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A study on various domination numbers related to Security process of Hexagonal star Network and comparison and non – comparison of this Network domination numbers

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1. Introduction:

Graph theory on networks plays a momentous role composed of operators (vertices) and its linked connections (edges). To scrutinize the properties, pathways and connections of networks, the tool like graph related parameters are imperative. Dominating set of a graph is subset of vertices, whose nodes are adjacent to remaining nodes belong to graph such that capturing these dominating nodes helps to find the total number of nodes by their degrees.

Securing such dominating nodes is more essential to protect the entire network from attackers like viruses, unstable current supplies, natural calamities, errors in arrangements etc. The dominating set basics are circumstantial[5]. Nodes on support of its neighbors guard the whole co – nodes within a fraction of second promptly. The requirement of defenders with minimum cardinality for standard graphs, equality between secure and co – secure dominating sets and their bounds are studied in[4]. In this paper, we enclosed the minimum cardinality of secure, co – secure, perfect secure and perfect co – secure domination number of some dimensional networks on base of increase in level. Network structures are discussed below with their commercial and confidential usage.

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Key words: secure, co – secure, strong, perfect secure, perfect co – secure domination number, Hexagonal star Networks.

2. Properties of Network:

Hexagonal Star Network $(\boldsymbol{HSN(p,q)})$:

Eshrag and Ali Ahmad represent a newly designed Hexagonal Star Network[7]. Star like networks are securing many comfort zone on behalf of their communication links. Hexagonal star is a structure where each hexagon's edges formed a base for triangle with 6 nodes addition to hexagon. That is, 6 triangles are bonded on base of hexagon. This network dimension extended in both column wise (p) and row wise (q) is the special feature. Star based network is sociable, feasible in coordinate with hexagon design stimulated us to work on arranging strong secure and strong co – secure domination number. This minimum cardinality featured the placement of defenders with maximum degree. Once defender attacked, the next step defender replaced the input to claim the result without any delay. A network topology like star network suffers from its vulnerability of having all computer nodes linked to a central

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computer node which can cause the entire network to fail to communicate once a failure occurs in the central computer node. To safeguard or to recover other nodes from attacks, it will be simple to end the process of one largest node rather than stopping the process of smaller branch nodes.

We observed and resulted the Equalities of strong secure domination numbers and strong co – secure domination numbers when relations vary on row wise and column wise. Here we compared and tabled $HSN(p,q); p \geq 2, q \leq 10, p \neq q$.

3. Materials and methods: we need following definitions and results.

3.1 Definition:[9] Let G be a loop less, interconnected, in directed, non – isolated graphs. A subset S of G is a *dominating set* if every vertex $u \in S$ is dominating every $v \in V - S$. The minimum cardinality is denoted by $\gamma(G)$.

3.2 Definition: [5] Let $G = (V, E)$ be a graph and let $S \subseteq V$ be a *secure dominating set of* G, with $v \in S$ and $u \in V \backslash S$. Then the vertex v of S defends u if and only if u is adjacent to each vertex of $\{v\} \cup (EPN(v, S) \backslash \{u\})$. The minimum cardinality of secure dominating set is secure domination number $\gamma_{sd}(G)$.

3.3 Definition: [4] A dominating set S of a graph $G = (V, E)$ is called a *co – secure dominating set CSDS*, if for each $u \in S$, there exists a vertex $v \in V \backslash S$ such that $v \in N(u)$ and $S \backslash \{u\} \cup \{v\}$ is a dominating set of G. The minimum cardinality of co – secure dominating set is co - secure domination number $\gamma_{csd}(G)$.

3.4 Definition: [12] A subset S of V of G is a *strong secure dominating set* if for every vertex in $v \in V - S$ there exists a vertex $u \in S$ such that v is adjacent to u and $S \backslash \{u\} \cup \{v\}$ is a dominating set and $\deg u \geq \deg v$. The minimum cardinality is denoted by $\gamma_{ssd}(G)$

3.5 Definition: A subset S is a *strong co – secure dominating set* if for every vertex $u \in S$, there exists a vertex $v \in V - S, degree\,(u) \geq degree\,(v)$, u and v are adjacent and $S \backslash \{u\} \cup \{v\}$ is a dominating set of G. The dominating set obtained is the minimal dominating set. ie) $S - \{u\}$ is not a dominating set. The minimum cardinality of this perfect strong co – secure dominating set is the perfect strong co-secure domination number $\gamma_{scsd}(G)$.

$$\gamma_{csd}(G) \leq \gamma_{scsd}(G);\ \gamma_{sd}(G) \leq \gamma_{ssd}(G)$$

4. HEXAGONAL STAR NETWORKS AND ITS EQUALITIES:

4.1 HEXAGONAL STAR NETWORKS:

This network is peculiar than hexagonal network as it is mainly based on their connections of nodes and development of graph by increasing stars on row and column alternatively. The subset is denoted as D of V is dominating set of G used here.

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Theorem 4.1.1: The strong domination number of $HSN(p,q)$ is

$$\gamma_s\big(HSN(p,q)\big) = \left\{\begin{matrix} 2p + pq + q - 1 \; if \; p \leq 3, q \leq 3 \\ p + 2pq - q + 1 \; if \; p > 3, q \leq 3 \\ p + 2pq - 2q + 2 \; if \; p > 3, q > 3 \end{matrix}\right\}$$ *where p is number of columns and q is number of rows of the graph.*

Proof: we prove this by three cases.

1. $p \leq 3, q \leq 3$

a) Let $p = q = 1$

Then the graph is single hexagonal star graph with cycle enclosed by triangles on base of edges of cycle. Here number of nodes of degree 4 is 6 and number of nodes of degree 2 is 6. Now let $D = \{u_1, u_3, u_5\}$ of degree 4. Also each node is independent and each node can dominate 4 nodes of its neighborhood. Thus $|D| = 3 = \gamma_s$

b) Let $p = 2, q = 1$

Here 2 hexagonal stars are connected at one common node of degree 4. Thus $|D| = 2 + 1 + 2 = 5 = \gamma_s$

c) Let $p = 2, q = 2$

Among 4 stars, upper 2 stars are connected to lower 2 stars by sharing 5 nodes in common of each column.

