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ON LOWER AND UPPER SEMI-CLIQUISH FUNCTIONS

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Abstract: This research article introduces lower and upper semi-cliquish functions on topological spaces. And investigates the relations between semi-continuous functions and semi-cliquish functions. Also, it is established that the sets of all real valued bounded lower and upper semi-cliquish functions on a topological space form commutative Banach algebras with identity under the supremum norm.

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Introduction:

The concept of *cliquishness* was first introduced by Lipinski, Gdansk W., and Salát T. In their research article [3] defined cliquish functions from a topological space into a metric space at a point. The authors also investigated the structure of the set of all points of cliquishness of such functions. In [5], the author has established that the set of all real valued bounded cliquish functions defined on a topological space forms a commutative Banach algebra with identity under supremum norm.

In what follows X and Y stand for a topological space and metric space respectively. We denote the real line with usual metric by the symbol $\mathbb{R}$.

1. Preliminaries

In this section we present a few definitions that are necessary for further study of this paper. Let d be a metric on Y and T_X be a topology on X.

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Definition 1.1: Let X be a topological space together with a topology T_X and let Y be a metric space with metric d. A function $f:(X,T_X)\to(Y,d)$ is said to be *cliquish* at a point $x_0\in X$ if for each $\varepsilon>0$ and each neighborhood $G(x_0)$ of the point $x_0\in X$ there exists a non-empty open set $U_0\subset G(x_0)$, such that

$$d\left(f(x),f(y)\right)<\varepsilon \quad \text{for each pair of points } x,y\in U_0.$$

Definition 1.2: Let X be a topological space together with a topology T_X and let Y be a metric space with metric d. A function $f:(X,T_X)\to(Y,d)$ is said to be *semi continuous* at a point $x_0\in X$ if for each $\varepsilon>0$ and each neighborhood $G(x_0)$ of the point $x_0\in X$ there exists a non-empty open set $U_0\subset G(x_0)$, such that

$$d\left(f(x),f(y)\right)<\varepsilon \quad \text{for each } x\in U_0.$$

Theorem 1.3: If $f:X\to Y$ is semi continuous at $x_0\in X$ then $f:X\to Y$ is cliquish at $x_0\in X$.

Remark 1.4: Every semi continuous function $f:X\to Y$ is Cliquish but a Cliquish function $f:X\to Y$ is not necessarily semi continuous. This fact is evident from the following example.

Example 1.5: Define $f:[-1,1]\to\mathbb{R}$ by $f(x)=\begin{cases}\frac{1}{x} & if \quad 0<x\le 1\\ 0 & if \quad -1\le x\le 0\end{cases}$

Then f is cliquish at $x_0=0$. This function $f:[-1,1]\to\mathbb{R}$ is semi continuous but not continuous $x_0=0$.

Example 1.6: Define $g:[-1,1]\to\mathbb{R}$ by $g(x)=\begin{cases}\frac{1}{x} & if \quad -1\le x<0\\ 0 & if \quad 0\le x\le 1\end{cases}$

Then g is cliquish at $x_0=0$.

Example 1.7: Define $h:[-1,1]\to\mathbb{R}$ by $h(x)=\begin{cases}1 & if \quad x \text{ is } rational\\ -1 & if \quad x \text{ is } irrational\end{cases}$

Then h is neither semi continuous nor cliquish at $x_0=0$.

Example 1.8: Define $f:[-1,1]\to\mathbb{R}$ by $f(x)=\begin{cases}\frac{1}{x} & if \quad 0<x\le 1\\ 2 & if \quad -1\le x\le 0\end{cases}$

Then f is semi continuous at every point of $[-1,1]$.

Example 1.9: Define $g:[-1,1]\to\mathbb{R}$ by $g(x)=\begin{cases}\frac{1}{x} & if \quad 0<x\le 1\\ 2 & if \quad x=0\\ 0 & if \quad -1\le x<0\end{cases}$

Put $x_0=0$. Let $\varepsilon>0$ be given

Suppose that $G(x_0)=(-\delta,\delta)$, for any such that $G(x_0)\subset[-1,1]$.

