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A Mathematical Model on Ecology Consisting of Two Hosts-One Commensal with Mortality Rate for the Second Species

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Abstract: The present investigation is a study on three species ecology consisting of two hosts and a commensal. The system comprises of two hosts S_1, S_2 and one commensal S_3 ie, S_1 and S_2 both gain S_3, without getting themselves impressed either positively or adversely. Further S_1 and S_2 are neutral. All the three species are having finite resources quantized by the respective carrying quantities. The fundamental equations for this scheme established as three first order non-linear accompanying accustomed differential equations. All accessible critical points are recognized based on the primary model equations and criteria for their consistency are explained. If all the latent roots of the peculiar equation are either negative or zero then the model would be balanced otherwise imbalanced. Curvatures of the perturbations upon the critical points are analyzed. Further, we explain the universal consistency by appropriate Liapunov's method and the expansion rates of the species are numerically calculable victimization Runge-Kutta fourth order scheme.

Keywords: Balanced, commensal, critical point, host, imbalanced, latent root.

INTRODUCTION

Ecology relates to the study of living being in relation to their habits and habitats. This discipline of knowledge is a branch of evolutionary biology purported to explain how or to what extent the living beings are regulated in nature. Allied to the problem of population regulation is the problem of species distribution- prey-predator, competition and so on. The subject of ecology can be broadly sub-divided as auto-ecology (the study of single species populations) and syn ecology (the study of two or more communities). Syn ecological studies lead to the concept of the eco-system. This concept is a direct outcome of the intensive work of several life scientists/biologists and botanists of many generations. An eco-system may be considered as a unit that includes animals, plants and the physical environment in which these live. Significant researches in the area of theoretical ecology have been discussed by Gillman [3] and by Kot [4]. Several ecologists and mathematicians contributed to the growth of this area of knowledge. Mathematical ecology can be broadly divided into two main sub-divisions, Autecology and

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Synecology, which are described in the treatises of Anna Sher [1], Arumugam [2] and Sharma [28].

Some real-life examples of commensalism are presented below.

(i) The clownfish shelters among the tentacles of the sea anemone, while the sea anemone is not effected.

(ii) Sucker fish (echeneis) gets attached to the under surface of sharks by its sucker. This provides easy transport for new feeding grounds and also food pieces falling from the sharks prey, to Echeneis.

(iii) A flatworm attached to the horse crab and eating the crab's food, while the crab is not put to any disadvantage.

(iv) The interaction between Euklonia maxima and patella compressa. The patella gets its food from the plant while the Euklonia, is not harmed or damaged in the process.

Mathematical models have become important tools in biological investigations with an iterative procedure of information collection. If such models are properly developed and used, they can provide insight into the relations between the physical variables and process influencing the system being studied. The resulting interplay between the experimental investigation and the theoretical model can be an essential factor in designing experiments and in the interpretation of data. There are various types of mathematical modelling. Since real-life systems are complex, mathematical formulations have been developed to reproduce the experimental results irrespective of the underlying mechanisms. Such models can be extremely useful in highlighting the performance of the biological systems, albeit the components of the model are not identifiable with the components and mechanisms of the real system. However, experimental results can be reproduced in such circumstances by arbitrarily adjusting the models to explore the relation among various systems. The insight obtained from studies of such models has proved to be of immense use in complex real-life systems. Several authors Ma [6], Moghadas [7], Murray [8] and Sze-Bi Hsu [30] were introduced the general concepts of Modeling in Biological Science. Srinivas [29] studied the competitive ecosystem of two species and three species with limited and unlimited resources. Later, Narayan [9] studied prey-predator ecological models with partial cover for the prey and alternate food for the predator. Further, Kumar [5] studied some mathematical models of ecological commensalism. The present author Prasad [10-27] investigated continuous and discrete models on two, three and four species syn-ecosystems.

The present investigation is an analytical and numerical study of three species (S_1, S_2, S_3) ecology with mortality rate for the second species. The system comprises of two hosts S_1, S_2 and

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one commensal S_3 ie, S_1 and S_2 both benefit S_3, without getting themselves affected either positively or adversely. Further S_1 and S_2 are neutral.