$\Rightarrow \gamma_s\big(HSN(2,2)\big) = 3 + 2 + 2 + 2 = 9$

d) Let $p = 2, q = 3$

In $HSN(2,2)$, add one star to each column leads to $HSN(2,3)$

$\Rightarrow \gamma_s\big(HSN(2,2)\big) = 2 + 3 + 3 + 4 = 12$

e) Let $p = 3, q = 2$

In $HSN(2,2)$, add one star to each row leads to $HSN(3,2)$

$\Rightarrow \gamma_s\big(HSN(3,2)\big) = 3 + 3 + 3 + 4 = 13$

f) Let $p = 3, q = 3$

In $HSN(2,2)$, add one star to each row and each column leads to $HSN(3,3)$

$\Rightarrow \gamma_s\big(HSN(3,3)\big) = 3 + 4 + 4 + 6 = 17$

In general, for $p \leq 3, q \leq 3$

$\gamma_s\big(HSN(p,q)\big) = p + (p+1)(q-1) + 2p$

$= p + pq - 1 + q - p + 2p$

$= 2p + pq + q - 1$

2) Let $p > 3, q \leq 3$

Number of stars in $p = 4, q = 3$ is 12. From this level, we find the strong dominating sets by separating the whole graph into row parts.

$\Rightarrow \gamma_s\big(HSN(4,3)\big) = 4 + 24 - 3 + 1 = 26$

In general, for $p > 3, q \leq 3, \gamma_s\big(HSN(p,q)\big) = p + (2p-1)(q-1) + 2p$

$= 3p + 2pq - q - 2p + 1$

$= 3p - 2p + 2pq - q + 1$

$= p + 2pq - q + 1$

3) Let $p > 3, q > 3$,

$\gamma_s\big(HSN(p,q)\big) = p + (2p-2)(q-1) + 2p$

$= p + 2pq - 2q + 2 - p + 2p$

$= p + 2pq - 2q + 2$

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Hence $\gamma_s(HSN(p,q)) = \begin{Bmatrix} 2p + pq + q - 1 \; if \; p \leq 3, q \leq 3 \\ p + 2pq - q + 1 \; if \; p > 3, q \leq 3 \\ p + 2pq - 2q + 2 \; if \; p > 3, q > 3 \end{Bmatrix}$

4.2 STRONG SECURE AND STRONG CO – SECURE DOMINATION NUMBER OF HEXAGONAL STAR NETWORKS

Theorem 4.2.1: The strong co – secure and strong secure domination number of $HSN(p,q)$ is

$$\gamma_{scsd}(HSN(p,q)) = 2(p + pq + 1)$$

$$\gamma_{ssd}(HSN(p,q)) = p(3 + 2q) = 3p + 2pq$$

Proof: By adding stars to the network in row wise and column wise, the strong secure domination number and strong co – secure domination number also varies. So in this result, we first fix p and change q from 2,3,4 .... such that row are separated to highlight the count of increase in defenders. The below tables are arranged for showing variation in increase of defenders for strong co – secure domination number of Hexagonal star network of (p,q).

Table 4.2(a): here we fix $p = 2$ and $q = 2,3,4 \ldots$

p,q	$I^{st}row$	$II^{nd}row$	$III^{rd}row$	$IV^{th}row$	$V^{th}row$	……	……	……	……	$pq^{th}row$
2,2	4	2	4	4						
2,3	4	2	2	2	4	4				
2,4	4	2	2	2	2	2	4	4		
$2,q$	4	2	2	2	2	2	2	2	…….	4

In general, $\gamma_{scsd}(HSN(2,q)) = 4 + 2(2q) + 2$

Table 4.2(b): here we fix $p = 3$ and $q = 2,3,4 \ldots$

p,q	$I^{st}row$	$II^{nd}row$	$III^{rd}row$	$IV^{th}row$	$V^{th}row$	……	……	……	……	$pq^{th}row$
3,2	6	2	6	6						
3,3	6	2	4	2	6	6				
3,4	6	2	4	2	4	2	6	6		
$3,q$	6	2	4	2	4	2	4	2	……	6

In general, $\gamma_{scsd}(HSN(3,q)) = 6 + 2(3q) + 2$

Table 4.2(c): here we fix $p = 4$ and $q = 2,3,4 \ldots$

p,q	$I^{st}row$	$II^{nd}row$	$III^{rd}row$	$IV^{th}row$	$V^{th}row$	……	……	……	……	$pq^{th}row$
4,2	8	2	8	8						
4,3	8	2	6	2	8	8				
4,4	8	2	6	2	6	2	8	8		
$4,q$	8	2	6	2	6	2	6	2	……	8

In general, $\gamma_{scsd}(HSN(4,q)) = 8 + 2(4q) + 2$

Hence from the above tables, we generalize the strong co – secure domination number with minimum cardinality.

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$$\gamma_{scsd}\big(HSN(p,q)\big) = 2p + 2(pq) + 2$$

Alternately in column wise addition, the strong co – secure domination number is

$\gamma_{scsd}\big(HSN(p,q)\big) = 6p + 2(q-1) + (2p-2)(q-2) = 2p + 2(pq) + 2$

The below tables are arranged for showing variation in increase of defenders for strong secure domination number of Hexagonal star network of (p,q).

