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Solvability of Fractional Delay Equations with Impulsive Effects

Piyusha Somvanshi

Poornima College of Engineering Jaipur India
E mail : piyusha.susheel@gmail.com

Abstract

This paper investigates the existence and uniqueness of solutions for a class of Cauchy-type initial value problems involving fractional-order impulsive differential equations with infinite delay. By employing the method of successive approximations and constructing an appropriate solution operator, we establish sufficient conditions for the existence and uniqueness of mild solutions. An illustrative example is provided to demonstrate the applicability of the theoretical results

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1. Introduction

Over recent decades, fractional calculus has experienced remarkable development in both pure mathematics and applied sciences. However, labeling it as a "young" discipline would be a misjudgment, as its roots trace back to the early foundations of classical calculus. The term "fractional calculus" may misleadingly suggest operations involving fractions, but it actually refers to integration and differentiation of arbitrary (non-integer) order, which form the essence of this field today (see [1–5] and references therein).

In contemporary research, there is a growing interest in the study of fractional differential equations, particularly those involving delays and impulses. Numerous studies have focused on the existence of solutions under various conditions (see [6–8] and references therein).

Motivated by some recent works, we consider the following fractional impulsive differential equations with infinite delay.

$$\begin{cases} D_C^q x(t) = f(t, x(t)), & t \in J = [0,T],\ t \neq t_k; \\ \Delta x(t_k) = I_k(x(t_k)), & k = 1,2,\ldots\ldots m; \\ x(t) = \phi(t), & t \in [-\infty, 0], \end{cases} \tag{1}$$

Where D_C^q is the Caputo fractional derivative of order , $0<q<1$, $0 = t_0<t_1<t_2<\ldots<t_m<t_{m+1} = T$, $f\in([0,T]\times\mathbb{R},\mathbb{R})$and $I_k\in \mathbb{C}(\mathbb{R},\mathbb{R})$ are given functions satisfying some assumptions that will be specified later. $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^+)$ and $x(t_k^-)$

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represents the right and left limits of x (t) at $t=t_k$ respectively, and they satisfy that $x(t_k^-) = x(t_k)$. Now define

$$x_t(\theta) = x(t+\theta), \qquad \theta \in (\infty, 0].$$

Here x_t, represents the history of the state up to present time t.

Further, the construction of paper is in following way. In section 2, we give some preliminaries. In Section 3, we study the existence and uniqueness of mild solutions for the problem (1). At last an example is given to demonstrate the applicability of results in Section 4.

2. Preliminaries

In this section we recall some basic facts, definitions and propositions of fractional calculus which will be needed in the paper. (See [9, 10]).

Definition 2.1.(See[11,12]). "*The fractional integral of order q with the lower limit zero for a function f is defined as*

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{f(s)}{(t-s)^{1-q}} ds, \quad t > 0, \quad q > 0.$$

Where Γ is the gamma function."

Definition 2.2.(See [11,12]). "*The Riemann-Liouville derivative of order q with the lower limit zero for a function f is defined as:*

$${}^{RL}D^q f(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-q-1} f(s) ds, \qquad n-1 < q < n, t > 0."$$

Definition 2.3.(See [11.12]). "*The Caputo derivative of order q with the lower limit zero for a functions f is defined as*

$$D_C^q = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^n(s)}{(t-s)^{q+1-n}} ds, \qquad t > 0, \qquad 0 < n-1 < q < n".$$

Let $J = [0,T]$ and $J' = J \setminus \{t_1, t_2, \ldots t_m\}$. We denote ℙℂ (J) = {u : $[0,T] \rightarrow$ ℝ | u ∈ ℂ (J,ℝ), and $u(t_k^-)$ exists and $u(t_k^-) = u(t_k), k = 1,2,\ldots,m\}$. Obviously ℙℂ (J) is a Banach space with the norm $\|u\| = \sup_{t \in J} |u(t)|$."