MATHEMATICAL MODEL

Notation Appropriated

$N_i(t)$: The population strength of S_i at time t, $i = 1,2,3$

t : Time instant

d_2 : Natural death rate of S_2

a_i : Natural growth rate of S_i, $i = 1,3$

a_{ii} : Self inhibition coefficients of S_i, $i = 1,2,3$

a_{13}, a_{23} : Interaction coefficients of S_1 due to S_3 and S_2 due to S_3

$e_2 = \frac{d_2}{a_{22}}$: Extinction coefficient of S_2

$k_i = \frac{a_i}{a_{ii}}$: Carrying capacities of S_i, $i = 1,3$

Fundamental Equations

The model equations for syn ecology is given by the following system of first order non-linear ordinary differential equations.

Equation for the first host (N_1):

$$\frac{dN_1}{dt} = N_1\left(a_1 - a_{11}N_1\right) \tag{1}$$

Equation for the second host (N_2):

$$\frac{dN_2}{dt} = N_2\left(-d_2 - a_{22}N_2\right) \tag{2}$$

Equation for the commensal (N_3):

$$\frac{dN_3}{dt} = N_3\left(a_3 - a_{33}N_3 + a_{13}N_1 + a_{23}N_2\right) \tag{3}$$

CRITICAL POINTS

The system under investigation has eight critical points at $\frac{dN_i}{dt} = 0, i = 1, 2, 3$ given by

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Fully washed out state.

$$E_1 : \bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = 0$$

States in which only two of the tree species are washed out while the other one is not.

$$E_2 : \bar{N}_1 = k_1, \bar{N}_2 = 0, \bar{N}_3 = 0$$

$$E_3 : \bar{N}_1 = 0, \bar{N}_2 = -e_2, \bar{N}_3 = 0$$

$$E_4 : \bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = k_3$$

States in which only one of the tree species is washed out while the other two are not.

$$E_5 : \bar{N}_1 = k_1, \bar{N}_2 = -e_2, \bar{N}_3 = 0$$

$$E_6 : \bar{N}_1 = k_1 , \bar{N}_2 = 0, \bar{N}_3 = k_3 + \frac{a_{13}k_1}{a_{33}}$$

$$E_7 : \bar{N}_1 = 0 , \bar{N}_2 = -e_2, \bar{N}_3 = k_3 - \frac{a_{23}e_2}{a_{33}}$$

The normal steady state.

$$E_8 : \bar{N}_1 = k_1, \bar{N}_2 = -e_2, \bar{N}_3 = k_3 + \frac{a_{13}k_1 - a_{23}e_2}{a_{33}}$$

CONSISTENCY ANALYSIS

Let $N = \left(N_1, N_2, N_3\right) = \bar{N} + U$

where $U = (u_1, u_2, u_3)^T$ is very small perturbation upon the critical point $\bar{N} = \left(\bar{N}_1, \bar{N}_2, \bar{N}_3\right)$.

The fundamental equations (1), (2) and (3) are quasi-linearized to obtain the equations for the perturbed state as,

$$\frac{dU}{dt} = AU$$

where

$$A = \begin{bmatrix} a_1 - 2a_{11}\bar{N}_1 & 0 & 0 \\ 0 & -d_2 - 2a_{22}\bar{N}_2 & 0 \\ a_{13}\bar{N}_3 & a_{23}\bar{N}_3 & b_{33} \end{bmatrix}$$

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with $b_{33} = a_3 - 2a_{33}\bar{N}_3 + a_{13}\bar{N}_1 + a_{23}\bar{N}_2$

The peculiar equation for the scheme is $|A - \lambda I| = 0$

If all the latent roots are either negative or zero then critical point is balanced otherwise imbalanced.

Consistency of $E_1 : \bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = 0$

In this case, we have $A_1 = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & -d_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$

The peculiar equation is given by

$$(\lambda - a_1)(\lambda + d_2)(\lambda - a_3) = 0$$

The latent roots are $a_1, -d_2, a_3$. Since only one root is negative. Hence the point is imbalanced and the solutions are

$$u_1 = u_{10}\, e^{a_1 t};\; u_2 = u_{20}\, e^{-d_2 t};\; u_3 = u_{30}\, e^{a_3 t} \qquad (4)$$

where u_{10}, u_{20}, u_{30} are the primary values of u_1, u_2, u_3 respectively.

