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Weighted Composition of Powers of Paranormal Operators

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Abstract

Let $T \in B(H)$ be an operator described as powers of paranormal operators if $\|Tx\|^{2p} \le \|T^{2p}x\|\|x\|$ for each $x \in H$. These classes include the classes of paranormal operators and class A operators. Here the chapter, composition and weighted composition of powers of paranormal operators on L^2 space are described.

Key words: Composition Operaor ,Paranormal Operators, Weighted Hardy Space

1. Introduction

The purpose of this paper is to study the weighted composition of powers of Paranormal Operators if $\|Tx\|^{2} \le \|T^{2p}x\|\|T^{3p}x\|$ for every $x \in H$.:S.C.Arora,G.Batt and S.Verma studies boundedness,compactness and closedness of ranges of weighted composition operatorson $L^{p,q}$. Demonstrated the powers of paranormal composition operators and similarly weighted composition of powers of paranormal operators on L^2 space and Hardy Space are described.

2. Properties of powers of paranormal operators

Theorem 2.1

Let $T \in B(H)$ be an operator described as powers of paranormal operators and if is the powers of paranormal operators, then T is unitarily equivalent of s.

oof:

While T is unitarily equivalent to s, here U is a unitarily operator such that $s = U^{*}TU$. we have to illustrate to $S^{*2p}S^{2p} - 2\lambda S^{*p}S^{p} + \lambda^{2}I \geq 0$. while T is powers of paranormal operator, we comprise $T^{*2p}T^{2p} + 2\lambda T^{*p}T^{p} + \lambda^{2}I \geq 0$.

Thus $S^{*2p}S^{2p} - 2\lambda S^{*p}S^{p} + \lambda^{2}I \geq 0$.

$$(U^{*}TU)^{*2p}(U^{*}TU)^{2p} - 2\lambda(U^{*}TU)^{*2p}(U^{*}TU)^{2p} + \lambda^{2}U^{*}U \geq 0$$

$$U^{*}T^{*2p}T^{2p}U - 2\lambda\left(U^{*}T^{*p}T^{p}U\right) + \lambda^{2}U^{*}U \geq 0$$

$$U^{*}\left(T^{*2p}T^{2p} - 2\lambda T^{*p}T^{p} + \lambda^{2}I\right)U \geq 0$$

$\therefore$ Hence S is powers of paranormal.

Theorem 2.2

Consider $T \in B(H)$ is a weighted shift with non-zero weighted $\{\alpha_n\}$, followed by T is powers of paranormal ifffor $n = 1, 2, 3, \ldots$

$$|\alpha_n| \leq |\alpha_{n+1}|$$

Proof:

Let $\{e_n\}_{n=0}^{\infty}$ is orthogonal origin of a Hilbert Space H. while $Te_n = \alpha_n e_{n+1}$ and

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$Te^*_n = \overline{\square}_{n-1} e_{n-1}$, we have,

$= \|Te_n\|^{2p}$

$\|Te_n\|^{2p} = \|\square_n e_{n+1}\|^{2p} = \|T(\square_n e_{n+1})\|^{2p}$

$\square |\square_n|^{2p} \|e_{n+1}\|^{2p}$

$\square |\square_n|^{2p} \|e_{n+1}\|^{2p}$

$\square |\square\square_n|^{2p} |\ _{n+1}|^{p} \|e_{n+2}\|^{2p}$

$\|Te_n\|^{2p} \square |\square\square_n|^{2p} |\ _{n+1}|^{2p}$

we observed that T is powers of paranormal iff $\|Tx\|^{2p} \square \|T^{2p} x\| \|x\|$ for all vector $x \square H$ ifffor all n = 1,2,3, … $\|Te_n\|^{2p} \square \|T^{2p} e_n\| \|e_n\|$ therefore T is powers of paranormal

ifffor

n = 1,2,3,… $|\square| \square |\square_n| \square |\ _{n+1}|$

3. Powers of paranormal composition operators

Consider $(X, \square, \square)$ is a $\square$ – finite measure space and $T: X \rightarrow X$ is a non-singular

measurable space. A linear operator $cf = f\,T$ on $L^2(X, \square, \square)$ is to be a composition operator by

T, tofind $\square T^{-1}$ is continuous and related to the measure f and the derivative

$d\, T\square^{-1k-1} / d\square = f$ is bounded.The measure $\square(T)$ is related to $\square$ and designated by

$f^{(K)}$, where T^k is the order of $T\,K-$ times.