Let $U_0=\left(-\frac{\delta}{2},0\right)$. Then U_0 a non empty open set in $[-1,1]$ such that $U_0\subset G(x_0)$.

For $x,y\in U_0$, we have $|g(x)-g(y)|=0<\varepsilon$

Thus for every $\varepsilon>0$ and for each neighborhood $G(x_0)$ of $x_0=0$ there is a non empty open set $U_0\subset G(x_0)$ such that

$$|g(x)-g(y)|<\varepsilon \quad \text{for } x,y\in U_0$$

$\Rightarrow$ g is cliquish at $x_0=0$.

But for any non empty open set $U_0\subset(-\delta,0)\subset G(x_0)$, we have

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$$|g(x)-g(x_0)|=|g(x)-g(0)|=|0-2|=2$$

For $x \in U_0$.

Similarly for any non-empty open set $U_0 \subset (0,\delta) \subset G(x_0)$ we have

$$|g(x)-g(x_0)|=\left|\frac{1}{x}-2\right| \quad \text{for } x \in U_0.$$

In any case, $|g(x)-g(x_0)|>\varepsilon$

$\Rightarrow$ g is not semi continuous function at $x_0=0$.

Hence g is cliquish but not semi continuous at $x_0=0$. ■

Remarks 1.10:

1. If $f: X \to Y$ and $g: X \to Y$ are semi-continuous at $x_0 \in X$ then $f+g$ is also semi-continuous at $x_0 \in X$.

2. If $f: X \to Y$ and $g: X \to Y$ are Cliquish at every point of X then $f+g$ is cliquish at every point of X.

3. In **[5],** It is proved that if $\{f_n\}$ is a sequence of real valued cliquish functions on X and if $f_n \to f$ uniformly on X. Then f is also cliquish on X. It is also established that the set $K(X)$ of all real valued bounded cliquish function defined on X forms a real commutative Banach algebra with identity under supremum norm.

2 Semi-Cliquishness & Half Semi-Continuity

In this section, we introduce the concepts of lower and upper semi-cliquishness, half semi-continuity and establish a few noteworthy results pertaining to these notions.

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Definition 2.1: Let X be a topological space together with a topology T_X. A function $f : X \to \mathbb{R}$ is said to be *lower semi-cliquish* at a point $x_0 \in X$ if for each $\varepsilon > 0$ and each neighborhood $G(x_0)$ of the point $x_0 \in X$ there exists a non-empty open set $U_0 \subset G(x_0)$, such that

$$f(x) - f(y) > -\varepsilon \quad \text{for each pair of points } x, y \in U_0 .$$

Definition 2.2: Let X be a topological space together with a topology T_X. A function $f : X \to \mathbb{R}$ is said to be *upper semi-cliquish* at a point $x_0 \in X$ if for each $\varepsilon > 0$ and each neighborhood $G(x_0)$ of the point $x_0 \in X$ there exists a non-empty open set $U_0 \subset G(x_0)$, such that

$$f(x) - f(y) < \varepsilon \quad \text{for each pair of points } x, y \in U_0 .$$

Definition 2.3: Let X be a topological space together with a topology T_X. A function $f : X \to \mathbb{R}$ is said to be *lower half semi-continuous* at a point $x_0 \in X$ if for each $\varepsilon > 0$ and each neighborhood $G(x_0)$ of the point $x_0 \in X$ there exists a non-empty open set $U_0 \subset G(x_0)$, such that

$$f(x) - f(x_0) > -\varepsilon \quad \text{for each } x \in U_0 .$$

Definition 2.4: Let X be a topological space together with a topology T_X. A function $f : X \to \mathbb{R}$ is said to be *upper half semi-continuous* at a point $x_0 \in X$ if for each $\varepsilon > 0$ and each neighborhood $G(x_0)$ of the point $x_0 \in X$ there exists a non-empty open set $U_0 \subset G(x_0)$, such that

$$f(x) - f(x_0) < \varepsilon \quad \text{for each } x \in U_0 .$$

Proposition 2.5: $f : X \to \mathbb{R}$ is lower semi-cliquish on X $\Leftrightarrow$ $-f : X \to \mathbb{R}$ is upper semi-cliquish on X.