The trajectories in $u_1 - u_2$ and $u_2 - u_3$ planes are

$$\left(\frac{u_1}{u_{10}}\right)^{\frac{1}{a_1}} = \left(\frac{u_2}{u_{20}}\right)^{-\frac{1}{d_2}} = \left(\frac{u_3}{u_{30}}\right)^{\frac{1}{a_3}}$$

Consistency of $E_2 : \bar{N}_1 = k_1, \bar{N}_2 = 0, \bar{N}_3 = 0$

At this critical point, we have $A_2 = \begin{bmatrix} -a_1 & 0 & 0 \\ 0 & -d_2 & 0 \\ 0 & 0 & a_3 + a_{13}k_1 \end{bmatrix}$

$-a_1,\ -d_2$ and $a_3 + a_{13}k_1$ are the latent roots. Here one of these three roots is positive, hence the point E_2 is imbalanced, the solutions curves are

$$u_1 = u_{10}e^{-a_1 t};\; u_2 = u_{20}e^{-d_2 t};\; u_3 = u_{30}e^{(a_3 + a_{13}k_1)t}$$

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And the trajectories of perturbations are given by

$$\left(\frac{u_1}{u_{10}}\right)^{\frac{1}{a_1}} = \left(\frac{u_2}{u_{20}}\right)^{\frac{1}{d_2}} = \left(\frac{u_3}{u_{30}}\right)^{-\frac{1}{a_3+a_{13}k_1}}$$

Consistency of $E_3 : \bar{N}_1 = 0, \bar{N}_2 = -e_2, \bar{N}_3 = 0$

Here, the matrix A_3 is given by

$$A_3 = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & a_3 - a_{23}e_2 \end{bmatrix}$$

The latent roots are a_1, d_2 and $a_3 - a_{23}e_2$. Since both a_1 and d_2 are greater than zero, hence the point is imbalanced. The equations yield the solutions,

$$u_1 = u_{10}e^{a_1 t};\ u_2 = u_{20}e^{d_2 t};\ u_3 = u_{30}e^{(a_3 - a_{23}e_2)t}$$

The curvatures of the perturbations of the above equations are

$$\left(\frac{u_1}{u_{10}}\right)^{\frac{1}{a_1}} = \left(\frac{u_2}{u_{20}}\right)^{\frac{1}{d_2}} = \left(\frac{u_3}{u_{30}}\right)^{-\frac{1}{a_3-a_{23}e_2}}$$

Consistency of $E_4 : \bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = k_3$

At this critical point the matrix A_4 is given by $A_4 = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & -d_2 & 0 \\ a_{13}k_3 & a_{23}k_3 & -a_3 \end{bmatrix}$

a_1, $-d_2$ and $-a_3$ are the latent roots. Here, one root a_1 is greater than zero, hence the point is imbalanced and the equations yield the solutions curves,

$$u_1 = u_{10}e^{a_1 t};\ u_2 = u_{20}e^{-d_2 t};\ u_3 = \alpha u_{10}e^{a_1 t} + \beta u_{20}e^{-d_2 t} + \gamma e^{-a_3 t}$$

where $\gamma = u_{30} - \alpha u_{10} - \beta u_{20}; \alpha = \dfrac{a_{13}k_3}{a_1 + a_3} > 0;\ \beta = \dfrac{a_{23}k_3}{a_3 - d_2}$; with $a_3 \neq d_2$

and

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$$\left(\frac{u_1}{u_{10}}\right)^{-d_2}=\left(\frac{u_2}{u_{20}}\right)^{a_1}\quad u_3=\gamma\left(\frac{u_2}{u_{20}}\right)^{\frac{a_3}{d_2}}+\alpha u_{10}\left(\frac{u_2}{u_{20}}\right)^{-\frac{a_1}{d_2}}+\beta u_2$$

are the curvatures of the perturbations in the u_1-u_2 and u_2-u_3 planes.