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In each basically bounded complex quantifiable function f □ stimulate the surrounded operator M_f on $L^2(□)$, which is distincted by $M_f = ff$ for all $f □ L^2(□)$; in addition $C C^* = M_f$ and $C^{*2} C^2 = M_f^{(2)}$.

Lemma 3.1

If p is the projection of L^2 on $\overline{R(C)}$, for all $f □ L^2$,

then $C C f^* = f$ □ $f, CC^* f = (f_0 T pf)$

and $\overline{R(C)} = □ f □ L^2 : f is T^{-1} □$ Measurable . □

Theorem 3.2

Consider $C □ B L(^2(□))$ andlet C is the powers of paranormal operator iff

$f_0^{2p} - 2□ f_0^{(2p)} + □^2 I □ 0$a.eand here p is a projection of L^2 on $\overline{R(C)}$.

Proof:

Where $C □ B L□□ ^2(□)□□$ be the power of paranormal operator iff

$C^{*2p} C^{2p} - 2□ C C^{*p} {}_p + □^2 I □ 0$

Therefore,

$\langle (C^{*2p} C^{2p} - 2□ C C^{*pp} + □ □ □^2 I)_{E,E} \rangle □ 0$

Given the function $□_E$ in □ being as □(E)□□. while $C C^{*2}$

$^2 = Mf_0(2)$ and $C C^* = M_{f_0}$, we may have

$\left\langle \left(M_{f_0}{}^{(2p)} - 2\square M p_{f_0}(\ ) + \square\ \square\ \square^2 \right) _{E,E} \right\rangle \square\ 0,$

$i\, e.\, ,\square_E\left(f_0{}^{(2p)} - 2\square f_0{}^{(p)} + \square\ \square^2 I\, d\right)\square\ 0$ for all E is $\square$.

Thus C is powers of paranormal operator'siff $f_0{}^{(2p)} - 2\square\ \square f_0{}^{(p)} + {}^2\ \square\ 0$ almost everywhere.

Corollary 3.3

Let $C\square B\, L\square\ {}^2(\square)\square\square$ by means of dense range and then T is powers of paranormal operator's iff $f_0{}^{(2p)} - 2\square\ \square f_0{}^{(p)} + {}^2 I\ \square 0$almost everywhere.

Example 3.4

Since $\square = N$ and $\square$is the counting measure.Describe$T\, N: \to N$ by $T(1) = 1, T\, n(\ +\ + m\ 1) = n\, m, = 0,1,2,...$ and $n\square N$. while $f_0{}^{(2p)} - 2\square f_0{}^{(p)} + \square^2 I\ \square\ 0$, C is powers paranormal composition operators.

Theorem 3.5

Let $C\square B\, L\square\ {}^2(\square)\square\square$. subsequently C^* is powers of paranormal operator iff

$\dfrac{\square (f_0 \circ T)^{2p}\ \square}{\square\qquad\square} p - 2\square\square\square(f\, T_0)^p p\square\square + \square^2 I\ \square\ 0\ a\ e.$, anywhere p is the projection of L^2 onto $R\, C(\ )$.

