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Alternating Sums of Fourth Powers of k-Fibonacci and k-Lucas Numbers

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Abstract: The Fibonacci sequence is one of the most studied sequences, and it has been generalized in many ways. The k-Fibonacci sequence is a one-parameter generalization of the Fibonacci sequence introduced by Falcon and Plaza. R.S. Melham also investigated the alternating sums of the fourth powers of the Fibonacci and Lucas numbers. In this paper, we study the alternating sums of the fourth powers of the k-Fibonacci and k-Lucas numbers. This study is heavily reliant on the well-known Binet's Formula.

Key words: k-Fibonacci number, k-Lucas number, Binet's Formula.
2010Mathematics subject classification: 11B37, 11B39

1. Introduction:

One of the more studied sequences is the Fibonacci sequence [1–3], and it has been generalized in many ways [4–10]. The k-Fibonacci sequence is the one –parameter generalization of the Fibonacci sequences that was introduced and studied extensively by Sergio Falcon and Angel Plaza [11-18]. In [12], many properties of these numbers are deduced and related with the so-called Pascal 2-triangle. Falcon and Plaza defined k-Fibonacci hyperbolic functions and deduced some properties of k-Fibonacci hyperbolic functions related with the analogous identities for the k-Fibonacci numbers. Falcon studied the k-Lucas numbers and proved various properties related with the k-Fibonacci numbers. Finally authors studied 3-dimensional k-Fibonacci spirals with a geometric point of view in [14]. Thongmoon[20] and Hoggatt [1] defined various identities for the Fibonacci and Lucas numbers. Gupta and Panwar [23], present generalized identities involving common factors of generalized Fibonacci, Jacobsthal and Jacobsthal-Lucas numbers and Binet's formula is used to obtain the identities. About two decades ago, motivated by the results of Clary and Hemenway [21] who obtained factored closed-form expressions for sums of the form $\sum_{k=1}^{n} F_{mk}^3$, R.S. Melham [19] also investigated the alternating sums of the fourth powers of the Fibonacci and Lucas numbers. Melhem obtained factored closed-form expressions for alternating sums of the form $\sum_{k=1}^{n}(-1)^{k-1}F_{mk}^4$. In 2017 Adegoke [22] investigated factored closed-form expressions for non alternating sums. The main results found by Adegoke were:
for any integers m and n with m not equal to zero:

$$25\sum_{k=1}^{n} F_{mk}^4 = \frac{F_{2mn+m}(L_{2mn+m} + 4(-1)^{mn-1}L_m)}{F_{2m}} + 6n + 3$$

and

$$\sum_{k=1}^{n} L_{mk}^4 = \frac{F_{2mn+m}(L_{2mn+m} + 4(-1)^{mn}L_m)}{F_{2m}} + 6n - 5$$

He also re-derived the alternating sums, in slightly different but equivalent forms to the results obtained by Melhem.

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$$\sum_{k=1}^{n}(-1)^{k-1}F_{mk}^{4}=\frac{F_{mn}F_{mn+m}\left\{\left\{(-1)^{n-1}L_m L_{mn}L_{mn+m}+(-1)^{n(m-1)}4L_{2m}\right\}\right\}}{5L_m L_{2m}}$$

and

$$\sum_{k=\left(1+(-1)^{n}\right)/2}^{n}(-1)^{k-1}L_{mk}^{4}=\frac{(-1)^{n-1}5F_{mn}F_{mn+m}\left\{L_m L_{mn}L_{mn+m}+(-1)^{mn}4L_{2m}\right\}}{L_m L_{2m}}$$

valid for all integers m and n with the help of following summation identities:

Lemma 1.1: If $f(k)$ is a real sequence and m and n are positive integers, then $\sum_{k=1}^{n}\left[f(mk+m)-f(mk)\right]=f(mn+m)-f(m)$

Lemma 1.2: If $f(k)$ is a real sequence and m and n are positive integers, then $\sum_{k=1}^{n}(-1)^{k-1}\left[f(mk+m)+f(mk)\right]=(-1)^{n-1}f(mn+m)+f(m)$

Lemma 1.3: If m and n are integers, then $F_m\sum_{k=1}^{n}(-1)^{mk-1}L_{2mk}=(-1)^{mn-1}F_{mn}L_{mn+m}$

Lemma 1.4: If m and n are integers, then $L_m\sum_{k=1}^{n}(-1)^{k(m-1)}L_{2mk}=(-1)^{n(m-1)}L_{2mn+m}-L_m$

Motivated by the results of Melham we study the alternating sums of the fourth powers of the k-Fibonacci and k-Lucas numbers. This work is heavily reliant on the well-known Binet's Formula. Here, we use the generalization introduced by Falcon and Plaza.

