

SOME PAIR QUOTIENT CORDIAL GRAPHS

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ABSTRACT:

Consider a (p, q) graph $G = (V, E)$. Define $\gamma = \begin{cases} \frac{p}{2}, & \text{if } p \text{ is even} \\ \frac{p-1}{2}, & \text{if } p \text{ is odd} \end{cases}$ and

$A = \{\pm 1, \pm 2, \pm 3, \dots, \pm \gamma\}$ called the set of labels. Consider a map $f : V \rightarrow A$. When γ is even, assign different labels in A to the p elements of V and when γ is odd, assign different labels in A to the $p - 1$ elements of V and repeat a label for the lone vertex. The above labeling is said to be a **pair quotient cordial labeling** if for each edge uv of G , there exists a labeling $\left| \frac{f(u)}{f(v)} \right|$ or $\left| \frac{f(v)}{f(u)} \right|$ based on $f(u) \geq f(v)$ or $f(v) > f(u)$ such that $|\delta_{f_1} - \delta_{f_1^c}| \leq 1$ where δ_{f_1} and $\delta_{f_1^c}$ denote the number of edges labeled with 1 and number of edges not labeled with 1 respectively. A graph which admits pair quotient cordial labeling is called a **pair quotient cordial graph**. In this paper, we look into the behaviour of pair quotient cordial labeling of Star, Bistar, Complete, Ladder and $K_2 + mK_1$ graphs.

Keywords: Star, Bistar, Complete, Ladder and Pair Quotient Cordial .

1. Introduction:

In 1987, the concept of cordial labeling was introduced by I. Cahit [2]. This concept was evolved into quotient cordial labeling in 2016 and pair difference cordial labeling in 2021 [5]. In this work, by considering finite, simple and undirected graphs we examine the pair quotient cordial labeling behaviour of Star, Bistar, Complete and Ladder graphs.

2. Preliminaries

Definition 2.1. [5]

The Ladder L_n is the product graph $P_n \times K_2$ with $2n$ vertices and $3n - 2$ edges.

Definition 2.2. [6]

Consider a (p, q) graph $G = (V, E)$. Define $\gamma = \begin{cases} \frac{p}{2}, & \text{if } p \text{ is even} \\ \frac{p-1}{2}, & \text{if } p \text{ is odd} \end{cases}$ and

$A = \{\pm 1, \pm 2, \pm 3, \dots, \pm \gamma\}$ called the label set. Consider a map $f : V \rightarrow A$. When γ is even, assign different labels in A to the p elements of V and when γ is odd, assign different labels in A to the $p - 1$ elements of V and repeat a label for the lone vertex. The above labeling is said to be a **pair quotient cordial labeling** if for each edge uv of G , there exists a labeling $\left| \frac{f(u)}{f(v)} \right|$ or $\left| \frac{f(v)}{f(u)} \right|$ based on $f(u) \geq f(v)$ or $f(v) > f(u)$ such that $|\delta_{f_1} - \delta_{f_1^c}| \leq 1$ where δ_{f_1} and $\delta_{f_1^c}$ denote the number of edges labeled with 1 and number of edges not labeled with 1 respectively. A graph which admits pair quotient cordial labeling is called a **pair quotient cordial graph**.

3. Pair Quotient Cordial Labeling**Theorem 3.1.**

The Complete graph K_n is pair quotient cordial only if $n = 1, 2$.

Proof:

Let $V(K_n) = \{u_1, u_2, \dots, u_n\}$.

Since $K_1 \cong P_1$ [6], it is pair quotient cordial.

Since $K_2 \cong P_2$ [6], it is pair quotient cordial.

Also Since $K_3 \cong C_3$ [6], it is not pair quotient cordial.

Case (i): n is even and $n > 2$.

Define $f(x_{2i-1}) = i, 1 \leq i \leq \frac{n}{2}$ and $f(x_{2i}) = -i, 1 \leq i \leq \frac{n}{2}$.

Here $\delta_{f_1} = \frac{n}{2}$ and $\delta_{f_1^c} = \frac{n(n-1)}{2} - \frac{n}{2} = \frac{n^2-2n}{2}$.

Then $|\delta_{f_1} - \delta_{f_1^c}| = \left| \frac{n^2-2n}{2} - \frac{n}{2} \right| = \frac{n^2-3n}{2} > 1$ as $n \geq 4$.

∴ The result is not true for n is even and $n > 2$

Case (ii): n is odd and $n > 3$.

Define $f(x_{2i-1}) = i, 1 \leq i \leq \frac{n-1}{2}, f(x_{2i}) = -i, 1 \leq i \leq \frac{n-1}{2}$ and $f(x_n) = 1$.