Proof

Let C^* be powers of paranormal operator iff$C\, C^{2p}\ {}^{*2p} - 2\square C\, C^{p\ *p} + \square^2 I\ \square\ 0$

$\left\langle \left(C^{2p} C^{*2p} - 2\square C\, C^{p\ *p} + \square^2 I f f \right), \right\rangle \square\ 0$

$\rangle\ \langle\ \circ\ \rangle$

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We include $\langle CC^* f, f\rangle = (f_0 T p f, f)$,
,everywhere projection of L^2 is p onto

$\overline{R\,C(\ )}$.Hence C^* is of powers of paranormal iff

$$\left\langle \left(\square\square(f_0 T)^{2p} p \square\square\right) f, f\right\rangle - \left\langle \left(2\square\square\square(f T_0)^p p \square\square\right) f, f\right\rangle + \square^2 \langle f, f \rangle \;\square\; 0$$

for every $f \square L^2$

$$\square\square(f_0 T)^{2p} p \square\square - 2\square\square\square(f T_0)^p p \square\square + \square^2 I \;\square\; 0 \; a.e.$$

Lemma 3.6

Let An operator $T \square B\,H(\ )$ is powers of paranormal iff $T^{*2p} T^{2p} - 2\square T^{*p} T^p + \square^2 I \square 0$ for all $\square\square$ 0.

Lemma 3.7

Let an operator $T^* \square B\,H(\ )$ is powers of paranormal iff $T^{2p} T^{*2p} - 2\square T^{2p} T^{*2p} + \square^2 I \square 0$ for every $\square\square$ 0.

4. Weighted powers of paranormal composition operators.

The weighted composition operator stimulated with linear and applying on $\square$ measurable functions f, defined as $Wf = \square(f\,T)$, $\square$ is a complex valued $\square$ measurable functions. While $\square = 1$, we state that W is a composition operator. Let W_K represent $\square\square(T)(\square T^2)\ldots(\square T^{K-1})$ so as to $W^K f = \square_K (f\,T)^K$. Lambert study the weighted composition operators successfully by related constrainted operator E through T as $E(T^{-1}\square) = E(\ )E\,f(\ )$is describedto all non-negative measurable function

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f □L^p (1□ p) and it is exclusively arrived the following axioms.

i. $E f(\ )$ is a T^{-1}□ measurable set

ii. If Bbelongs to T^{-1}□ measurable and in thiscase we get □$_B$ fd□=□$_B$ $E\ f\ d(\ )$ □.

The projection operator E on L^p be uniqueness iffT^{-1} □ □= .

Proposition 4.1

For □□ 0,

i. $W\ Wf^* = f_0$ □□E(□^{2})□□ $T^{-1} f$.

ii. $WW^* f =$□($f_0\ T\ E$) (□f).

Theorem 4.2

W is powers of paranormal iff

□$_{02p}$□□E(□$_{22p}$)□□ T_{-2p} − 2□□$_{0(p)}$□□E(□$_{p2}$)□□ T_{-p} +□$_2 I$ □ 0 $a\ e.$.

Proof

As W is a power of paranormal,

$W^{*2p} W^{2p}$ −2□$W^{*p} W^p$ +□$^2 I$ □ 0and

Thus

$\langle (W^{*2p} W^{2p} - 2$□$W^{*p} W^p +$□$^2 I f f)\ ,\ \rangle$ □ 0 for all f □L^2

Since, $W^K f =$□$_K$ ($f\ T^K$) and $W^{*Kp} f =$□$_0^{(K)}$ E(□$_K f$) T^{-K} ,

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$W^*_K W_K = \square_{(K)} E(\square_{K2}) T_{-K} f$ and we have $W Wf^* = f \square\square E(\square_2) T_{-1} f \square$ for all

00 □

$E \square\square$ and so,

2p 2 □ T_{-2p} − 2□ (p)

$\square_0 \square\square E(\square_{2p})\square \quad \square_0 \square\square E(\square_{p2})\square\square T_{-p} + \square_2 I \square 0$ *a e.* .

Theorem 4.3

Let $T^{-1}\square\square = .$ in that case W is of powers of paranormal iff

$\square_0^{(2p)} (\square_{2p})^2 T^{-2p} - 2\square\square \square_0^{(p)} (\ _p)^2 T^{-p} + \square^2 I \square 0$ *a e.* .