Proof: Suppose that $f : X \to \mathbb{R}$ is lower semi-cliquish on X and let $x_0 \in X$.

$\Leftrightarrow$ given $\varepsilon > 0$, and for each neighborhood $G(x_0)$ of $x_0 \in X$. Then there exists a non-empty open set $U_0 \subset G(x_0)$ such that

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$$f(x)-f(y)>-\varepsilon \text{ for each pair of points } x,y\in U_0 .$$

$\Leftrightarrow$ $(-f)(x)-(-f)(y)<\varepsilon$ for each pair of points $x,y\in U_0$.

$\Leftrightarrow$ $(-f)$ is upper semi-cliquish on X. ■

The proofs of the following propositions 2.6, 2.7 and 2.8 are easy to verify and hence can be omitted.

Proposition 2.6: $f:X\to\mathbb{R}$ is lower half semi-continuous on X $\Leftrightarrow$ $-f:X\to\mathbb{R}$ is upper half semi-continuous on X.

Proposition 2.7: $f:X\to\mathbb{R}$ is both lower and upper semi-cliquish on X $\Leftrightarrow$ f is cliquish on X.

Proposition 2.8: $f:X\to\mathbb{R}$ is both lower and upper half semi-continuous on X $\Leftrightarrow$ f is semi continuous on X.

Example 2.9: Define $f:[-1,1]\to\mathbb{R}$ by $f(t)=\begin{cases}0 & if \quad x=0\\ 2 & if \quad x\, is\ rational\ and\ x\neq 0\\ 1 & if \quad x\ is\ irrational\end{cases}$

This function $f:[-1,1]\to\mathbb{R}$ is lower half semi-continuous at $x_0=0$. But, this function is neither lower semi-cliquish at $x_0=0$ nor upper half semi-continuous at $x_0=0$. This example shows that a lower half semi-continuous function at a point x_0 is not necessarily lower semi-cliquish at $x_0=0$. This function $f:[-1,1]\to\mathbb{R}$ is neither cliquish nor a upper semi cliquish function at $x_0=0$.

3 Algebraic Properties Semi-Cliquish Functions

This section is devoted to a few algebraic properties of lower and upper semi-cliquish functions. In this section, we establish that $f+g$, $f\vee g$, $f\wedge g$ are lower (upper) semi-cliquish

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provided f and g are lower (upper) semi-cliquish on X. We also prove that cf is lower (upper) semi-cliquish on X when f lower (upper) semi-cliquish on X, where c is any positive real number.

Proposition 3.1: If $f: X \to \mathbb{R}$ and $g: X \to \mathbb{R}$ are lower semi-cliquish on X. Then $f+g$ is lower semi-cliquish on X.

Proof: Let $x_0 \in X$ and let $\varepsilon > 0$ be given. Let $G(x_0)$ be a neighborhood of x_0 in X. Then there exists a non-empty open set $U_0 \subset G(x_0)$ such that

$$f(x) - f(y) > -\frac{\varepsilon}{2} \text{ for each pair of points } x, y \in U_0.$$

Let $x_1 \in U_0$. Since $g: X \to \mathbb{R}$ is lower semi-cliquish at $x_1 \in U_0$, then there exists a non-empty open set $V_0 \subset U_0$ such that

$$g(x) - g(y) > -\frac{\varepsilon}{2} \text{ for each pair of points } x, y \in V_0.$$

Now, $x, y \in V_0$ we have

$$(f+g)(x) - (f+g)(y) = \big(f(x) - f(y)\big) + \big(g(x) - g(y)\big) > -\frac{\varepsilon}{2} - \frac{\varepsilon}{2} = -\varepsilon$$

$(f+g)(x) - (f+g)(y) > -\varepsilon$ for each pair of points $x, y \in V_0$

$\Rightarrow (f+g)$ is lower semi-cliquish on X. ■

Proposition 3.2: If c is a positive real number, and if $f: X \to \mathbb{R}$ is a lower semi-cliquish function at $x_0 \in X$. Then cf is lower semi-cliquish at $x_0 \in X$.