Consistency of $E_5:\bar{N}_1=k_1,\bar{N}_2=-e_2,\bar{N}_3=0$

In this case, the matrix A_5 is given by

$$A_5=\begin{bmatrix}-a_1 & 0 & 0\\ 0 & d_2 & 0\\ 0 & 0 & a_3-a_{23}e_2+a_{13}k_1\end{bmatrix}$$

The latent roots are $-a_1$, d_2 and $a_3+a_{13}k_1-a_{23}e_2$. Since one of these three roots is positive, hence the point is imbalanced. The solutions curves are given by

$$u_1=u_{10}e^{-a_1t};u_2=u_{20}e^{d_2t};u_3=u_{30}e^{(a_3+a_{13}k_1-a_{23}e_2)t}$$

The curvatures of the perturbations of perturbed species are given by

$$\left(\frac{u_1}{u_{10}}\right)^{-\frac{1}{a_1}}=\left(\frac{u_2}{u_{20}}\right)^{\frac{1}{d_2}}=\left(\frac{u_3}{u_{30}}\right)^{\frac{1}{a_3+a_{13}k_1-a_{23}e_2}}$$

Consistency of critical point $E_6:\bar{N}_1=k_1,\ \bar{N}_2=0,\bar{N}_3=k_3+\frac{a_{13}k_1}{a_{33}}$

At this point, we have

$$A_6=\begin{bmatrix}-a_1 & 0 & 0\\ 0 & -d_2 & 0\\ a_{13}k_3+\frac{a_{13}^2k_1}{a_{33}} & a_{23}k_3+\frac{a_{13}a_{23}k_1}{a_{33}} & -(a_3+a_{13}k_1)\end{bmatrix}$$

Here, all the three roots $-a_1$, $-d_2$ and $-(a_3+a_{13}k_1)$ are negative, hence the point is balanced and the solutions curves are

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$$u_1 = u_{10}e^{-a_1 t}; u_2 = u_{20}e^{-d_2 t}; u_3 = \alpha_1 u_{10}e^{-a_1 t} + \beta_1 u_{20}e^{-d_2 t} + \alpha_{30}e^{-(a_3 + a_{13}k_1)t}$$

where $\alpha_{30} = u_{30} - \alpha_1 u_{10} - \beta_1 u_{20}; \alpha_1 = \dfrac{a_{13}(a_3 + a_{13}k_1)}{a_{33}(a_3 + a_{13}k_1 - a_1)}$; $\beta_1 = \dfrac{a_{23}(a_3 + a_{13}k_1)}{a_{33}(a_3 + a_{13}k_1 - d_2)}$

with $a_3 + a_{13}k_1 \neq a_1$; $a_3 + a_{13}k_1 \neq d_2$

The curvatures of the perturbations are given by

$$\left(\frac{u_1}{u_{10}}\right)^{d_2} = \left(\frac{u_2}{u_{20}}\right)^{a_1} \quad u_3 = \alpha_{30}\left(\frac{u_2}{u_{20}}\right)^{\frac{a_{13}k_1 + a_3}{d_2}} + \alpha_1 u_{10}\left(\frac{u_2}{u_{20}}\right)^{\frac{a_1}{d_2}} + \beta_1 u_2$$

Consistency of critical point $E_7 : \bar{N}_1 = 0$, $\bar{N}_2 = -e_2$, $\bar{N}_3 = k_3 - \dfrac{a_{23}e_2}{a_{33}}$

At this critical point the matrix A_7 is given by

$$A_7 = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & d_2 & 0 \\ a_{13}k_3 - \dfrac{a_{13}a_{23}e_2}{a_{33}} & a_{23}k_3 - \dfrac{a_{23}^2 e_2}{a_{33}} & a_{23}e_2 - a_3 \end{bmatrix}$$