Proof

Since T is powers of paranormal operators,

$T^{*2p} T^{2p} - 2\square T^{*p} T^p + \square^2 I \square 0$, Then

W is powers of paranormal operators,

$W^{*2p} W^{2p} - 2\square W^{*p} W^p + \square^2 I \square 0$

and therefore $W^*_K W_K = \square_{0(K)}\square\square E(\square_{K2})\square\square T_{-1}$,

$W W^* = \square_0 \square\square E(\square^2)\square\square T^{-1}$ and Let $T^{-1}\square\ \square = .$

$\langle (W^{*2p} W^{2p} - 2\square W^{*p} W^p + \square^2 I \square 0) ff, \rangle \square 0$ for all $f \square L^2$.

$\square\ \square_\square \square\ _{0(2p)}\square\square E(\square_{22p})\square\square T_{-2p} - 2\square\square_{0(2p)}\square\square E(\square_{p2})\square\square T_{-p} + \square\ \square_2 I\, d\square \quad \square 0$

For all $E \square\square$. and so,

$\square_{0(2p)}\square\square E(\square_{22p})\square\square T_{-2p} - 2\square\square_{0(p)}\square\square E(\square_{p2})\square\square T_{-p} + \square_2 I \square 0$ *a e.* .

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Let T^{-1}□ □=

$$□□_0 \left(□_{2p}\right)^{(2p)} \mathring{T}^{2} {}^{-2p} - 2□□□_0 \left({}_p\right)^{(p)} \mathring{T}^{2} {}^{-p} + □ I^{2} □ 0 \; a\, e. .$$

Theorem 4.4

W *is powers of paranormal operator iff

$$□_{2p}\left(f_{02p} \mathring{T}_{2p}\right)□□E\left(□_{2p}\right) f□□ - 2□□_p \left(f_{0(p)} \mathring{T}_p \right)□□E\left(□_p\right) f□□ + □_2 I □ 0 .$$

Proof

While W * is powers of paranormal operator,

$W W^{2p} *^{2p} - 2□W W^p *^p + □^2 I □ 0$, and thus,

$$\left\langle \left(W^{2p} W^{*2p} - 2□W W^{p*p} + □^2 I □ 0\right) f f, \right\rangle □ 0 \text{for all} f □ L^2.$$

While $WW^* = □\left(f_0 \mathring{T}\right)□□E\left(□f\right)□□$, $W W_K$

$*_K = □_k \left(f_{0(K)} \mathring{T}_K \right)□□E\left(□_K\right) f□□$. □4□

$$□□□_{2p}\left(f_{0(2p)} \mathring{T}_{(2p)}\right)□□E\left(□_{2p}\right) f□□ - 2□□_{(p)}\left(f_{0(p)} \mathring{T}_p \right)□□E\left(□_p\right) f□□ + □ □_2 I\, d□ \; □0$$

□

and for all E □□,

$$□_{2p}\left(f_{0(2p)} \mathring{T}_{2p}\right)□□E\left(□_{2p}\right) f□□ - 2□□_p \left(f_{0(p)} \mathring{T}_p \right)□□E\left(□_p\right) f□□ + □_2 I □ 0 .$$

Theorem 4.5

Let T^{-1}□ □= . then W * is powers of paranormal operator iff

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$$\square_{2p}\left(f_{0(2p)}\overset{\circ}{T}_{2p}\right)\left(\square_{2p}f\right)-2\square\square_{p}\left(f_{0(p)}\overset{\circ}{T}_{p}\right)\left(\square_{2p}f\right)+\square_{2}I\ \square\ 0\ a\ e..$$

Proof

Let T^* is powers of paranormal,

$$T^{2p}T^{*2p}-2\square T\,T^{p\,*p}+\square^{2}I\ \square\ 0$$

and W^* is powers of paranormal,

$$W\,W^{2p\,*2p}-2\square W\,W^{p\,*p}+\square^{2}I\ \square\ 0$$

since

$$WW^{*}=\square\left(f_{0}\overset{\circ}{T}E\right)(\square)$$

$$W\,W_{K}\ {}^{*}{}_{K}=\square_{k}\left(f_{0(K)}\overset{\circ}{T}_{K}\right)E\left(\square_{K}\right)$$

$$\left\langle\left(W^{2p}W^{*2p}-2\square W\,W^{p\,*p}+\square^{2}Iff\right),\right\rangle\ \square\ 0 \text{ for all } f\,\square L^{2}$$

$$\square\square\square_{2p}\left(f_{0(2p)}\overset{\circ}{T}_{2p}\right)E\left(\square_{K}f\right)-2\square\square_{p}\left(f_{0(p)}\overset{\circ}{T}_{p}\right)E\left(\square\ \square\ \square_{p}f\right)+{}_{2}I\,d\square\ \ \square 0$$