Proof: Suppose $c > 0$. Let $\varepsilon > 0$ be given and let $G(x_0)$ be a neighborhood of x_0 in X. Then there exists a non-empty open set $U_0 \subset G(x_0)$ such that

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$$\left(f(x)-f(y)\right)>-\frac{\varepsilon}{c} \text{ for each pair of points } x,y\in U_0$$

$$\Rightarrow \quad (cf)(x)-(cf)(y)>-\varepsilon \text{ for each pair of points } x,y\in U_0.$$

Hence cf is lower semi-cliquish at $x_0 \in X$. ■

Proposition 3.3: If $f: X \to \mathbb{R}$ and $g: X \to \mathbb{R}$ are lower semi-cliquish functions on X. Then $f \vee g$ is also lower semi-cliquish on X.

Proof: Suppose $f: X \to \mathbb{R}$ and $g: X \to \mathbb{R}$ are lower semi-cliquish functions on X. Let $\varepsilon > 0$ be given and $x_0 \in X$. Then for each neighborhood $G(x_0)$ of x_0 in X, there exists a non-empty open set $U_0 \subset G(x_0)$ such that

$$f(x)-f(y)>-\frac{\varepsilon}{2} \text{ for each pair of points } x,y\in U_0.$$

$$\Rightarrow \quad f(x)>-\frac{\varepsilon}{2}+f(y) \text{ for each pair of points } x,y\in U_0.$$

Let $x_1 \in U_0$. Since $g: X \to \mathbb{R}$ is lower semi-cliquish at $x_1 \in U_0$, then there exists a non-empty open set $V_0 \subset U_0$ such that

$$g(x)-g(y)>-\frac{\varepsilon}{2} \text{ for each pair of points } x,y\in V_0.$$

$$\Rightarrow \quad g(x)>-\frac{\varepsilon}{2}+g(y) \text{ for each pair of points } x,y\in V_0.$$

Let $x, y \in V_0$ then we have $f(x)>-\frac{\varepsilon}{2}+f(y)$ and $g(x)>-\frac{\varepsilon}{2}+g(y)$

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$\Rightarrow\ (f\vee g)(x)\geq f(x)>-\frac{\varepsilon}{2}+f(y)$ and $(f\vee g)(x)\geq g(x)>-\frac{\varepsilon}{2}+g(y)$

$\Rightarrow\ (f\vee g)(x)+\frac{\varepsilon}{2}>f(y)$ and $(f\vee g)(x)+\frac{\varepsilon}{2}>g(y)$

$\Rightarrow\ (f\vee g)(x)+\frac{\varepsilon}{2}\geq(f\vee g)(y)$

$\Rightarrow\ (f\vee g)(x)+\varepsilon>(f\vee g)(x)+\frac{\varepsilon}{2}\geq(f\vee g)(y)$

$\Rightarrow\ (f\vee g)(x)+\varepsilon>(f\vee g)(y)$

$\Rightarrow\ (f\vee g)(x)-(f\vee g)(y)>-\varepsilon$

$\Rightarrow$ $f\vee g$ is lower semi-cliquish on X.

The following propositions from 3.4 to 3.8 are easy to prove. And hence the proofs are omitted.

Proposition 3.4: If $f:X\to\mathbb{R}$ and $g:X\to\mathbb{R}$ are lower semi-cliquish functions on X. Then $f\wedge g$ is also lower semi-cliquish on X.

Proposition 3.5: If $f:X\to\mathbb{R}$ and $g:X\to\mathbb{R}$ are upper semi-cliquish on X. Then $f+g$ is upper semi-cliquish on X.

Proposition 3.6: If c is a positive real number, and if $f:X\to\mathbb{R}$ is a upper semi-cliquish function at $x_0\in X$. Then cf is upper semi-cliquish at $x_0\in X$.

Proposition 3.7: If $f:X\to\mathbb{R}$ and $g:X\to\mathbb{R}$ are upper semi-cliquish functions on X. Then $f\vee g$ is also upper semi-cliquish on X.