The latent roots a_1 and d_2 are greater than zero, hence the point is imbalanced. The solutions are given by

$$u_1 = u_{10}e^{a_1 t}; u_2 = u_{20}e^{d_2 t}; u_3 = \alpha_2 u_{10}e^{a_1 t} + \beta_2 u_{20}e^{d_2 t} + \beta_{30}e^{(a_{23}e_2 - a_3)t}$$

where $\beta_{30} = u_{30} - \alpha_2 u_{10} - \beta_2 u_{20}; \alpha_2 = \dfrac{a_{13}(a_3 - a_{23}e_2)}{a_{33}(a_1 + a_3 - a_{23}e_2)}$; $\beta_2 = \dfrac{a_{23}(a_3 - a_{23}e_2)}{a_{33}(d_2 + a_3 - a_{23}e_2)}$

with $a_1 + a_3 \neq a_{23}e_2$; $d_2 + a_3 \neq a_{23}e_2$

and

$$\left(\frac{u_1}{u_{10}}\right)^{a_2} = \left(\frac{u_2}{u_{20}}\right)^{d_1} \quad u_3 = \beta_{30}\left(\frac{u_2}{u_{20}}\right)^{\frac{a_{23}e_2 - a_3}{d_2}} + \alpha_2 u_{10}\left(\frac{u_2}{u_{20}}\right)^{\frac{a_1}{d_2}} + \beta_2 u_2$$

are the trajectories of perturbed species.

Consistency of critical point $E_8 : \bar{N}_1 = k_1$, $\bar{N}_2 = -e_2$, $\bar{N}_3 = k_3 + \dfrac{a_{13}k_1 - a_{23}e_2}{a_{33}}$

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In this critical point, we get

$$A_8 = \begin{bmatrix} -a_1 & 0 & 0 \\ 0 & d_2 & 0 \\ \frac{a_{13}d_{33}}{a_{33}} & \frac{a_{23}d_{33}}{a_{33}} & -d_{33} \end{bmatrix}$$

where $d_{33} = a_3 + a_{13}k_1 - a_{23}e_2$

The latent roots are $-a_1$, d_2 and $-d_{33}$. Since one of these three roots is greater than zero, hence the point is imbalanced and the solutions curves are

$$u_1 = u_{10}e^{-a_1 t}; u_2 = u_{20}e^{d_2 t}; u_3 = \alpha_3 u_{10}e^{-a_1 t} + \beta_3 u_{20}e^{d_2 t} + \gamma_{30}e^{-d_{33} t}$$

where $\gamma_{30} = u_{30} - \alpha_3 u_{10} - \beta_3 u_{20}; \alpha_3 = \dfrac{a_{13}(a_3 + a_{13}k_1 - a_{23}e_2)}{a_{33}(a_{23}e_2 - a_1 - a_3 - a_{13}k_1)}; \beta_3 = \dfrac{a_{23}(a_3 + a_{13}k_1 - a_{23}e_2)}{a_{33}(d_2 + a_{23}e_2 - a_3 - a_{13}k_1)}$

with $a_{23}e_2 \neq a_1 + a_3 + a_{13}k_1;\ d_2 + a_{23}e_2 \neq a_3 + a_{13}k_1$

The trajectories of perturbed species are given by

$$\left(\frac{u_1}{u_{10}}\right)^{d_2} = \left(\frac{u_2}{u_{20}}\right)^{-a_1} \quad u_3 = \gamma_{30}\left(\frac{u_2}{u_{20}}\right)^{\frac{a_{23}e_2 - a_3 - a_{13}k_1}{d_2}} + \alpha_3 u_{10}\left(\frac{u_2}{u_{20}}\right)^{-\frac{a_1}{d_2}} + \beta_3 u_2$$

LIAPUNOV'S METHOD OF UNIVERSAL CONSISTENCY

We discussed the local consistency of all eight critical points. From which only one point $E_6\left(\bar{N}_1, 0, \bar{N}_3\right)$ is balanced and rest of them are imbalanced. We now examine the universal consistency of dynamical system (1), (2) and (3) at this state by suitable Liapunov's function.

Theorem: *The critical point E_6 is universally asymptotical balanced.*

Proof: Assuming the subsequent Liapunov's function

$$L\left(N_1, N_3\right) = N_1 - \bar{N}_1 - \bar{N}_1 \ln\left(\frac{N_1}{\bar{N}_1}\right) + l_1\left[N_3 - \bar{N}_3 - \bar{N}_3 \ln\left(\frac{N_3}{\bar{N}_3}\right)\right]$$

where l_1 is appropriate constants that have to be computed as in the following steps.