Table 4.2(d): here we fix $p = 2$ and $q = 2,3,4$ ...

p,q	$I^{st}row$	$II^{nd}row$	$III^{rd}row$	$IV^{th}row$	$V^{th}row$	……	……	……	……	$pq^{th}row$
2,2	6	2	6							
2,3	6	2	4	6						
2,4	6	2	4	4	6					
$2,q$	6	2	4	4	4	4	4	4	……	6

In general, $\gamma_{ssd}\big(HSN(2,q)\big) = 2(3+2q)$

Table 4.2(e): here we fix $p = 3$ and $q = 2,3,4$ ...

p,q	$I^{st}row$	$II^{nd}row$	$III^{rd}row$	$IV^{th}row$	$V^{th}row$	……	……	……	……	$pq^{th}row$
3,2	9	3	9							
3,3	9	3	6	9						
3,4	9	3	6	6	9					
$3,q$	9	3	6	6	6	6	6	6	……	9

In general, $\gamma_{ssd}\big(HSN(3,q)\big) = 3(3+2q)$

Table 4.2(f): here we fix $p = 4$ and $q = 2,3,4$ ...

p,q	$I^{st}row$	$II^{nd}row$	$III^{rd}row$	$IV^{th}row$	$V^{th}row$	……	……	……	……	$pq^{th}row$
4,2	12	4	12							
4,3	12	4	8	12						
4,4	12	4	8	8	12					
$4,q$	12	4	8	8	8	8	8	8	……	12

In general, $\gamma_{ssd}\big(HSN(4,q)\big) = 4(3+2q)$

Hence from the above tables, we generalize the strong secure domination number with minimum cardinality.

$\gamma_{ssd}\big(HSN(p,q)\big) = p(3+2q) = 3p + 2pq$

4.3 EQUAL STRONG CO – SECURE DOMINATION NUMBER OF $HSN(p,q)$ WHERE $p \neq q$

Result 4.3.1: $\gamma_{scsd}\big(HSN(2,6n+2)\big) = \gamma_{scsd}\big(HSN(3,4n+1)\big)$

Proof: From above theorem, the minimum cardinality of strong co – secure domination number is resulted and here the comparison are stated for $p = 2$ and 3.

$\gamma_{scsd}\big(HSN(2,8)\big) = \gamma_{scsd}\big(HSN(3,5)\big) = 38$

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$\gamma_{scsd}\big(HSN(2,14)\big) = \gamma_{scsd}\big(HSN(3,9)\big) = 62$

$\gamma_{scsd}\big(HSN(2,20)\big) = \gamma_{scsd}\big(HSN(3,13)\big) = 86$

$\gamma_{scsd}\big(HSN(2,26)\big) = \gamma_{scsd}\big(HSN(3,17)\big) = 110$

Continuing in this way we get, $\gamma_{scsd}\big(HSN(2,6n+2)\big) = \gamma_{scsd}\big(HSN(3,4n+1)\big)$

Result 4.3.2: The strong co – secure domination number of $HSN(2,q), HSN(3,q), HSN(4,q)$ are equal with this required condition,

$$\gamma_{scsd}\big(HSN(2,6n-1)\big) = \gamma_{scsd}\big(HSN(3,4n-1)\big) = \gamma_{scsd}\big(HSN(4,3n-1)\big)$$

Proof: The minimum cardinality of strong co – secure domination number of $HSN(p,q)$ where $p = 2,3,4$ are compared here.

$\gamma_{scsd}\big(HSN(2,5)\big) = \gamma_{scsd}\big(HSN(3,3)\big) = \gamma_{scsd}\big(HSN(4,2)\big) = 26$

$\gamma_{scsd}\big(HSN(2,11)\big) = \gamma_{scsd}\big(HSN(3,7)\big) = \gamma_{scsd}\big(HSN(4,5)\big) = 50$

$\gamma_{scsd}\big(HSN(2,17)\big) = \gamma_{scsd}\big(HSN(3,11)\big) = \gamma_{scsd}\big(HSN(4,8)\big) = 74$

$\gamma_{scsd}\big(HSN(2,23)\big) = \gamma_{scsd}\big(HSN(3,15)\big) = \gamma_{scsd}\big(HSN(4,11)\big) = 98$

Continuing in this way, we get

$\gamma_{scsd}\big(HSN(2,6n-1)\big) = \gamma_{scsd}\big(HSN(3,4n-1)\big) = \gamma_{scsd}\big(HSN(4,3n-1)\big)$

Result 4.3.3: Strong co – secure domination number of $HSN(2,q)$ & $HSN(4,q)$ are equal when $p = 2, q = 2n+1$ & $p = 4, q = n$ where $n = 2,3$ ....

The strong co – secure domination number of $HSN(p,q)$ can be determined with variation of increase in p or increase in q or increase in both p & q.

Let $p = 2, q = 4$; $p = 4, q = 2$ where number of hexagonal stars in it is 8. Though, number of stars equal, the result $\gamma_{scsd}\big(HSN(2,4)\big) \neq \gamma_{scsd}\big(HSN(4,2)\big)$ implicates us to check out the comparison of strong co – secure domination number. Here we tabled the results of comparison from $HSN(2,q)$ to $HSN(10,q)$ where $2 \leq p \leq 10, q \geq 2$. Some $\gamma_{scsd}\big(HSN(p,q)\big)$ are equal to some $\gamma_{scsd}\big(HSN(p,q)\big)$ and some $\gamma_{scsd}\big(HSN(p,q)\big)$ is not equal to $\gamma_{scsd}\big(HSN(p,q)\big)$.

Theorem 4.3.4: $\gamma_{scsd}\big(HSN(p,2n+1)\big) = \gamma_{scsd}\big(HSN(2p,n)\big)\ where\ p,n = 2,3,4$ ...

Proof: Let $p = 2, n = 2,3,4,$ ... then

$\gamma_{scsd}\big(HSN(2,5)\big) = \gamma_{scsd}\big(HSN(4,2)\big)$

$\gamma_{scsd}\big(HSN(2,7)\big) = \gamma_{scsd}\big(HSN(4,3)\big)$

$\gamma_{scsd}\big(HSN(2,9)\big) = \gamma_{scsd}\big(HSN(4,4)\big)$

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$\gamma_{scsd}\big(HSN(2,11)\big) = \gamma_{scsd}\big(HSN(4,5)\big)$

$\gamma_{scsd}\big(HSN(2,13)\big) = \gamma_{scsd}\big(HSN(4,6)\big) \ldots\ldots\ \gamma_{scsd}\big(HSN(2,2n+1)\big) = \gamma_{scsd}\big(HSN(4,n)\big)$