In this case, $\delta_{f_1} = \frac{n+3}{2}$ and $\delta_{f_1^c} = \frac{n(n-1)}{2} - \frac{n+3}{2} = \frac{n^2-2n-3}{2}$

∴ $|\delta_{f_1} - \delta_{f_1^c}| = \left| \frac{n+3}{2} - \frac{n^2-2n-3}{2} \right| = \frac{n^2-3n-6}{2} > 1$ as $n \geq 5$.

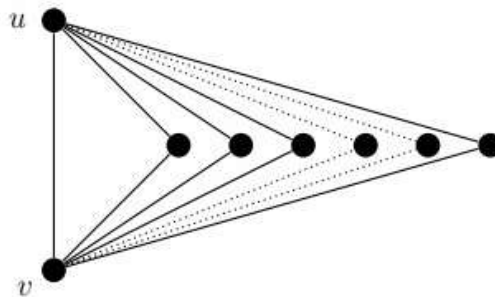
Hence, the Complete graph K_n is pair quotient cordial only if $n = 1, 2$.

Theorem 3.2.

The graph $K_2 + mK_1$ is pair quotient cordial only if $m = 2, 3$.

Proof:

To set the edge with label 1, the adjacent vertices should receive the labels k and $-k$.



Assume $f(u) = k_1$ and $f(v) = k_2$.

For n is odd, assign the lone vertex as repeated label k_1 .

This implies, $\delta_{f_1} = 1 + 1 = 2$ and $\delta_{f_1} = (1 + 1) + 1 = 3$ according as n is even or odd.

For even, $\delta_{f_1^c} = (2m + 1) - 2 = 2m - 1$

∴ $|\delta_{f_1} - \delta_{f_1^c}| = |2m - 1 - 2| = |2m - 3| \leq 1 \Rightarrow m \leq 3$.

For odd and $m > 1, \delta_{f_1^c} = (2m + 1) - 3 = 2m - 2$.

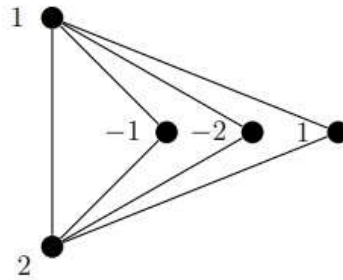
∴ $|\delta_{f_1} - \delta_{f_1^c}| = |2m - 2 - 3| = |2m - 5| \leq 1 \Rightarrow 2m \leq 4 \Rightarrow m \leq 2$.

If $m = 1$, then $K_2 + mK_1 \cong C_3$.

\therefore The graph $K_2 + mK_1$ is pair quotient cordial only if $m = 2,3$.

Illustration:

The pair quotient cordial labeling of $K_2 + 3K_1$



Theorem 3.3.

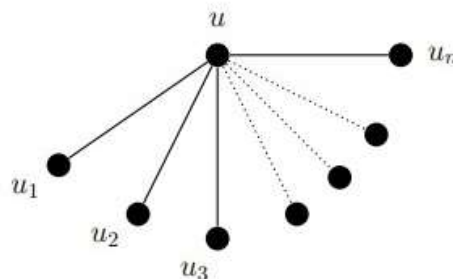
The Star graph $K_{1,n}$ is pair quotient cordial for $n = 1, 3, 4$

Proof:

Let $f(u) = k$.

To set the edge label 1, the only possibility is that the pendant vertices receive the label $-k$. This implies that $\delta_{f_1} = 1$ or 2 according as n is odd or even.

Then $\delta_{f_1^c} = n - 1$ or $n - 2$.



Suppose f is a pair quotient cordial labeling, then $|\delta_{f_1} - \delta_{f_1^c}| \leq 1$.

Case (i): n is odd.

Here $\delta_{f_1} = 1$ and $\delta_{f_1^c} = n - 1$

$$\Rightarrow |\delta_{f_1} - \delta_{f_1^c}| = |n - 1 - 1| = |n - 2|$$

Suppose f is pair quotient cordial, then $|n - 2| \leq 1$.

$$\Rightarrow n \leq 3$$

Therefore the result is true for $n = 1, 3$.

Case (ii): n is even.

Subcase (i): $n = 2$

Label the center vertex u as 1 and a pendant vertex as -1 .

Then label 1 to the remaining pendant vertex.

$$\therefore \delta_{f_1} = 2 \text{ and } \delta_{f_1^c} = 0.$$

$$\Rightarrow |\delta_{f_1} - \delta_{f_1^c}| = 2 > 1, \text{ a contradiction.}$$

Subcase (ii): $n > 2$

Here $\delta_{f_1} = 2$ and $\delta_{f_1^c} = n - 2$

$$\Rightarrow |\delta_{f_1} - \delta_{f_1^c}| = |n - 2 - 2| = |n - 4|$$

Suppose f is pair quotient cordial.