Lemma 2.4. According to X. Zhang (See [13]), we give following Lemma:

"*Assume that y ∈ ℂ ([0,T], ℝ) then a function x ∈ PC (J) is a solution of Cauchy problem*

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$$\begin{cases} D_C^q x(t) = y(t, x(t)), & t \in J = [0, T], t \neq t_k; \\ \Delta x(t_k) = I_k(x(t_k)), & k = 1,2,\dots\dots m; \\ x(t) = \phi(t), & t \in [-\infty, 0], \end{cases} \tag{2}$$

Iff x satisfies following equation

$$x(t) = \begin{cases} \phi(t), \quad t \in [-\infty, 0]; \\ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \; y(s)ds + \sum_{j=1}^{k} I_j x(t_j) \\ + \sum_{i=0}^{k-1} \frac{1}{\Gamma(q)} \int_{t_i}^{t_i+1} (t_{i+1} - s)^{q-1} y(s)ds, \quad t \in (t_k, t_{k+1}], k = 0,1,2,\dots\dots m. \end{cases} \tag{3}$$

Proof. Assume that x satisfies the integral equation (3). We have ϕ (0) = 0 and

$$x(t) = \frac{1}{\Gamma(q)} \int_{0}^{t} (t-s)^{q-1} y(s)ds, \qquad t \in [t_0, t_1].$$

since $x(t_1^+) - x(t_1^-) = I_1(x(t_1))$, hence we get

$$x(t_k^+) = I_1(x(t_1)) + \frac{1}{\Gamma(q)} \int_{0}^{t_1} (t_1 - s)^{q-1} y(s)ds.$$

It follows that for t∊ (t1,t2],

$$x(t) = x(t_1^+) + \frac{1}{\Gamma(q)} \int_{t_1}^{t} (t-s)^{q-1} y(s)ds$$

$$= \frac{1}{\Gamma(q)} \int_{t_1}^{t} (t-s)^{q-1} y(s)ds + \frac{1}{\Gamma q} \int_{0}^{t_1} (t-s)^{q-1} y(s)ds + I_1(x(t_1))$$

In consequence, we can see, by means of, $x(t_2^+) = x(t_2^-) + I_2(x(t_2))$, that

$$x(t_2^+) = \sum_{i=0}^{1} \frac{1}{\Gamma(q)} \int_{t_1}^{t_{i+1}} (t_{i+1} - s)^{q-1} y(s)ds + \sum_{j=1}^{2} I_j(x(t_j)),$$

Which shows that for t∊ (t2,t3].

$$x(t) = \frac{1}{\Gamma(q)} \int_{t_2}^{t} (t-s)^{q-1} y(s)ds + \sum_{i=0}^{k-1} \frac{1}{\Gamma(q)} \int_{t_i}^{t_{i+1}} (t_{i+1} - s)^{q-1} y(s)ds + \sum_{j=1}^{k} I_j(x(t_j)),$$

By iteration, the solution x(t) for t∈ (t_k, t_{k+1}] can be written as

$$x(t)=\frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}y(s)ds+\sum_{i=0}^{k-1}\frac{1}{\Gamma(q)}\int_{t_i}^{t_{i+1}}(t_{i+1}-s)^{q-1}y(s)ds+\sum_{j=1}^{k}I_j\,(x(t_j)),$$

Conversely, if x is a solution of problem (1), then it can be easily seen by direct computation, that $D^q x(t)=y(t)$, $t\neq t_k$, $t\in[0,T]$ and $\Delta x(t)=x(t_k^+)-x(t_k^-)=I_k(x(t_k))$, where

$$x(t_k^+)=\sum_{i=0}^{k-1}\frac{1}{\Gamma(q)}\int_{t_i}^{t_{i+1}}(t_{i+1}-s)^{q-1}y(s)ds+\sum_{j=1}^{k}I_j\,(x(t_j)),$$

and

$$x(t_k^-)=\frac{1}{\Gamma(q)}\int_{t_{k-1}}^{t_k}(t-s)^{q-1}y(s)ds+\sum_{i=0}^{k-2}\frac{1}{\Gamma(q)}\int_{t_i}^{t_{i+1}}(t_{i+1}-s)^{q-1}y(s)ds+\sum_{j=1}^{k-1}I_j(x(t_j)).$$

In this way the proof of the Lemma is completed.”