Let $p = 4, n = 2,3,4,\ldots$ then

$\gamma_{scsd}\big(HSN(4,5)\big) = \gamma_{scsd}\big(HSN(8,2)\big)$

$\gamma_{scsd}\big(HSN(4,7)\big) = \gamma_{scsd}\big(HSN(8,3)\big)$

$\gamma_{scsd}\big(HSN(4,9)\big) = \gamma_{scsd}\big(HSN(8,4)\big)$

$\gamma_{scsd}\big(HSN(4,11)\big) = \gamma_{scsd}\big(HSN(8,5)\big) \ldots\ldots \gamma_{scsd}\big(HSN(4,2n+1)\big) = \gamma_{scsd}\big(HSN(8,n)\big)$

Let $p = 3, n = 2,3,4,\ldots$ then

$\gamma_{scsd}\big(HSN(3,5)\big) = \gamma_{scsd}\big(HSN(6,2)\big)$

$\gamma_{scsd}\big(HSN(3,7)\big) = \gamma_{scsd}\big(HSN(6,3)\big)$

$\gamma_{scsd}\big(HSN(3,9)\big) = \gamma_{scsd}\big(HSN(6,4)\big)$

$\gamma_{scsd}\big(HSN(3,11)\big) = \gamma_{scsd}\big(HSN(6,5)\big) \ldots\ldots \gamma_{scsd}\big(HSN(3,2n+1)\big) = \gamma_{scsd}\big(HSN(6,n)\big)$

Let $p = 5, n = 2,3,4,\ldots$ then

$\gamma_{scsd}\big(HSN(5,5)\big) = \gamma_{scsd}\big(HSN(10,2)\big)$

$\gamma_{scsd}\big(HSN(5,7)\big) = \gamma_{scsd}\big(HSN(10,3)\big)$

$\gamma_{scsd}\big(HSN(5,9)\big) = \gamma_{scsd}\big(HSN(10,4)\big)$

$\gamma_{scsd}\big(HSN(5,11)\big) = \gamma_{scsd}\big(HSN(10,5)\big) \ldots\ldots \gamma_{scsd}\big(HSN(5,2n+1)\big) = \gamma_{scsd}\big(HSN(10,n)\big)$

From these results, we get

$$\gamma_{scsd}\big(HSN(p,2n+1)\big) = \gamma_{scsd}\big(HSN(2p,n)\big)\ where\ p,n = 2,3,4\ldots \&\ 2n+1, n \in q$$

Table 4.3(a): Comparison of $\gamma_{scsd}\big(HSN(p,q)\big)$ with $\gamma_{scsd}\big(HSN(p,q)\big)$ where $p \neq q$

$\boldsymbol{p,q}$	$\boldsymbol{2,q}$	$\boldsymbol{3,q}$	$\boldsymbol{4,q}$	$\boldsymbol{5,q}$
$3,q$	$(2,3n+2) = (3,2n+1)$			
$4,q$	$(2,2n+1) = (4,n)$	$(3,2n+1) = (4,n)$		
$5,q$	$(2,5n+4) = (5,2n+1)$	$(3,5n-1) = (5,3n-1)$	$(4,5n-1) = (5,2n+1)$	
$6,q$	$(2,3n+2) = (6,n)$	$(3,2n+1) = (6,n)$	$(4,3n+2) = (6,2n+1)$	$(5,6n-1) = (6,5n-1)$
$7,q$	$(2,7n-1) = (7,2n-1)$	$(3,7n-1) = (7,3n-1)$	$(4,7n-1) = (7,4n-1)$	$(5,7n-1) = (7,5n-1)$
$8,q$	$(2,4n+3) = (8,n)$	$(3,8n-1) = (8,3n-1)$	$(4,2n+3) = (8,n+1)$	$(5,8n-1) = (8,5n-1)$
$9,q$	$(2,9n-1) = (9,2n-1)$	$(3,3n+2) = (9,n)$	$(4,9n-1) = (9,4n-1)$	$(5,9n-1) = (9,5n-1)$
$10,q$	$(2,5n+4) = (10,n)$	$(3,10n-1) = (10,3n+1)$	$(4,5n-1) = (10,2n-1)$	$(5,2n+1) = (10,n)$

$\boldsymbol{p,q}$	$\boldsymbol{6,q}$	$\boldsymbol{7,q}$	$\boldsymbol{8,q}$	$\boldsymbol{9,q}$

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$3, q$				
$4, q$				
$5, q$				
$6, q$				
$7, q$	$(6,7n-1)=(7,6n-1)$			
$8, q$	$(6,4n-1)=(8,3n-1)$	$(7,8n-1)=(8,7n-1)$		
$9, q$	$(6,3n+2)=(9,2n+1)$	$(7,9n-1)=(9,7n-1)$	$(8,9n-1)=(9,8n-1)$	
$10, q$	$(6,5n-1)=(10,3n-1)$	$(7,10n-1)=(10,7n-1)$	$(8,5n-1)=(10,4n-1)$	$(9,5n-1)=(10,9n-1)$

5. COMPARISON OF STRONG SECURE DOMINATION NUMBER OF $HSN(p, q)$ WITH $HSN(p, q)$ WHERE $p \neq q$

Secure domination number of $HSN(p, q)$ are also equal to some $HSN(p, q)$ but are not relevant as strong co – secure domination number. Here the theorems below are comparison of $HSN(2, q)$, $HSN(3, q)$, ... with other $HSN(p, q)$

Theorem 5.1: $\gamma_{ssd}\big(HSN(2, q)\big) = \gamma_{ssd}\big(HSN(4k + 2, q)\big)$

$$\Rightarrow \gamma_{ssd}\big(HSN(2, [2k + 1]n + k - 1)\big) = \gamma_{ssd}\big(HSN(4k + 2, n - 1)\big)$$

(This theorem is comparison of $HSN(2, q)$ to other $HSN(p, q)$ with some required conditions)

Proof: By theorem 4.2.1, $\gamma_{ssd}\big(HSN(p, q)\big) = 3p + 2pq \ldots (A)$

Let $p = 2$, then $\gamma_{ssd}\big(HSN(2, q)\big) = 6 + 4q = 2(3 + 2q) \ldots . (1)$

$$\gamma_{ssd}\big(HSN(4k + 2, q)\big) = 3(4k + 2) + 2(4k + 2)q$$

$$= 12k + 6 + 8kq + 4q$$

Let $k = 1$,then $\gamma_{ssd}\big(HSN(6, q)\big) = 12 + 6 + 8q + 4q = 18 + 12q = 6(3 + 2q) \ldots (2)$

From (1)&(2), we get $\gamma_{ssd}\big(HSN(2, q)\big) = \gamma_{ssd}\big(HSN(4k + 2, q)\big)$

Now let vary q by means of n & k.