$\square$

For all $E\ \square\square$.

$$\square_{2p}\left(f_{0(2p)}\overset{\circ}{T}_{2p}\right)E\left(\square_{p}f\right)-2\square\square_{p}\left(f_{0(p)}\overset{\circ}{T}_{p}\right)E\left(\square\ \square_{p}f\right)+{}_{2}I\ \square\ 0.$$

and for all $T^{-1}\square\ \square=$.

$$\square_{2p}\left(f_{0(2p)}\overset{\circ}{T}_{2p}\right)\left(\square_{p}f\right)-2\square\square_{p}\left(f_{0(p)}\overset{\circ}{T}_{p}\right)\left(\square\ \square_{p}f\right)+{}_{2}I\ \square\ 0\ a\ e..$$

And then it is proved.

The Aulthge transformation of T is the operator T offer as $\tilde{T}=\ T\,U\,T^{\frac{1}{2}\,\frac{1}{2}}$ is established by Aluthge. In general, it could be constructed the family of operator

$\square T_{s}: 0\ \square\ \square s 1\square$. The polar decomposition is specified as $C=U\,|C|$, and

$|\ \ |\ \ \sqrt{C}f=f\qquad f$ and $Uf=\ {}_{s}\ \ \ {}_{1-s}$

$C_s = C U C$ and $C U f_s \frac{1}{\sqrt{f_0 \circ T}} f \circ T.$ =

Where C_s is weighted composition operator, maybe simple and explain to $\left(\frac{1}{\sqrt{f_0 \circ T}}\right)^{s/2} f \circ T.$

$| \; | \; \sqrt{\quad} \; {}^{2\,-1} f$ and $C^*_s f = VE\, Vf$, Where $V = C \frac{\sqrt{f_0 \circ T}}{\left(E\left(\sqrt{f_0 \circ T}\right)^2\right)^{1/4}}.$

$f_s = f_0 E(\,) \; T$

Also We get,

i. $C f_s^K = {}_K \left(f \; T^K \right)$ ii. $C_{s^*K} f =$

$f_{0(K)} E \, {}_K f \; T_{-K}$ iii. $C_{s^*K} C f_{sK} =$

$f_{0(K)} E \, {}_{K2} f \; T_{-K} f$

Theorem 4.6

Let $C_s \in B\, L(\; ^2(\,))$ be powers of paranormal iff

$${}_0^{(2p)} E\left({}_{2p}\right)^2 T^{-2p} f - 2\, {}_0^{(p)} E\left({}_p\right)^2 T^{-p} f + {}^2 I \; 0 \; a\, e..$$

Proof

Let T is powers of paranormal,

$T^{*2p} T^{2p} - 2 T^{*p} T^p + {}^2 I \; 0,$

and as C_s is powers of paranormal,

$C_{s^*2p} C_{s2p} - 2 C C_{s^*p\, s} + {}_2 I \; 0$

Since $C C f_s^{*}{}_s = {}_0 E\left({}^2\right) T^{-1} f,$

$^{*K} \quad K \quad (K) \quad 2 \quad -K$

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$$C_s C f_s = \square_0 \quad E(\square)^{\circ} T \quad f.$$

$$\langle (C_s^{*2p} C_s^{2p} - 2\square C\, C_s^{*p} {}_s^{p} + \square^2 I f f) , \square\, 0 \text{ for all } f \square L^2$$

$$(2p)$$

$$\square \; \square_B \square \; {}_0 E(\square_{22p})^{\circ} \, T^{-2p} f - 2^{\square\,\square} \square_{0(p)} E({}_{2p})^{\circ} \, T^{-1} f + \square\, \square_2 I\, d \square \qquad \square\, 0.$$