Definition 1.5 (Falcon and Plaza): For any positive real number k-Fibonacci sequence $\left\{F_{k,n}\right\}_{n\in N}$ is defined recurrently by $F_{k,0}=0, F_{k,1}=1$ and $F_{k,n+1}=kF_{k,n}+F_{k,n-1}$ for $n\geq 1$.

The k-Lucas sequence is defined by $L_{k,n+1}=kL_{k,n}+L_{k,n-1}$ with $L_{k,0}=2, L_{k,1}=k$ for $n\geq 1$. Note that for $k=1$ the classical Fibonacci sequence is obtained while for $k=2$ we obtain the Pell's sequence. Here we are using only Binet's formula to find the required sums.

Binet's Formula 1.6: The n^{th} k-Fibonacci and k-Lucas number is given by $F_{k,n}=\frac{r_1^n-r_2^n}{r_1-r_2}$ and $L_{k,n}=r_1^n+r_2^n$ where r_1 *and* r_2 are the roots of the characteristic equation $r^2=kr+1$ and $r_1>r_2$. Evidently

$$r_1+r_2=k,\ r_1r_2=-1,\ r_1-r_2=\sqrt{k^2+4},\ r_1^2-1=kr_1,\ r_2^2-1=kr_2$$

2. Preliminary Results:

We require the following results, which can be proved using Binet's formula for $F_{k,n}$ and $L_{k,n}$

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$$F_{k,n+m}+F_{k,n-m}=F_{k,n}L_{k,m} \qquad m \text{ even} \tag{2.1}$$
$$F_{k,n+m}+F_{k,n-m}=L_{k,n}F_{k,m} \qquad m \text{ odd} \tag{2.2}$$
$$F_{k,n+m}-F_{k,n-m}=F_{k,n}L_{k,m} \qquad m \text{ odd} \tag{2.3}$$
$$F_{k,n+m}-F_{k,n-m}=L_{k,n}F_{k,m} \qquad m \text{ even} \tag{2.4}$$
$$L_{k,n+m}+L_{k,n-m}=L_{k,n}L_{k,m} \qquad m \text{ even} \tag{2.5}$$
$$L_{k,n+m}+L_{k,n-m}=(k^2+4)F_{k,n}F_{k,m} \qquad m \text{ odd} \tag{2.6}$$
$$L_{k,n+m}-L_{k,n-m}=L_{k,n}L_{k,m} \qquad m \text{ odd} \tag{2.7}$$
$$L_{k,n+m}-L_{k,n-m}=(k^2+4)F_{k,n}F_{k,m} \qquad m \text{ even} \tag{2.8}$$
$$L_{k,2m}-2=L_{k,m}^2 \qquad m \text{ odd} \tag{2.9}$$
$$L_{k,2m}+2=L_{k,m}^2 \qquad m \text{ even} \tag{2.10}$$
$$L_{k,2m}+(-1)^{m+1}2=(k^2+4)F_{k,m}^2 \tag{2.11}$$

Throughout this paper $m \neq 0$ is an integer. Four sums involving Lucas numbers with even subscripts are also used to help in our proofs. If m is odd, we have

$$\sum_{i=1}^{n} L_{k,2mi}=\begin{cases} \dfrac{(k^2+4)F_{k,mn}F_{k,m(n+1)}}{L_{k,m}} & n \text{ even} \\ \dfrac{L_{k,mn}L_{k,m(n+1)}}{L_{k,m}} & n \text{ odd} \end{cases} \tag{2.12}$$

And

$$\sum_{i=0}^{n} L_{k,2mi}=\begin{cases} \dfrac{L_{k,mn}L_{k,m(n+1)}}{L_{k,m}} & n \text{ even} \\ \dfrac{(k^2+4)F_{k,mn}F_{k,m(n+1)}}{L_{k,m}} & n \text{ odd} \end{cases} \tag{2.13}$$

On the right sides of (2.12) and (2.13), the even and odd cases are reversed. If m is even, then also we have found the following similar surprising results.

$$\sum_{i=1}^{n} (-1)^i L_{k,2mi}=\begin{cases} \dfrac{(k^2+4)F_{k,mn}F_{k,m(n+1)}}{L_{k,m}}, & n \text{ even} \\ -\dfrac{L_{k,mn}L_{k,m(n+1)}}{L_{k,m}} & n \text{ odd} \end{cases} \tag{2.14}$$