LHS $= \gamma_{ssd}\big(HSN(2, [2k + 1]n + k - 1)\big)$

$= 2(3 + 2(2k + 1)n + k - 1)$ $(since\ by\ (A))$

$= 2(3 + (4k - 2)n + k - 1) = 2(3 + 4kn + 2n + k - 1)$

Fix $k = 1$ and let $n = 3$, then

$\gamma_{ssd}\big(HSN(2,9)\big) = 2(3 + 4n + 2n) = 6 + 36 = 42 \ldots (a)$

RHS $= \gamma_{ssd}\big(HSN(4k + 2, n - 1)\big)$

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$= (4k + 2)\big(3 + 2(n - 1)\big)\ (since\ by\ (A))$

$= 4k + 2 + 8kn + 4n$

$k = 1, n = 3$ in this equation leads to $\gamma_{ssd}\big(HSN(6, 2)\big) = 6 + 36 = 42 \dots (b)$

From $(a)\&(b)$, we get $\gamma_{ssd}\big(HSN(2,9)\big) = \gamma_{ssd}\big(HSN(6,2)\big) = 42$

Fix $k = 1$ and let $n = 4$, then

LHS $= \gamma_{ssd}\big(HSN(2, [2k + 1]n + k - 1)\big)$

$= \gamma_{ssd}\big(HSN(2,12)\big) = p(3 + 2q) = 2(3 + 24) = 48 \dots (c)$

RHS $= \gamma_{ssd}\big(HSN(4k + 2, n - 1)\big)$

$= \gamma_{ssd}\big(HSN(6, 3)\big) = 6(3 + 6) = 48 \dots (d)$

From $(c)\&(d)$, we get $\gamma_{ssd}\big(HSN(2,12)\big) = \gamma_{ssd}\big(HSN(6,3)\big) = 48$

Fix $k = 1$ and let $n = 4$, then $\gamma_{ssd}\big(HSN(2,15)\big) = \gamma_{ssd}\big(HSN(6,4)\big)$

Fix $k = 1$ and let $n = 5$, then $\gamma_{ssd}\big(HSN(2,18)\big) = \gamma_{ssd}\big(HSN(6,5)\big)$

Continuing in this way, the solution obtained that $\gamma_{ssd}\big(HSN(2,3n)\big) = \gamma_{ssd}\big(HSN(6, n - 1)\big)$ is the result of comparison of strong secure domination number of $HSN(p, q)$ where $p = 2$ is compared with $p = 6$.

Fix $k = 2$ and let $n \geq 3$, then

LHS $= \gamma_{ssd}\big(HSN(2, [2k + 1]n + k - 1)\big)$

$= \gamma_{ssd}\big(HSN(2,16)\big) = p(3 + 2q) = 2(3 + 32) = 70 \dots (e)$

RHS $= \gamma_{ssd}\big(HSN(4k + 2, n - 1)\big)$

$= \gamma_{ssd}\big(HSN(10, 2)\big) = 10(7) = 70 \dots (f)$

From $(e)\&(f)$, we get $\gamma_{ssd}\big(HSN(2,16)\big) = \gamma_{ssd}\big(HSN(10,2)\big) = 70$

Fix $k = 2$ and let $n = 4$

LHS $= \gamma_{ssd}\big(HSN(2, [2k + 1]n + k - 1)\big)$

$= \gamma_{ssd}\big(HSN(2,21)\big) = p(3 + 2q) = 2(3 + 42) = 90 \dots (g)$

RHS $= \gamma_{ssd}\big(HSN(4k + 2, n - 1)\big)$

$= \gamma_{ssd}\big(HSN(10, 3)\big) = 10(3 + 6) = 90 \dots (h)$

From $(g)\&(h)$, we get $\gamma_{ssd}\big(HSN(2,21)\big) = \gamma_{ssd}\big(HSN(10,3)\big) = 90$

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Continuing in this way, we obtained that $\gamma_{ssd}(HSN(2,5n+1)) = \gamma_{ssd}(HSN(10,n-1))$ is the result of comparison of strong secure domination number of $HSN(p,q)$ where $p = 2$ is compared with $p = 10$.

Also, $\gamma_{ssd}(HSN(2,7n+2)) = \gamma_{ssd}(HSN(14,n-1))$

Continuing in this way by fixing k and varying n where $k = 1,2,3 \ldots$; $n \geq 3$, we get the result on comparison of $HSN(2,q)$ with $HSN(6,q), HSN(10,q), HSN(14,q) \ldots$

Hence $\gamma_{ssd}(HSN(2,q)) = \gamma_{ssd}(HSN(4k+2,q))$

$\Rightarrow \gamma_{ssd}(HSN(2,[2k+1]n+k-1)) = \gamma_{ssd}(HSN(4k+2,n-1))$ is proved.

Theorem 5.2: $\gamma_{ssd}(HSN(3,q)) = \gamma_{ssd}(HSN(2k+1,q))$

$$\Rightarrow \gamma_{ssd}(HSN(3,[2k+1]n+k-1)) = \gamma_{ssd}(HSN(2k+1,3n))$$

(This theorem is comparison of $HSN(3,q)$ to other $HSN(p,q)$ with some required conditions)

Proof: By theorem 4.2.1, $\gamma_{ssd}(HSN(p,q)) = 3p + 2pq \ldots (A)$

Let $p = 3$, then $\gamma_{ssd}(HSN(3,q)) = 3(3+2q) \ldots . (1)$

$\gamma_{ssd}(HSN(2k+1,q)) = 3(2k+1) + 2(2k+1)q = 16k + 3 + 4kq + 2q$

$k = 1$ is not considered here since $\gamma_{ssd}(HSN(2k+1,q)) = \gamma_{ssd}(HSN(3,q))$ promotes to $p = q$ which is not possible. Therefore start from $k = 2$.