Proposition 3.8: If $f:X\to\mathbb{R}$ and $g:X\to\mathbb{R}$ are upper semi-cliquish functions on X. Then $f\wedge g$ is also upper semi-cliquish on X.

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4 Algebraic Properties of Half Semi-Continuous Functions

The proofs of the following propositions are similar to algebraic properties of semi-cliquish functions.

Proposition 4.1: If $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are lower half semi-continuous on X. Then $f + g$ is lower half semi-continuous on X.

Proposition 4.2: If $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are upper half semi-continuous on X. Then $f + g$ is upper half semi-continuous on X.

Proposition 4.3: If c is a positive real number, and if $f : X \to \mathbb{R}$ is a lower half semi-continuous function at $x_0 \in X$. Then cf is lower half semi-continuous at $x_0 \in X$.

Proposition 4.4: If c is a positive real number, and if $f : X \to \mathbb{R}$ is a upper half semi-continuous function at $x_0 \in X$. Then cf is upper half semi-continuous at $x_0 \in X$.

Proposition 4.5: If $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are lower half semi-continuous functions on X. Then $f \vee g$ is also lower half semi-continuous on X.

Proposition 4.6: If $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are upper half semi-continuous functions on X. Then $f \vee g$ is also upper half semi-continuous on X.

Proposition 4.7: If $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are lower half semi-continuous functions on X. Then $f \wedge g$ is also lower half semi-continuous on X.

Proposition 4.8: If $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are upper half semi-continuous functions on X. Then $f \wedge g$ is also upper half semi-continuous on X.

5 Uniform Limits

This section demonstrates that the uniform limit of a sequence of real-valued lower (upper) semi-cliquish functions on X is itself lower (upper) semi-cliquish on X. These results are further extended to include lower (upper) half semi-continuous functions on X.

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Proposition 5.1: If $f_n : X \to \mathbb{R}$, $n = 1, 2, 3, ...$ is a lower semi-cliquish function at $x_0 \in X$ and $f_n \to f$ uniformly on X then f is lower semi-cliquish function at x_0.

Proof: Let $\varepsilon > 0$ be given and let $G(x_0)$ be a neighborhood of x_0. in X. Since $f_n : X \to \mathbb{R}$ uniformly on X, there exists an integer N such that

$$n \geq N \Rightarrow |f_n(x) - f(x)| < \frac{\varepsilon}{3} \quad \forall \; x \in X$$

$$\Rightarrow -\frac{\varepsilon}{3} < f_n(x) - f(x) < \frac{\varepsilon}{3} \quad \forall \; x \in X$$

Since f_N is lower semi-cliquish function at x_0., there exists a non-empty open set $U_0 \subset G(x_0)$ such that

$$f_N(x) - f_N(y) > -\frac{\varepsilon}{3} \;\; \forall \;\; x, y \in U_0.$$

Now $x, y \in U_0. \Rightarrow f(x) - f(y) = f(x) - f_N(x) + f_N(x) - f_N(y) + f_n(y) - f(y) > -\frac{\varepsilon}{3} - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} = -\varepsilon$

$$\Rightarrow f(x) - f(y) > -\varepsilon$$

Hence f is lower semi cliquish at $x_0 \in X$.

The proofs of the following propositions 5.2, 5.3 and 5.4 follows analogous to previous proposition 5.1.

Proposition 5.2: If $f_n : X \to \mathbb{R}$, $n = 1, 2, 3, ...$ is a upper semi-cliquish function at $x_0 \in X$ and $f_n \to f$ uniformly on X then f is upper semi-cliquish function at x_0

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Proposition 5.3: If $f_n : X \to \mathbb{R}$, $n \in \mathbb{N}$ is a lower half semi-continuous function at $x_0 \in X$ and $f_n \to f$ uniformly on X then f is lower half semi-continuous function at x_0.

Proposition 5.4: If $f_n : X \to \mathbb{R}$, $n = 1, 2, 3, ...$ is a upper half semi-continuous function at $x_0 \in X$ and $f_n \to f$ uniformly on X then f is upper half semi-continuous function at x_0.

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