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And

$$\sum_{i=0}^{n}(-1)^i L_{k,2mi}=\begin{cases}\dfrac{L_{k,mn}L_{k,m(n+1)}}{L_{k,m}} & n \ \text{even}\\ -\dfrac{(k^2+4)F_{k,mn}F_{k,m(n+1)}}{L_{k,m}} & n \ \text{odd}\end{cases} \qquad (2.15)$$

The proofs of (2.12)—(2.15) are similar. We illustrate the method by proving (2.13).
Proof of (2.13): Since $L_{k,2mi}=r_1^{2mi}+r_2^{2mi}$ and summing the resulting geometric progressions, we obtain

$$\begin{aligned}\sum_{i=0}^{n}L_{k,2mi}&=\frac{r_1^{2mn+2m}-1}{r_1^{2m}-1}+\frac{r_2^{2mn+2m}-1}{r_2^{2m}-1}\\&=\frac{L_{k,2mn+2m}-L_{k,2mn}+L_{k,2m}-2}{L_{k,2m}-2}\\&=\frac{L_{k,(2mn+m)+m}-L_{k,(2mn+m)-m}+L_{k,m}^2}{L_{k,m}^2}\quad [by\,(2.9)]\\&=\frac{L_{k,2mn+m}L_{k,m}+L_{k,m}^2}{L_{k,m}^2}\quad [by\,(2.7)]\\&=\frac{L_{k,(mn+m)+mn}+L_{k,(mn+m)-mn}}{L_{k,m}}\end{aligned}$$

Since m is odd, so when n is even then mn is even and when n is odd then mn is odd

Thus (2.13) follows from (2.5) and (2.6).

3. The Main Results:

We now present our main results. If m is even, then

$$\sum_{i=1}^{n}(-1)^i F_{k,mi}^4=\frac{(-1)^n F_{k,mn}F_{k,m(n+1)}[L_{k,m}L_{k,mn}L_{k,m(n+1)}-4L_{k,2m}]}{(k^2+4)L_{k,m}L_{k,2m}} \qquad (3.1)$$

$$\sum_{i=1}^{n}(-1)^i L_{k,mi}^4=\frac{(k^2+4)F_{k,mn}F_{k,m(n+1)}[L_{k,m}L_{k,mn}L_{k,m(n+1)}+4L_{k,2m}]}{L_{k,m}L_{k,2m}}, \qquad n \ \text{even} \quad (3.2)$$

$$\sum_{i=0}^{n}(-1)^i L_{k,mi}^4=-\frac{(k^2+4)F_{k,mn}F_{k,m(n+1)}[L_{k,m}L_{k,mn}L_{k,m(n+1)}+4L_{k,2m}]}{L_{k,m}L_{k,2m}}, \qquad n \ \text{odd} \quad (3.3)$$

We mention that (3.2) and (3.3) can be combined in a single sum as

$$\sum_{i=1}^{n}(-1)^i F_{k,mi}^4=\frac{(-1)^n(k^2+4)F_{k,mn}F_{k,m(n+1)}[L_{k,m}L_{k,mn}L_{k,m(n+1)}+4L_{k,2m}]}{L_{k,m}L_{k,2m}}-8(1+(-1)^{n+1})$$

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On the other hand, if m is odd, then

$$\sum_{i=1}^{n}(-1)^i F_{k,mi}^4 = \frac{(-1)^n F_{k,mn}F_{k,m(n+1)}[L_{k,m}L_{k,mn}L_{k,m(n+1)} + 4(-1)^{n+1}L_{k,2m}]}{(k^2+4)L_{k,m}L_{k,2m}} \tag{3.4}$$

$$\sum_{i=1}^{n}(-1)^i L_{k,mi}^4 = \frac{(k^2+4)F_{k,mn}F_{k,m(n+1)}[L_{k,m}L_{k,mn}L_{k,m(n+1)} + 4L_{k,2m}]}{L_{k,m}L_{k,2m}}, \quad n \text{ even} \tag{3.5}$$

$$\sum_{i=0}^{n}(-1)^i L_{k,mi}^4 = -\frac{(k^2+4)F_{k,mn}F_{k,m(n+1)}[L_{k,m}L_{k,mn}L_{k,m(n+1)} - 4L_{k,2m}]}{L_{k,m}L_{k,2m}}, \quad n \text{ odd} \tag{3.6}$$

As before, (3.5) and (3.6) can be expressed as a single sum, but we choose to write them separately in order to present the right sides in factored form. This is the reason for the appearance of the zero lower limit.

4. The method of Proof:

To illustrate the method, we prove (3.4).