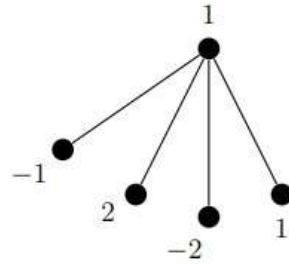
Then $|n - 4| \leq 1$.

$$\Rightarrow n \leq 5 .$$

Therefore, the result is true for $n = 4$.

Illustration:

The pair quotient cordial labeling of $K_{1,4}$.



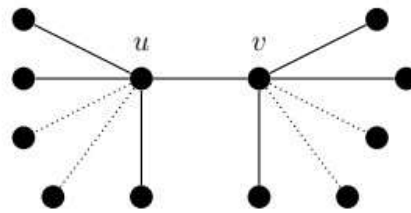
Theorem 3.4.

The Bistar graph $B_{m,n}$, $m, n \geq 1$ is pair quotient cordial iff $2 \leq m + n \leq 5$.

Proof:

Let $f(u) = k$ and $f(v) = l$.

To set the edge label 1, the only possibility is that the pendant vertices of u should receive $-k$ and v should receive $-l$.



Case (i): $m + n$ is even.

Here $\delta_{f_1} = 2$ and $\delta_{f_1^c} = m + n + 1 - 2 = m + n - 1$.

$$\Rightarrow |\delta_{f_1} - \delta_{f_1^c}| = |m + n - 3|$$

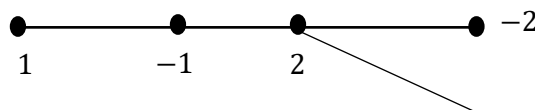
Suppose f is pair quotient cordial, then $|m + n - 3| \leq 1$

$$\Rightarrow m + n \leq 4$$

\therefore If $m + n > 4$, then $B_{m,n}$ is not pair quotient cordial.

Case (ii): $m + n$ is odd.

Subcase (i): $m + n = 3$



● 1

Here $\delta_{f_1} = 2$ and $\delta_{f_1^c} = m + n + 1 - 2 = m + n - 1$

$$\Rightarrow |\delta_{f_1} - \delta_{f_1^c}| = |m + n - 3| = 0$$

$\therefore B_{1,2}$ and $B_{2,1}$ are pair quotient cordial graphs.

Subcase (ii): $m + n > 3$

Then $\delta_{f_1} = 3$ and $\delta_{f_1^c} = m + n + 1 - 3 = m + n - 2$

$$\Rightarrow |\delta_{f_1} - \delta_{f_1^c}| = |m + n - 5|$$

Suppose f is pair quotient cordial, then $|m + n - 5| \leq 1$.

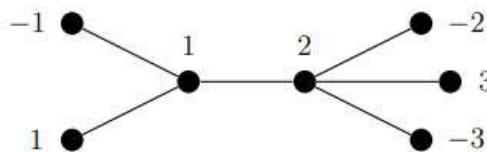
$$\Rightarrow m + n \leq 6 .$$

Since $m + n$ is odd, $m + n \leq 5$.

Hence the result is true for $2 \leq m + n \leq 5$.

Illustration:

The pair quotient cordial labeling of $B_{2,3}$.



Result:

The Bistar $B_{n,n}$ is pair quotient cordial only if $n = 1, 2$.

Theorem 3.5.

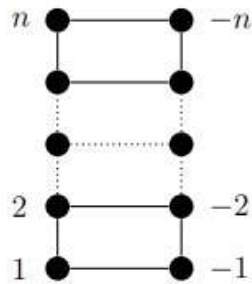
The Ladder graph L_n is pair quotient cordial for $n \leq 3$.

Proof:

Let $V(L_n) = \{u_j, v_j: 1 \leq j \leq n\}$ and

$E(L_n) = \{u_j u_{j+1}, v_j v_{j+1}: 1 \leq j \leq n - 1\} \cup \{u_j v_j: 1 \leq j \leq n\}$.

Define a map $f: V(L_n) \rightarrow \{\pm 1, \pm 2, \dots, \pm n\}$ by $f(u_j) = j, 1 \leq j \leq n$ and $f(v_j) = -j, 1 \leq j \leq n$.



Then $\delta_{f_1} = n$ and $\delta_{f_1^c} = 3n - 2 - n = 2(n - 1)$.

$\Rightarrow |\delta_{f_1} - \delta_{f_1^c}| = |2n - 2 - n| = |n - 2|$

Suppose f is pair quotient cordial.

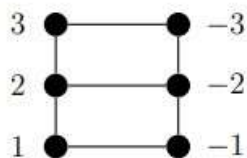
Then $|n - 2| \leq 1$.

$\Rightarrow n \leq 3$

Hence the Ladder graph L_n is pair quotient cordial for $n \leq 3$.

Illustration:

The pair quotient cordial labeling of L_3 .



Conclusion:

The behavior of Complete, Star, Bistar, $K_2 + mK_1$ and Ladder graphs were examined in this article.

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