Given E □□. and so,

$$\square_0^{(2p)} \quad E(\square_{2p})^{\circ\,2} \, T^{-2p} \quad f - 2^{\square} \square_0^{(p)} E(\square_p)^{\circ\,2} \, T^{-p} \quad +\square\, I^2 \square\, 0 \; a\, e..$$

Theorem 4.7

Let C_s □B L□□ ^{2}(□)□□ be powers of paranormal operator iff

$$\square_0^{(2p)} \quad (\square_{2p})^{\circ\,2} \; T^{-2p} \quad f - 2^{\square\,\square} \square_0^{(p)} (\,{}_p)^{\circ\,2} \; T^{-p} \quad f + \square\, I^2 \square\, 0.$$

Proof

By theorem 1.4.6,

$$\square_0^{(2p)\circ} \quad E(\square_{2p})^{\circ\,2} \; T^{-2p} \quad f - 2^{\square} \square_0^{(p)} E(\square_p)^{\circ\,2} \; T^{-p} \quad f + \square\, I^2 \square\, 0.$$

and let T^{-1}□□= .

$$\square_0^{(2p)} \quad (\square_{2p})^{\circ\,2} \; T^{-2p} \quad f - 2^{\square\,\square} \square_0^{(p)} (\,{}_p)^{2} \; T^{-p} \quad f + \square\, I^2 \square\, 0$$

Theorem 4.8

Let C_s^*□B L□□ ^{2}(□)□□ be powers of paranormal iff

$$\square_4 (f_0^{2p} \overset{\circ}{T}{}^{2p}) E(\square_{2p} f) - 2\square\square_p (f_0^p \overset{\circ}{T}{}^p) E(\square_p f) + \square^2 I \,\square\, 0$$

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Proof

Let T^* is powers of paranormal operator iff $T^{2p}T^{*2p} - 2□T T^{p*p} + □^2 I □ 0$.

While C_s^* is powers of paranormal,

$C_{s2p}C_{s*2p} - 2□C\, C_{sps*p} + □_2 I □ 0$ and,

$C\, C_{s\,s}^{*} = □\left(\mathring{f_0}\, T\, E\right)\left(□f\right)$,

$C\, C_{sKs*K} = □_K\left(\mathring{f_{0K}}\, T_K\right)E\left(□_K f\right)$.

$\left\langle\left(C_{s2p}C_{s*2p} - 2□C\, C_{sps*p} + □_2 I f f\right), \right\rangle □ 0.$

$□_B □ □_{2p}\left(\mathring{f_0}^{2p} T^{2p}\right)E\left(□_{2p} f\right) - 2□□_p\left(\mathring{f_0}^{p} T^{p}\right)E\left(□\, □_p f\right) + {}^2 I □ □ 0.$

For all $E □ □$, and so,

$□_{2p}\left(\mathring{f_{02p}}\, T_{2p}\right)E\left(□_{2p} f\right) - 2□□_p\left(\mathring{f_{0p}}\, T_p\right)E\left(□\, □_p f\right) + {}_2 I □ 0$

Theorem 4.9

If $T^{-1}□\ □ = , C_s^* □ B\, L□□\ ^2(□)□□$ is powers of paranormal iff

$□_{2p}\left(\mathring{f_{0(2p)}}\, T_{2p}\right)\left(\mathring{□}_{2p} f\right) - 2□□_p\left(\mathring{f_{0(p)}}\, T_p\right)\left(□\, □_□ f\right) + {}_2 I □ 0$

Proof

From the theorem 1.4.8, C_s^* is powers of paranormal iff,

$□_{2p}\left(\mathring{f_{0(2p)}}\, T_{2p}\right)E\left(□_{2p} f\right) - 2□□_p\left(\mathring{f_{0(p)}}\, T_p\right)E\left(□\, □_p f\right) + {}_2 I □ 0$