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Now, the time derivative of L, along with solutions of (1) and (3) can be written as

$$\frac{dL}{dt}=\left(\frac{N_1-\bar{N}_1}{N_1}\right)\frac{dN_1}{dt}+l_1\left(\frac{N_3-\bar{N}_3}{N_3}\right)\frac{dN_3}{dt}$$

$$\frac{dL}{dt}=-a_{11}\left(N_1-\bar{N}_1\right)^2+l_1a_{23}\left(N_1-\bar{N}_1\right)\left(N_3-\bar{N}_3\right)+l_1\left[-a_{33}\left(N_3-\bar{N}_3\right)^2\right]$$

$$\frac{dL}{dt}=-\left[\sqrt{a_{11}}\left(N_1-\bar{N}_1\right)-\sqrt{l_1a_{33}}\left(N_3-\bar{N}_3\right)\right]^2-\left(2\sqrt{l_1a_{11}a_{33}}-l_1a_{23}\right)\left(N_1-\bar{N}_1\right)\left(N_3-\bar{N}_3\right)$$

The positive constant l_1 as so chosen that, the coefficient of $\left(N_1-\bar{N}_1\right)\left(N_3-\bar{N}_3\right)$ in the above equation vanish.

Then we have $l_1=\frac{4a_{11}a_{33}}{a_{23}^2}>0$ and, with this choice of the constant l_1

$$\frac{dL}{dt}=-a_{11}\left[\left(N_1-\bar{N}_1\right)-\frac{2a_{33}}{a_{23}}\left(N_3-\bar{N}_3\right)\right]^2<0$$

Hence, the steady state is universally asymptotical balanced.

NUMERICAL APPROACH

The numerical solutions of the growth rate equations computed employing the fourth order Runge-Kutta method for specific values of the various parameters that characterize the model and the initial conditions. The results are illustrated in Figures 1, 2,3 and 4.

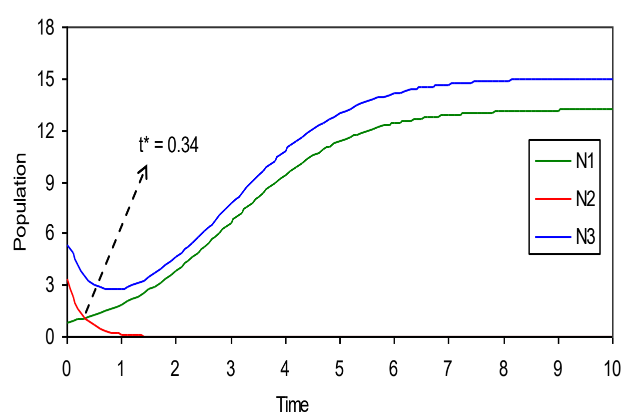

Figure 1: Variation of N_1, N_2, N_3 against time (t) for $a_1 = 0.9$, $a_{11} = 0.068$, $a_{13} = 2.96$, $d_2 = 2.65$, $a_{22} = 0.4$, $a_{23} = 3.03$, $a_3 = 1.8$, $a_{33} = 2.72$, $N_1 = 0.84$, $N_2 = 3.4$, $N_3 = 5.4$.

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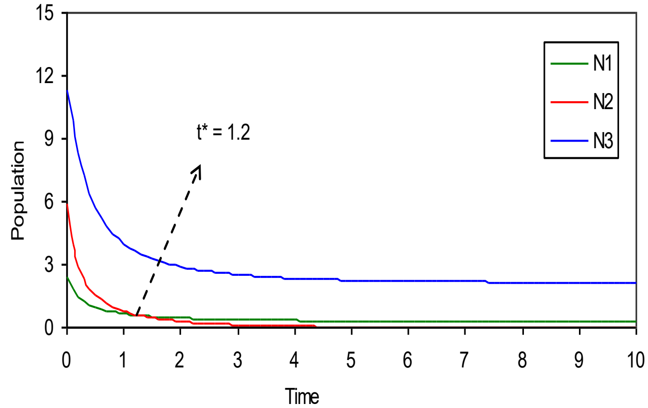

Figure 2: Variation of N_1, N_2, N_3 against time (t) for $a_1 = 0.47$, $a_{11} = 1.6$, $a_{13} = 2$, $d_2 = 0.65$, $a_{22} = 0.7$, $a_{23} = 0.54$, $a_3 = 1.3$, $a_{33} = 0.88$, $N_1 = 2.38$, $N_2 = 5.9$, $N_3 = 11.28$.

