper subgraph $K_{1,3}$ with u as central vertex and x, y, z as terminal vertices. Then there exists atleast one vertex $w \neq x, y, x, u$ in G so that $\{x, w\}, \{y, w\}, \{z, w\}, \{u, w\}$ form a claw in $U_2(G)$ with $\{u, w\}$ as central vertex.

Hence the theorem. ■

Theorem 3.8. *A vertex $\{x, y\}$ of $U_2(G)$ is a pendant vertex if and only if one of the two vertices x, y is a pendant vertex and other is either a vertex of degree two or an isolated vertex.*

Proof. Suppose the pendant vertex $\{x, y\}$ is a line pair in $U_2(G)$ then $deg\{x, y\} = degx + degy - 2 = 1$ which implies $degx + degy = 3$ that is one of the vertex is pendant and other is of degree two. If $\{x, y\}$ is a non line pair then $deg\{x, y\} = degx + degy = 1$ which implies one of the vertex is isolated and other one is pendant.

Conversly, if $deg\, x = 1, deg\, y = 2$ and x, y are adjacent in G then $deg\{x, y\} = 1 + 2 - 2 = 1$ and if $deg\, x = 0, deg\, y = 1$ then $deg\{x, y\} = 1$.

Hence the theorem. ■

3.References

[1] G.Chartrand, Ping Zhang, Introduction to graph theory,TataMc.Graw Hill Edition,2006

[2]Y.Alavi, D.R.Lick,J.Liu,Survey of double vertex graphs,Graph.Combin,18(4) (2002),709-715

[3]Y.Alavi,Behzad.M,Erdos.P,Lick.D.R, Double vertex graphs,J.Comb.Inf.Sys.Sci.16,1991,37-50

[4]J.Jacob,W.Goddard, R.Laskar, Double vertex graphs and complete double vertex graphs, Cong.Numer.,188 (2007),161 -174