3. Main results

Firstly, set $\mathbb{C}_0=\{z|z\in\mathbb{C}([0,T],\mathbb{R}),z(0)=0\}$. For each z ∈ $\mathbb{C}_0$, we denote by z the function defined by

$$\bar{z}(t)=z(t),\ 0\le t\le T, and\ \bar{z}\,(t)=0,\quad -\infty\le t\le 0. \tag{4}$$

If x is solution of (1), then x (.) can be decomposed as$x$(t) = $\bar{z}$ (t) +φ(t) for - ∞ ≤ t ≤ T, which implies that $x_t = \bar{z}_t + \varphi_t$for 0 ≤ t ≤ T, where

$$\varphi(t)=0,\ 0\le t\le T, and\ \varphi(t)=\varphi(t),\quad -\infty\le t\le 0. \tag{5}$$

Therefore, the problem (1) can be transformed into the following fixed point problem of the operator F : $\mathbb{C}_0\rightarrow\mathbb{R}$,

$$\begin{aligned}Fz(t)\ &=\frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}\,f\,(s,\bar{z}_s+\varphi_s)ds\\ &+\sum_{i=0}^{k-1}\frac{1}{\Gamma(q)}\int_{t_i}^{t_{i+1}}(t_{i+1}-s)^{q-1}\,f\,(s,\bar{z}_s+\varphi_s)ds\\ &+\sum_{j=1}^{k}I_j\,(\bar{z}(t_j)),\ t\in(t_k,t_k+1],\ k=0,1,......m.\end{aligned} \tag{6}$$

Now, let us present our main result.

Theorem 3.1. *For the functions $f \in \mathbb{C}\ ([0,T]\ \times \mathbb{R},\mathbb{R})$ and $I_k : \mathbb{R} \to \mathbb{R}$, assume the following conditions hold.*

(H1) There exists a continuous function $a : [0,T] \to \mathbb{R}^+$ satisfying

$$|f(t,u_t)-f(t,v_t)| \le a(t) \sup_{s\in[0,t]} |u(s)-v(s)|,\ u,v\in R,\quad t\in[0.T];$$

(H2) There exists a constant $L_k > 0$ such that $|I_k(u) - I_k(v)| \le L_k|u-v|$, $k=1,2,\ldots.,m$;

(H3) $\displaystyle\sum_{i=1}^{m+1} \frac{a_i T^q}{\Gamma(q+1)} + \sum_{j=1}^{m} L_j < 1, where\ a_k\ =\ \sup_{t\in(i_k,t_{k+1})} a(t);$

(H4) There exists a constant $M > 0$ such that $|f(t,\varphi t)| \le M$, where φ is defined in (5).

Proof . We complete the proof, via method of successive approximations. Define a sequence of functions $z_n : [0,T] \to \mathbb{R}$, n = 0,1,2,…. as follows:

$$z_0(t)=0,\qquad z_n(t)=Fz_{n-1}(t). \tag{7}$$

Since $z_0(t)= 0$, it is easy to see from (4) that $(\bar{z}_0)_s = 0$ for $s \in [0,T]$. Thus we have

$$\begin{aligned}
|z_1(t)-z_0(t)| \ &\le\ \frac{1}{\Gamma q}\int_{t_k}^{t}(t-s)^{q-1}|f(s,\varphi_s)|\,ds+\sum_{j=1}^{k} I_j(0) \\
&+\sum_{i=0}^{k-1}\ \frac{1}{\Gamma q}\int_{t_i}^{t_{i+1}}(t_{i+1}-s)^{q-1}|f(s,\varphi_s)|\,ds \\
&\le \frac{M(t-t_k)^q}{\Gamma(q+1)}+\sum_{i=1}^{k}\ \frac{M(t_i-t_{i-1})^q}{\Gamma(q+1)}\ +\sum_{j=1}^{k} I_j(0)| \\
&\le \sum_{i=1}^{m+1}\ \frac{M(t_i\ -t_{i-1})^q}{\Gamma(q+1)}\ +\sum_{j=1}^{m}|I_j(0)| := N_0,\quad k\ =1,2,\ldots,m,
\end{aligned}$$

it follows that $\|z_1 - z_0\| \le N_0$. Furthermore,

$$\begin{aligned}
&|z_n(t)-z_{n-1}(t)| \\
\le\ &\ \frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}|f(s,(\bar{z}_{n-1})_s+\varphi_s)-f(s,(\bar{z}_{n-2})_s+\varphi_s)|\,ds \\
&+\ \sum_{i=0}^{k-1}\ \frac{1}{\Gamma(q)}\int_{t_i}^{t_{i+1}}(t_{i+1}-s)^{q-1}|f(s,(\bar{z}_{n-1})_s+\varphi_s)-f(s,(\bar{z}_{n-2})_s+\varphi_s)|\,ds \\
&+\ \sum_{j=1}^{k}|I_j(\bar{z}_{n-1}(t_j))-I_j\ (\bar{z}_{n-2}(t_j))|
\end{aligned}$$