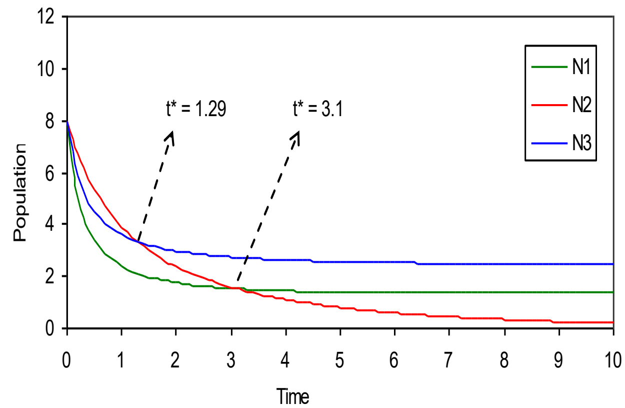

Figure 3: Variation of N_1, N_2, N_3 against time (t) for $a_1 = 0.65$, $a_{11} = 0.48$, $a_{13} = 4.08$, $d_2 = 0.24$, $a_{22} = 0.086$, $a_{23} = 0.54$, $a_3 = 8.57$, $a_{33} = 5.75$, $N_1 = N_2 = N_3 = 8$.

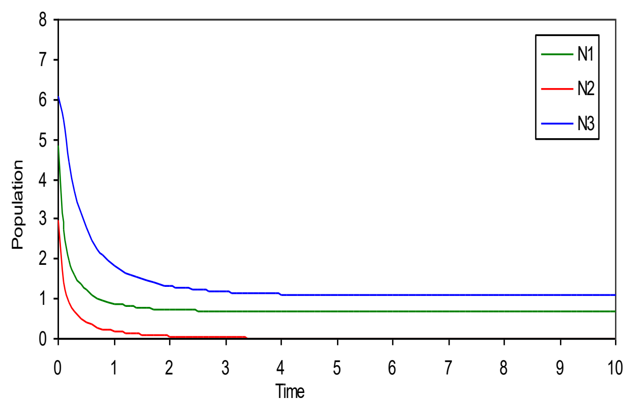

Figure 4: Variation of N_1, N_2, N_3 against time (t) for $a_1 = 1.36$, $a_{11} = 1.98$, $a_{13} = 0.7$, $d_2 = 0.66$, $a_{22} = 3.24$, $a_{23} = 1.78$, $a_3 = 0.98$, $a_{33} = 1.36$, $N_1 = 4.84$, $N_2 = 2.96$, $N_3 = 6.06$.

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Observations of the graphs

Case 1: In this case the first species has the least initial value. The S_2 dominates over the S_1 initially up to the time instant $t^* = 0.34$ after which the dominance is reversed. Further the initial conditions of the first, second and third species are in increasing order. This is illustrated in Figure 1.

Case 2: The species S_3 has the highest initial value and the second species dominates over the first initially up to the time instant $t^* = 1.2$ after which the dominance is reversed. Further it is evident that all the three species asymptotically converge to the equilibrium point as shown in Figure 2.

Case 3: The initial conditions of all the three species are identical. The first species has the least natural birth rate. Further we notice that the second species has the least self inhibition coefficient. This is shown in Figure 2.

Case 4: Here all the three species decrease initially. The second species dominates over the other two throughout. In course of time we notice a steady variation with no appreciable growth rate in all the three species. (Figure 4).

CONCLUSION

The present paper deals with an investigation on the consistency of a syn ecology consisting of two hosts and one commensal with mortality rate for the second species. In this paper we established all possible critical points. It is conclude that, in all eight critical points, only one state E_6 is balanced. Further the universal consistency is established with the help of suitable Liapunov's function and the numerical solutions are computed using Runge-Kutta fourth order method.

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