$\gamma_{ssd}(HSN(2k+1,q)) = \gamma_{ssd}(HSN(5,q)) = 15 + 10q = 5(3+2q) \ldots (2)$

From (1)&(2), we get $\gamma_{ssd}(HSN(3,q)) = \gamma_{ssd}(HSN(4k+2,q))$ where $p = 3$ is compared with $p = 5$.

Now let vary q by means of n & k.

LHS $= \gamma_{ssd}(HSN(3,[2k+1]n+k-1))$

Fix $k = 2$ and let $n = 3$, then

$\gamma_{ssd}(HSN(3,16)) = 3(3+32) = 105 \ldots (a)$

RHS $= \gamma_{ssd}(HSN(2k+1,3n))$

$k = 2, n = 3$ in this equation leads to $\gamma_{ssd}(HSN(5,9)) = 15 + 90 = 105 \ldots (b)$

From (a)&(b), we get $\gamma_{ssd}(HSN(3,16)) = \gamma_{ssd}(HSN(5,9))$

Fix $k = 2$ and let $n = 4$, then

LHS $= \gamma_{ssd}(HSN(3,[2k+1]n+k-1))$

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$= \gamma_{ssd}\big(HSN(3{,}21)\big) = p(3+2q) = 3(3+42) = 135 \ldots (c)$

RHS $= \gamma_{ssd}\big(HSN(2k+1{,}3n)\big)$

$= \gamma_{ssd}\big(HSN(5,12)\big) = 5(3+24) = 135 \ldots (d)$

From $(c)\&(d)$, we get $\gamma_{ssd}\big(HSN(3{,}21)\big) = \gamma_{ssd}\big(HSN(5{,}12)\big)$

Fix $k = 2$ and let $n = 5$, then $\gamma_{ssd}\big(HSN(3{,}26)\big) = \gamma_{ssd}\big(HSN(5{,}15)\big)$

Fix $k = 2$ and let $n = 6$, then $\gamma_{ssd}\big(HSN(3{,}31)\big) = \gamma_{ssd}\big(HSN(5{,}18)\big)$

Continuing in this way, the solution obtained that $\gamma_{ssd}\big(HSN(3{,}5n+1)\big) = \gamma_{ssd}\big(HSN(5{,}3n)\big)$ is the result of comparison of strong secure domination number of $HSN(p,q)$ where $p = 3$ is compared with $p = 5$.

Fix $k = 3$ and let $n = 3$, then

LHS $= \gamma_{ssd}\big(HSN(3{,}7n+2)\big)$

$= \gamma_{ssd}\big(HSN(3{,}23)\big) = p(3+2q) = 3(3+46) = 147 \ldots (e)$

RHS $= \gamma_{ssd}\big(HSN(2k+1{,}3n)\big)$

$= \gamma_{ssd}\big(HSN(7{,}3n)\big) = 7(21) = 147 \ldots (f)$

From $(e)\&(f)$, we get $\gamma_{ssd}\big(HSN(3{,}23)\big) = \gamma_{ssd}\big(HSN(7{,}9)\big)$

Fix $k = 3$ and let $n = 4$

LHS $= \gamma_{ssd}\big(HSN(3,[2k+1]n+k-1)\big)$

$= \gamma_{ssd}\big(HSN(3{,}30)\big) = p(3+2q) = 3(3+60) = 189 \ldots (g)$

RHS $= \gamma_{ssd}\big(HSN(2k+1{,}3n)\big)$

$= \gamma_{ssd}\big(HSN(7,12)\big) = 7(3+24) = 189 \ldots (h)$

From $(g)\&(h)$, we get $\gamma_{ssd}\big(HSN(3{,}30)\big) = \gamma_{ssd}\big(HSN(7{,}12)\big)$

Continuing in this way, we obtained that $\gamma_{ssd}\big(HSN(3{,}7n+2)\big) = \gamma_{ssd}\big(HSN(7{,}3n)\big)$ is the result of comparison of strong secure domination number of $HSN(p,q)$ where $p = 3$ is compared with $p = 7$.

Continuing in this way by fixing k and varying n where $k = 2{,}3{,}4 \ldots$; $n \geq 3$, we get the result on comparison of $HSN(3,q)$ with $HSN(5,q), HSN(7,q)$

Hence $\gamma_{ssd}\big(HSN(3,q)\big) = \gamma_{ssd}\big(HSN(2k+1,q)\big)$

$\Rightarrow \gamma_{ssd}\big(HSN(3,[2k+1]n+k-1)\big) = \gamma_{ssd}\big(HSN(2k+1{,}3n)\big)$ is proved.

Theorem 5.3: $\gamma_{ssd}\big(HSN(4,q)\big) = \gamma_{ssd}\big(HSN(8k+4,q)\big)$

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$$\Rightarrow \gamma_{ssd}\big(HSN(4,[2k+1]n+k-1)\big) = \gamma_{ssd}\big(HSN(8k+4,n-1)\big)$$

(This theorem is comparison of $HSN(4,q)$ to other $HSN(p,q)$ with some required conditions)

Proof: By theorem 4.2.1, $\gamma_{ssd}\big(HSN(p,q)\big) = 3p + 2pq \ldots (A)$

Let $p = 4$, then $\gamma_{ssd}\big(HSN(4,q)\big) = 4(3+2q) \ldots . (1)$

$$\gamma_{ssd}\big(HSN(8k+4,q)\big) = 4(8k+4) + 2(8k+4)q$$

$$= 32k + 16 + 16kq + 8q$$

Fix $k = 1$, $\gamma_{ssd}\big(HSN(8k+4,q)\big) = \gamma_{ssd}\big(HSN(12,q)\big) = 12(3+2q) \ldots (2)$

From (1)&(2), we get $\gamma_{ssd}\big(HSN(4,q)\big) = \gamma_{ssd}\big(HSN(8k+4,q)\big)$ where $p = 4$ is compared with $p = 12$.