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$$\leq \quad \frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}a(s)\underset{r\in[0,s]}{SUP}|\bar{z}_{n-1}(r) \;-\bar{z}_{n-2}(r)|\,ds$$

$$+\sum_{i=0}^{k-1}\quad\frac{1}{\Gamma(q)}\int_{t_i}^{t_{i+1}}(t_{i+1}-s)^{q-1}a(s)\underset{r\in[0,s]}{SUP}|\bar{z}_{n-1}(r) \;-\bar{z}_{n-2}(r)|\,ds$$

$$+\sum_{j=1}^{k}|L_j(\bar{z}_{n-1}(t_j)-\bar{z}_{n-2}(t_j))|$$

$$\leq \left(a_k\frac{(t-t_k)^q}{\Gamma(q+1)}+\sum_{i=1}^{k}a_i\frac{(t_i-t_{i-1})^q}{\Gamma(q+1)}+\sum_{J=1}^{K}L_j\right)||\bar{z}_{n-1}-z_{n-2}||$$

$$\leq \left(\sum_{i=1}^{m+1}a_i\frac{T^q}{\Gamma(q+1)}+\sum_{J=1}^{m}L_j\right)||\bar{z}_{n-1}-z_{n-2}||$$

$$:= \quad N||z_{n-1}-z_{n-2}||, \tag{8}$$

which implies that $||z_n\text{-}z_{n-1}|| \leq N\ ||\ z_{n-1}\text{-}\ z_{n-2}||$ with $N < 1$. Note that for any $m > n > 0$, we have

$$||z_n=z_n|| \;\leq\; ||z_{n+1}-z_n||+||z_{n+2}-z_{n+1}||+....+||z_m=z_{m-1}||$$

$$\leq\;(N^n+N^{n+1}+....+N^{m-1})||z_1=z_0||$$

$$\leq\;\frac{N^n}{1-N}||z_1=z_0||. \tag{9}$$

If m,n are sufficiently large numbers then it follows form the above inequalities with $N < 1$ that $||z_m\text{-}z_n|| \to 0$. Thus $\{z_n(t)\}$ is a Cauchy sequence in $\mathbb{PC}$ *(J)*. Since $\mathbb{PC}$ *(J)* is a complete Banach space, then $||z_n\text{-}z|| \to 0$ as $n\to\infty$ for some $z \in \mathbb{PC}$ *(J)*, which means the z_n(t) is uniformly convergent to $z(t)$ with respect to t.

Thus, we will reach to the conclusion that z(t) is a solution of the equation (1). Observe that

$$\left|\frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}\;f(s,(\bar{z}_n)_s+\varphi_s)\,ds-\frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}\;f(s,\bar{z}_s+\varphi_s)ds\right|$$

$$\leq \frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}|f(s,(\bar{z}_n)_s+\varphi_s)-f(s,\bar{z}_s+\varphi_s)|\,ds$$

$$\leq \frac{1}{\Gamma(q)}\int_{t_k}^{t}a(t)(t-s)^{q-1}\sup_{r\in[0,s]}|\bar{z}_n(r)-\bar{z}\;(r)|\,ds$$

$$= \frac{1}{\Gamma(q)}\int_{t_k}^{t}a(t)(t-s)^{q-1}\sup_{r\in[0,s]}|\bar{z}_n(r)-z\;(r)|\,ds$$

Since $z_n(t) \to z(t)$ as $n \to +\infty$, for any$\varepsilon > 0$, there exists a sufficiently large *number* $n_0 > 0$ such that for all $n > n_0$, we have

$$|z_n(r) - z(r)| < \min \{ \frac{\Gamma(q+1)}{\sum_{i=0}^{m} a_i T^q}\varepsilon, \frac{1}{\sum_{j=1}^{m} L_j}\varepsilon \}.$$