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Let $T^{-1}\square = E$,

$$\square_{2p}\left(f_{0(2p)}\overset{\circ}{T}_{2p}\right)\left(\square_{2p}f\right) - 2\square\square_{p}\left(f_{0(p)}\overset{\circ}{T}_{p}\right)\left(\square_{p}f\right) + \square_{2}I \square 0\ a\,e..$$

The subsequent Aluthge transformation of T expressed by puggal, is specified by

$\tilde{T}\ \tilde{T}\ V\tilde{T} = |\ |^{1/2}\ |\hat{\ }|^{1/2}$, where $\tilde{T} = V|\hat{T}|$ is the polar decomposition of $\hat{T}$.

D.Senthilkumar et.al, discussed that the operator $\tilde{C} = |C_s|^{1/2} V |C_s|^{1/2}$, here $C_s = V|C_s|$ is the polar decomposition of the general Alutge transformation C_s, $0 \square \square s\ 1$ is weighted

$\frac{1}{1}{}^{4}\ \square\square \sup \overline{J\ J/\square\square}$. where $J = \square_0 E(\square^2)\ \overset{\square}{T}_0{}^{-1}$. composition

operator with weight $\square = J\ \square\ \frac{1}{\square J^4}\ \square$

Theorem 4.10

Let $\tilde{C} \square B\ L \square\square\ ^2(\square)\square\square$ is powers of paranormal operator iff

$$\square_0 E\left(\square_{2p}\right)^{2p} \overset{\circ}{T}{}^{\frac{1}{2}-2p} - 2^{\square\ \square}\square_0 E\left({}_{p}\right)^{(p)} \overset{\circ}{T}{}^{\frac{1}{2}-p} + \square^{2} I \square 0\ a\,e..$$

Proof

Let T is powers of paranormal iff

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$T^{*2p} T^{2p} - 2\lambda T^{*p} T^{p} + \lambda^2 I \geq 0.$

Since $\tilde{C}$ is powers of paranormal iff

$\tilde{C}^{*2p} \tilde{C}^{2p} - 2\lambda \tilde{C}^{*p} \tilde{C}^{p} + \lambda^2 I \geq 0.$

Let $\tilde{C}\tilde{C}^* = h_0 E(\omega^1) \circ T^{-1}$

$\tilde{C}^{*K} \tilde{C}^K f = h_0^{(k)} E(\omega_K^{1/2}) \circ T^{-K} f$

$\left\langle \left(\tilde{C}^{*2p} \tilde{C}^{2p} - 2\lambda \tilde{C}^{*p} \tilde{C}^{p} + \lambda^2 I\right) f, f \right\rangle \geq 0$ for all $f \in L^2$

$\left[h_0^{(2p)} E(\omega_{2p}^{1/2}) \circ T^{-2p} f - 2\lambda h_0^{(p)} E(\omega_p^{1/2}) \circ T^{-p} f + \lambda^2 I\right] \geq 0$

For all $E \in \Sigma$. and,

$h_0^{(2p)} E(\omega_{2p}^{1/2}) \circ T^{-2p} f - 2\lambda h_0^{(p)} E(\omega_p^{1/2}) f + \lambda^2 I \geq 0$

and proved.

Theorem 4.11

If $T^{-1}\Sigma = \Sigma$, $\tilde{C} \in B(L^2(\mu))$ is powers of paranormal iff

$h_0^{(2p)} (\omega_{2p})^{1/2} \circ T^{-2p} - 2\lambda h_0^{p} (\omega_p)^{1/2} \circ T^{-p} + \lambda^2 I \geq 0 \; a.e.$

Proof

Since C is a weighted composition operator and weight $\omega = J^{1/4} \left(\dfrac{1 + J}{\sup J^{1/4}} \right) \circ T$, it shows that C is powers of paranormal operator but and only if

$h_0^{(2p)} (\omega_{2p})^{1/2} \circ T^{-2p} - 2\lambda h_0^{p} (\omega_p)^{1/2} \circ T^{-p} + \lambda^2 I \geq 0 \; a.e.$

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5. Powers of paranormal composition operators on weighted hardy space.