Now let vary q by means of n & k.

LHS = $\gamma_{ssd}\big(HSN(4,[2k+1]n+k-1)\big)$

Fix $k = 1$ and let $n = 3$, then

$\gamma_{ssd}\big(HSN(4,3n)\big) = 4(3+18) = 84 \ldots (a)$

RHS = $\gamma_{ssd}\big(HSN(8k+4,n-1)\big)$

$k = 1, n = 3$ in this equation leads to $\gamma_{ssd}\big(HSN(12,2)\big) = 36 + 48 = 84 \ldots (b)$

From (a)&(b), we get $\gamma_{ssd}\big(HSN(4,9)\big) = \gamma_{ssd}\big(HSN(12,2)\big)$

Fix $k = 1$ and let $n = 4$, then

LHS = $\gamma_{ssd}\big(HSN(4,[2k+1]n+k-1)\big)$

$= \gamma_{ssd}\big(HSN(4,3n)\big) = 4(3+6n) = 4(3+24) = 108 \ldots (c)$

RHS = $\gamma_{ssd}\big(HSN(8k+4,n-1)\big)$

$= \gamma_{ssd}\big(HSN(12,3)\big) = 12(3+6) = 108 \ldots (d)$

From (c)&(d), we get $\gamma_{ssd}\big(HSN(4,12)\big) = \gamma_{ssd}\big(HSN(12,3)\big)$

Fix $k = 1$ and let $n = 5$, then $\gamma_{ssd}\big(HSN(4,15)\big) = \gamma_{ssd}\big(HSN(12,4)\big)$

Fix $k = 1$ and let $n = 6$, then $\gamma_{ssd}\big(HSN(4,18)\big) = \gamma_{ssd}\big(HSN(12,5)\big)$

Continuing in this way, the solution obtained that $\gamma_{ssd}\big(HSN(4,3n)\big) = \gamma_{ssd}\big(HSN(12,n-1)\big)$ is the result of comparison of strong secure domination number of $HSN(p,q)$ where $p = 4$ is compared with $p = 12$.

Fix $k = 2$ and let $n = 3$, then

LHS $= \gamma_{ssd}\big(HSN(4,5n+1)\big)$

$= \gamma_{ssd}\big(HSN(4,16)\big) = p(3+2q) = 4(3+32) = 140 \ldots (e)$

RHS $= \gamma_{ssd}\big(HSN(8k+4, n-1)\big)$

$= \gamma_{ssd}\big(HSN(20,2)\big) = 20(3+4) = 140 \ldots (f)$

From $(e)\&(f)$, we get $\gamma_{ssd}\big(HSN(4,16)\big) = \gamma_{ssd}\big(HSN(20,2)\big)$

Fix $k = 2$ and let $n = 4$

LHS $= \gamma_{ssd}\big(HSN(4, [2k+1]n + k - 1)\big)$

$= \gamma_{ssd}\big(HSN(4,21)\big) = 180 \ldots (g)$

RHS $= \gamma_{ssd}\big(HSN(8k+4, n-1)\big)$

$= \gamma_{ssd}\big(HSN(20,3)\big) = 180 \ldots (h)$

From $(g)\&(h)$, we get $\gamma_{ssd}\big(HSN(4,21)\big) = \gamma_{ssd}\big(HSN(20,3)\big)$

Continuing in this way, we obtained that $\gamma_{ssd}\big(HSN(4,5n+1)\big) = \gamma_{ssd}\big(HSN(20, n-1)\big)$ is the result of comparison of strong secure domination number of $HSN(p,q)$ where $p = 4$ is compared with $p = 20$.

Continuing in this way by fixing k and varying n where $k = 1,2,3 \ldots$; $n \geq 3$, we get the result on comparison of $HSN(4,q)$ with $HSN(12,q), HSN(20,q), \ldots$

Hence $\gamma_{ssd}\big(HSN(4,q)\big) = \gamma_{ssd}\big(HSN(8k+4,q)\big)$

$\Rightarrow \gamma_{ssd}\big(HSN(4,[2k+1]n+k-1)\big) = \gamma_{ssd}\big(HSN(8k+4, n-1)\big)$ is proved.

Comparison of strong secure domination number of $HSN(6,q), HSN(7,q), HSN(8,q), HSN(9,q)$ to other $HSN(p,q)$ are resulted here.

Result 5.4: $\gamma_{ssd}\big(HSN(5,q)\big) = \gamma_{ssd}\big(HSN(2k+1,q)\big)$

$\Rightarrow \gamma_{ssd}\big(HSN(5,[2k+1]n+k-1)\big) = \gamma_{ssd}\big(HSN(2k+1, 5n+1)\big)$

Result 5.5: $\gamma_{ssd}\big(HSN(6,q)\big) = \gamma_{ssd}\big(HSN(4k+2,q)\big)$

$\Rightarrow \gamma_{ssd}\big(HSN(6,[2k+1]n+k-1)\big) = \gamma_{ssd}\big(HSN(4k+2, 3n)\big)$

Result 5.6: $\gamma_{ssd}\big(HSN(7,q)\big) = \gamma_{ssd}\big(HSN(2k+1,q)\big)$

$\Rightarrow \gamma_{ssd}\big(HSN(7,[2k+1]n-k-2)\big) = \gamma_{ssd}\big(HSN(2k+1, 7n-5)\big)$

Result 5.7: $\gamma_{ssd}\big(HSN(9,q)\big) = \gamma_{ssd}\big(HSN(4k+3,q)\big)$

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$$\Rightarrow \gamma_{ssd}\big(HSN(9,[4k+3]n-2k-3)\big) = \gamma_{ssd}\big(HSN(4k+3,9n-6)\big)$$