Therefore,

$$\left| \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} f(s,(\bar{z}_n)_s, +\varphi_s)ds - \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} f(s,\bar{z}_s + \varphi_s)ds \right| < \varepsilon, \tag{10}$$

and

$$\begin{aligned} & \Bigg| \sum_{i=0}^{k-1} \frac{1}{\Gamma(q)} \int_{ti}^{t_{i+1}} (t_{i+1}-s)^{q-1} f(s,(\bar{z}_n)_s, +\varphi_s)ds \\ & -\sum_{i=0}^{k-1} \frac{1}{\Gamma(q)} \int_{ti}^{t_{i+1}} (t_{i+1}-s)^{q-1} f(s,(\bar{z}_s)_s, +\varphi_s)ds \Bigg| \\ \leq & \sum_{i=0}^{k-1} \frac{1}{\Gamma(q)} \int_{ti}^{t_{i+1}} (t_{i+1}-s)^{q-1} \, |f(s,(\bar{z}_n)_s, +\varphi_s) - f(s,(\bar{z}_s + \varphi_s)| ds \\ \leq & \sum_{i=0}^{k-1} a(t_i) \frac{(t_i - t_{i-1})^q}{\Gamma(q+1)} \sup_{r\in[0,s]} |z_n(r) - z(r)| ds < \varepsilon, \end{aligned} \tag{11}$$

and

$$\begin{aligned} & \Bigg| \sum_{j=1}^{k} I_j(\bar{z}_n(t_j)) - \sum_{j=1}^{k} \, |I_j(\bar{z}(t_j)) \Bigg| \\ \leq & \sum_{j=1}^{k} L_j(\bar{z}_n(t_j) - \bar{z}(t_j)| \\ = & \sum_{j=1}^{k} L_j |z_n(t_j) - z(t_j)\|| < \varepsilon. \end{aligned} \tag{12}$$

In lieu of the same, for a sufficiently large number $n > n_0$.

$$\begin{aligned} & |z(t) - Fz(t)| \\ \leq & \; |z(t) - z_{n+1}(t)| + |z_{n+1}(t) - Fz_n(t)| + |Fz_n(t) - Fz(t)| \\ \leq & \; |z(t) - z_{n+1}(t)| + \Bigg| \; z_{n+1}(t) - \Bigg[\; \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} f(s,(\bar{z}_n)_s + \varphi_s)ds \end{aligned}$$

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$$+\sum_{i=0}^{k-1}\frac{1}{\Gamma(q)}\int_{t_i}^{t_{i+1}}(t_{i+1}-s)^{q-1}\, f(s,(\bar{z}_n)_s+\varphi_s)ds+\sum_{j=1}^{k}I_j(\bar{z}_n(t_j))\ \Bigg]\Bigg\|$$

$$+\left|\ \frac{1}{\Gamma(q)}\int_{t_k}^{t}\ (t-s)^{q-1}\, f(s,(\bar{z}_n+\varphi_s)ds-\frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}f(s,(\bar{z}_n)_s+\varphi_s)ds\ \right|$$

$$+\ \left|\sum_{i=0}^{k-1}\ \frac{1}{\Gamma(q)}\int_{t_i}^{t_{i+1}}\ (t_{i+1}-s)^{q-1}\, f(s,(\bar{z}_s+\varphi_s)ds\right.$$

$$\left.-\sum_{i=0}^{k-1}\ \frac{1}{\Gamma(q)}\int_{t_i}^{t_{i+1}}\ (t_{i+1}-s)^{q-1}\, f(s,(\bar{z}_s)_s+\varphi_s)ds\ \right|$$

$$+\left|\ \sum_{j=1}^{k}I_j(\bar{z}_n(t_j))-\sum_{j=1}^{k}I_j\ (\bar{z}_n(t_j))\ \right|$$

Thus, as per convergence of the two previous and equations (10)-(12), one obtains that $|z(t)-Fz(t)|\to 0$, which implies that z is a solution of (1).