The set $H^2(\beta)$ of formal complex power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ such that $\|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 < \infty$ is the common Hardy space of functions analytic in the write disc with inner product $\langle f, g \rangle = \sum_{n=0}^{\infty} a_n \overline{b_n} \beta_n^2$ for f as above and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ and $\beta = (\beta_n)_{n=0}^{\infty}$ are series of positive numbers with $\beta_0 = 1$ as $n \to \infty$. If φ is an analytic function mapping the unit disc D into D, then we defined composition operator C_φ on the space $H^2(\beta)$ by $C_\varphi f = f_0 \varphi$.

Even if the operator C_φ is defined all over on the classical Hardy space H^2, they are not essentially defined on each $H^2(\beta)$. The composition operator C_φ is defined on $H^2(\beta)$ only when the function φ is analytic on a few open set containing the closed unit disc housing superemumnorm firmly lesser than one.

The properties of composition operator on the common Hardy spaces $H^2(\beta)$ are discussed.

The properties of powers of paranormal composition operators on general Hardy space $H^2(\beta)$ are analysed. For order β as above and a point α in D. Let

$\square 1 \quad n \quad 2$

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$K_{□}□(z) = □ \underset{n=0□2}{n(□_{z})}$ then the function $K_{□}□$ is a point evaluation for $H(□)$, for f in

$H^2(□)$. $\langle f K'_{□}□ \rangle = f(□)$ then $K_0□ = 1$ and $C K_{□□}^{*}□ = K_{□□()}□$.

Theorem 5.1

If $C_{□}$ is powers of paranormal operator on $H^2(□)$, then $□ = 1$.

Proof

Let $C_{□}$ be powers of paranormal on $H^2(□)$. By the definition of powers of paranormal,

$$C_{□□^{*}2p}C_{2p} - □C\, C_{□□^{*}pp} + □_2 I □ 0$$

$$\left(C_{□} \left\langle _{□^{*}2p}C_{2p} - □C\, C_{□□^{*}p\ p} + □_2 I f \right\rangle □ 0 \text{ given any } f □ H(□)\ f \right),\ {}^{2}$$

$$\left\langle \left(C_{□□^{*}2p}C_{2p} \right) ff, \right\rangle - 2□ \left\langle \left(C\ C_{□□^{*}p\ p} \right) ff, \right\rangle + □^2 \langle ff, \rangle □ 0$$

$$\left\langle C_{□^{*}2p} f\, C,_{□2p} f \right\rangle - 2□ \left\langle C_{□^{*}p} f\, C f,_{□p} \right\rangle + □^2 \langle ff, \rangle □ 0$$

$$\left\| C_{□2p} f \right\|^2 - 2□ \left\| C f_{□p} \right\|^2 + □_2 \left\| f \right\|^2 □ 0$$

Let $f = K_0□$ therefore we get,

$$\left\| C_{□□}\left(C f \right)_p \right\|^2 - 2□ \left\| C f_{□p} \right\|^2_2 + □ \left\| f \right\|^2 □ 0_p$$

$$\left\| C_{□p}\left(K_{0p}□ \right) \right\|^2 - 2□ \left\| C_{□p}\left(K_0□\ □\ □ \right) \right\|^2 + {}_2 \left\| K_0□ \right\|^2 □ 0$$

$$\left\| K_0□\ □\ □\ □\ □ \right\|_p^2 - 2 \left\| K_0 \right\|_p^2 + {}_2 \left\| K_0□ \right\| □ 0$$

Let $K_0□ = 1$

$$1 - +2□\ □^2 □ 0$$

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$\square \ \square^2 - \ +21\square \ 0$

$(\square - 1)^2 \ \square \ 0$

$\square\square 1$

$\square = 1$ and proved.

6. Conclusion

Illustrated the powers of paranormal composition operators and similarly weighted composition of powers of paranormal operators on L^2 space and Hardy Space are described.

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