Result 5.8: $\gamma_{ssd}\big(HSN(10,q)\big) = \gamma_{ssd}\big(HSN(4k+2,q)\big)$

$$\Rightarrow \gamma_{ssd}\big(HSN(10,[2k+1]n+k-1)\big) = \gamma_{ssd}\big(HSN(4k+2,5n+1)\big)$$

Theorem 5.9: For $p=5t, q \geq 3$ where $t=1,2,3\ldots$ the strong secure domination number is equal to other strong domination number of $HSN(p,q)$ under the condition

$$\gamma_{ssd}\big(HSN(5t,[2k+1]n+k-1)\big) = \gamma_{ssd}\big(HSN(t(2k+1),5n+1)\big)$$

Proof: Let $t=1, k=1, n=1$

LHS: $\gamma_{ssd}\big(HSN(5,[2k+1]n+k-1)\big) = \gamma_{ssd}\big(HSN(5,3n)\big) = 5(3+6n) = 45 \ldots (1)$

RHS: $\gamma_{ssd}\big(HSN(t[2k+1],5n+1)\big) = \gamma_{ssd}\big(HSN(3,6)\big) = 3(3+12) = 45 \ldots (2)$

Let $t=1, k=1, n=2$

LHS: $\gamma_{ssd}\big(HSN(5,[2k+1]n+k-1)\big) = \gamma_{ssd}\big(HSN(5,6)\big) = 5(3+12) = 75 \ldots (3)$

RHS: $\gamma_{ssd}\big(HSN(t[2k+1],5n+1)\big) = \gamma_{ssd}\big(HSN(3,11)\big) = 3(3+22) = 75 \ldots (4)$

By fixing $t=1, k=1$ and changing $n=1,2,3\ldots$ we forward to continue the comparison of $\gamma_{ssd}\big(HSN(5,q)\big)$ with $\gamma_{ssd}\big(HSN(3,q)\big)$

Let $t=2, k=1, n=3$

LHS: $\gamma_{ssd}\big(HSN(10,[2k+1]n+k-1)\big) = \gamma_{ssd}\big(HSN(10,9)\big) = 10(21) = 210 \ldots (5)$

RHS: $\gamma_{ssd}\big(HSN(t[2k+1],5n+1)\big) = \gamma_{ssd}\big(HSN(14,6)\big) = 14(15) = 210 \ldots (4)$

By fixing $t=2, k=1$ and changing $n=3,4,5\ldots$ we forward to continue the comparison of $\gamma_{ssd}\big(HSN(10,q)\big)$ with $\gamma_{ssd}\big(HSN(14,q)\big)$

Thus, by using $\gamma_{ssd}\big(HSN(5t,[2k+1]n+k-1)\big) = \gamma_{ssd}\big(HSN(t(2k+1),5n+1)\big)$ we obtain the result of comparison of $\gamma_{ssd}\big(HSN(5t,q)\big), t=1,2,3..$ with other $\gamma_{ssd}\big(HSN(p,q)\big)$.

Theorem 5.10:

If $p=8, q\geq 2$, then $\gamma_{ssd}\big(HSN(8,q)\big) \neq \gamma_{ssd}\big(HSN(p,q)\big)$ where $p \geq 2, q \geq 2$.

Note 5.11: The strong secure domination number of $HSN(3,q), HSN(9,q) HSN(15,q)$ are equal to one under the condition

$$\gamma_{ssd}\big(HSN(3,q)\big) = \gamma_{ssd}\big(HSN(9,q)\big) = \gamma_{ssd}\big(HSN(15,q)\big)$$

$$\Rightarrow \gamma_{ssd}\big(HSN(3,15n+6)\big) = \gamma_{ssd}\big(HSN(9,5n+1)\big) = \gamma_{ssd}\big(HSN(15,4n-2)\big)$$

That is, $\gamma_{ssd}\big(HSN(3,21)\big) = \gamma_{ssd}\big(HSN(9,6)\big) = \gamma_{ssd}\big(HSN(15,2)\big) = 135$

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$$\gamma_{ssd}\big(HSN(3,36)\big) = \gamma_{ssd}\big(HSN(9,11)\big) = \gamma_{ssd}\big(HSN(15,6)\big) = 225$$

Observation 5.12: Non – comparison of $\gamma_{ssd}\big(HSN(p,q)\big)$ with $\gamma_{ssd}\big(HSN(p,q)\big)$ where $2 \le p \le 10, 2 \le q \le 11$ are

1) $\gamma_{ssd}\big(HSN(2,q)\big) \neq \gamma_{ssd}\big(HSN(p,q)\big)$ where $p = 3,4,5,7,8,9,11$
2) $\gamma_{ssd}\big(HSN(3,q)\big) \neq \gamma_{ssd}\big(HSN(p,q)\big)$ where $p = 2,4,6,8,10$
3) $\gamma_{ssd}\big(HSN(4,q)\big) \neq \gamma_{ssd}\big(HSN(p,q)\big)$ where $p = 5,6,7,8,9,10,11$
4) $\gamma_{ssd}\big(HSN(5,q)\big) \neq \gamma_{ssd}\big(HSN(p,q)\big)$ where $p = 2,4,6,8,10$
5) $\gamma_{ssd}\big(HSN(6,q)\big) \neq \gamma_{ssd}\big(HSN(p,q)\big)$ where $p = 3,4,5,7,8,9,11$
6) $\gamma_{ssd}\big(HSN(7,q)\big) \neq \gamma_{ssd}\big(HSN(p,q)\big)$ where $p = 2,4,6,8,10$
7) $\gamma_{ssd}\big(HSN(8,q)\big) \neq \gamma_{ssd}\big(HSN(p,q)\big)$ where $p = 2,3,4 \ldots n$
8) $\gamma_{ssd}\big(HSN(9,q)\big) \neq \gamma_{ssd}\big(HSN(p,q)\big)$ where $p = 2,4,6,8,10$

Conclusion:

In this paper, we introduced the concept of domination of secure, strong secure, co – secure, strong co – secure sets of chemical networks on behave of their uniqueness in structure. Furthermore, the scope of this paper is in finding secure nodes of the development of electrical and different dimensional extension of peculiar networks.

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