Finally, we prove the uniqueness of the solution . Assume that $z_1,z_2 :[0,T]\to \mathbb{R}$ are two solution of (1) , Note that

$$|\,z_1(t)-z_2(t)\,|$$

$$\leq\frac{1}{\Gamma(q)}\int_{t_k}^{t}\ (t-s)^{q-1}\ a(s)\ \sup_{r\in[0,s]}\ |\,\bar{z}_1\ (r)-\bar{z}_2\ (r)\,|\,ds$$

$$+\sum_{i=0}^{k-1}\frac{1}{\Gamma(q)}\int_{t_i}^{t_{i+1}}(t_{i+1}-s)^{q-1}\ a(s)\ \sup_{r\in[0,s]}\ |\,\bar{z}_1\ (r)-\bar{z}_2\ (r)\,|\,ds+\sum_{j=1}^{k}L_j\ |\bar{z}_1(t_j)-\bar{z}_2(t_j)\,|$$

$$\leq\left(\ \sum_{i=i}^{m+1}\ \frac{a_iT^q}{\Gamma(q+1)}+\sum_{j=1}^{m}L_j\ \right)\ .\ \ \|\,z_1-z_2\,\|\,.$$

According to the condition (H3), the uniqueness of the problem (1) follows immediately, which completes the proof.

4. An Example

In this section we give an example to illustrate the above results. Consider the following impulsive partial hyperbolic fractional differential equations with infinite delay.

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$$\begin{cases} D^q u(t) = \dfrac{e^t}{(9+e^t)} \dfrac{|u_t|}{(1+|u_t|)} & t \in [0,1],\ t \notin \tfrac{1}{2},\ 0 < q < 1; \\ \Delta u\left(\tfrac{1}{2}\right) \ = \ \dfrac{|u(\tfrac{1}{2}^{-})|}{1/4 + |u(\tfrac{1}{2}^{-})|} & \\ u(t) \ = \ \phi(t) = \dfrac{e^{-t}-1}{2}, & -\infty \le t \le 0, \end{cases} \tag{13}$$

where $0 < q < 1$, $\Gamma\,(q+1) < \dfrac{3}{10}$.

Set,

$$f(t,u) \ = \ \frac{e^{-t}u}{(9+e^t)(1+u)},\ I(u) = \frac{u}{1/4+u},\ \text{for } (t,u) \in [0,1] \times [0,+\infty]$$

It is clear that the functions *f* and *I* are continuous. Now we have .

$$|f(t,u_t) - f(t,v_t)| = \frac{e^{-t}}{(9+e^t)} \frac{(|u_t| - |v_t|)}{(1+|v_t|)(1+|v_t|)}$$

$$\le \frac{e^{-t}}{(9+e^t)} |u_t - v_t|$$

$$\le \ a(t) \sup_{s\in[0,1]} |u(s) - v(s)|,$$

Where $a(t) = \dfrac{1}{10}$. So the condition (H1) of Theorem 3.1 is satisfied. also we have

$$|I(u) - I(v)| = \left| \ \frac{u}{\frac{1}{4}+u} - \frac{v}{\frac{1}{4}+v} \ \right|$$

$$= \frac{1}{4} \left| \ \frac{|u| - |v|}{(\frac{1}{4}+u)\,(\frac{1}{4}+v)} \ \right|$$

$$= \frac{1}{4} |u - v|,$$

Where $L = \dfrac{1}{4}$. So the condition (H2) is also satisfied. Now it is easy to conclude

$$\sum_{i=1}^{m+1} \frac{a_1 T^q}{\Gamma(q+1)} + \sum_{j=1}^{m} L_j = \frac{1}{10} \frac{1}{\Gamma(q+1)} \frac{1}{4} < 1$$

and

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$$| f(t,u_t) | = \frac{e^{-t}}{(9+e^{t})} \frac{| u_t |}{(1+| u_t |)} \leq \frac{e^{-t}}{(9+e^{t})} \leq \frac{1}{10}, t \in [0, 1].$$

Thus, the equation (4) satisfies all the conditions given in Theorem 3.1, which implies that the (13) has a unique solution.

5. Conclusion

In this paper, existence and uniqueness for a class of Cauchy initial value problem with impulses and infinite delay is discusses. A better and simple method is set to get criteria of existence and uniqueness of the solution to such problems by using successive approximation . Such type of equations arise in real world phenomena like oscillation with discontinuities, etc. One can deduce slightly different results by taking different time variables.

6. Competing interest

Both the authors do not have any competing interests.

7. Author's contribution

Both the researchers have given equal input in preparing the paper.

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