First, let n be even. Since m is odd and $r_1 r_2 = -1$, then $(r_1 r_2)^{mi} = (-1)^i$. Now

$$\begin{aligned}
\sum_{i=1}^{n}(-1)^i F_{k,mi}^4 &= \frac{1}{(r_1-r_2)^4}\sum_{i=1}^{n}(-1)^i (r_1^{mi} - r_2^{mi})^4 \\
&= \frac{1}{(r_1-r_2)^4}\sum_{i=1}^{n}(-1)^i (L_{k,4mi} - 4(-1)^i L_{k,2mi} + 6) \\
&= \frac{1}{(r_1-r_2)^4}\sum_{i=1}^{n}\{(-1)^i L_{k,4mi} - 4L_{k,2mi}\} + 6\frac{1}{(r_1-r_2)^4}\sum_{i=1}^{n}(-1)^i \\
&= \frac{1}{(r_1-r_2)^4}\sum_{i=1}^{n}\{(-1)^i L_{k,4mi} - 4L_{k,2mi}\} + 0 \quad \text{since } n \text{ is even.} \\
&= \frac{1}{(k^2+4)^2}\sum_{i=1}^{n}\{(-1)^i L_{k,4mi} - 4L_{k,2mi}\}
\end{aligned}$$

With the use of (2.12) and (2.14), this becomes

$$\begin{aligned}
&\frac{1}{(k^2+4)^2}\left[\frac{(k^2+4)F_{k,2mn}F_{k,2m(n+1)}}{L_{k,2m}} - \frac{4(k^2+4)F_{k,mn}F_{k,m(n+1)}}{L_{k,m}}\right] \\
&= \frac{1}{(k^2+4)}\left[\frac{F_{k,mn}L_{k,mn}F_{k,m(n+1)}L_{k,m(n+1)}}{L_{k,2m}} - \frac{4F_{k,mn}F_{k,m(n+1)}}{L_{k,m}}\right] \\
&= \left[\frac{F_{k,mn}F_{k,m(n+1)}\left[L_{k,m}L_{k,mn}L_{k,m(n+1)} - 4L_{k,2m}\right]}{(k^2+4)L_{k,m}L_{k,2m}}\right]
\end{aligned}$$

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If n is odd, then we have

$$\sum_{i=1}^{n}(-1)^i F_{k,mi}^4 = \sum_{i=0}^{n}(-1)^i F_{k,mi}^4 \ (\text{since } F_{k,0}=0)$$

$$= \frac{1}{(k^2+4)^2}\sum_{i=0}^{n}((-1)^i L_{k,4mi} - 4L_{k,2mi} + 6(-1)^i)$$

$$= \frac{1}{(k^2+4)^2}\sum_{i=0}^{n}\{(-1)^i L_{k,4mi} - 4L_{k,2mi}\} \text{ since } n \text{ is odd}$$

With the aid of (2.13) and (2.15), this sum becomes

$$\frac{1}{(k^2+4)^2}\left[\frac{-(k^2+4)F_{k,2mn}F_{k,2m(n+1)}}{L_{k,2m}} - \frac{4(k^2+4)F_{k,mn}F_{k,m(n+1)}}{L_{k,m}}\right]$$

$$= -\frac{1}{(k^2+4)}\left[\frac{F_{k,mn}L_{k,mn}F_{k,m(n+1)}L_{k,m(n+1)}}{L_{k,2m}} + \frac{4F_{k,mn}F_{k,m(n+1)}}{L_{k,m}}\right]$$

$$= -\left[\frac{F_{k,mn}F_{k,m(n+1)}\left[L_{k,m}L_{k,mn}L_{k,m(n+1)} + 4L_{k,2m}\right]}{(k^2+4)L_{k,m}L_{k,2m}}\right]$$

This completes the proof.

We remark that the proof of (3.1) is similar since the parities of n must be considered separately, but the proofs of the results in section 3 are more straightforward.

Conclusion:

We can mention here two more results which are in similar fashion.

If m is odd, then

$$\sum_{i=1}^{n}(-1)^i L_{k,2mi} = \frac{(-1)^n F_{k,mn}L_{k,m(n+1)}}{F_{k,m}} \tag{5.1}$$

and

$$\sum_{i=0}^{n}(-1)^i L_{k,2mi} = \frac{(-1)^n L_{k,mn}F_{k,m(n+1)}}{F_{k,m}} \tag{5.2}$$

If m is even, then

$$\sum_{i=1}^{n} L_{k,2mi} = \frac{F_{k,mn}L_{k,m(n+1)}}{F_{k,m}} \tag{5.3}$$

and

$$\sum_{i=0}^{n} L_{k,2mi} = \frac{L_{k,mn}F_{k,m(n+1)}}{F_{k,m}} \tag{5.4}$$

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