Journal of Computational Analysis and Applications **VOL. 33, NO. 8, 2024**
10.48047/jocaaa.2024.33.08.205

ON THE FORCING SIGNAL NUMBER OF A GRAPH

Dr J Christabel Sudha[+], **Dr S Elizabeth Bernie**[*]

[+]Assistant Professor of Mathematics

Pioneer Kumaraswamy College at Nagercoil

Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli – 627012, Tamilnadu, India.

Tamil Nadu, India.

sudhachristabel@gmail.com

[*]Assistant Professor of Humanities and Science,

St. Xavier's Catholic College of Engineering (Autonomous) at Chunkankadai

Affiliated to Anna University, Chennai, Tamilnadu, Inida

elizabeth@sxcce.edu.in

Abstract

For two vertices x and y of a graph G, the set $L[x, y]$ consists of x and y and all vertices lying on some $x - y$ geosig of G and for a non-empty set $S \subseteq V(G), L[S] = \cup_{x,y \in S} L[x, y]$.

A set $S \subseteq V(G)$ is said to be a signal set of G if $L[S] = V(G)$. The minimum cardinality of a signal set is known as signal number and is denoted by $S(G)$. A subset T of a minimum signal set S is called a forcing subset for S if S is the unique minimum signal set containing T. The forcing signal number $fs_G(S)$ of S is the minimum cardinality among the forcing subsets of S and the forcing signal number $f_s(G)$ of G is the minimum forcing signal number among all minimum signal sets of G. In this paper, the forcing signal number of several classes of graphs are determined some its general properties also studied.

Keywords: Signal set, Signal number, forcing signal number.

AMS Subject Classification number: 05C12

1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by m and n respectively. For basic definitions and terminologies, we refer to [1]. For any vertex v in $V(G)$, the open neighbourhood $N(v)$ is the set of all vertices adjacent to that v and $N[v] = N(v) \cup \{v\}$ is the closed neighbourhood of v. Let $\Delta = \Delta(G)$ and $\delta = \delta(G)$ denote the maximum and minimum degree of G, respectively. A vertex v is said to be an extreme vertex of G, if its neighbourhood $N(v)$ induces a complete sub graph of G. If G is a connected graph, then the distance denoted by $d(x, y)$ is the length of a shortest $x - y$ path in G.

A set $S \subseteq V(G)$ is said to be a signal set of G if $L[S] = V(G)$. The minimum cardinality of a signal set is known as signal number and is denoted by $S_n(G)$.

We present some basic information in this area that help in the creation of the paper. In section 2, we defined and demonstrated the forcing signal number of a graph. Section 3 contains the paper's conclusion. In the sequel, the following results are used.

Theorem 1.2 [6]

$S_n(G) = 2$ if and only if there exist vertices u, v such that v is an $u -$ signal vertex of G.

Theorem 1.3 [2]

For any connected graph $G, 2 \leq S_n(G) \leq n$.

2. Forcing Signal number of a graph

In this section we define the forcing signal number $f_s(G)$ of a graph and initiate a study of this parameter.

Definition: 2.1

Let G be a connected graph and S be a minimum signal set of G. A subset $T \subseteq S$ is called a forcing subset of S, if S is the unique minimum signal set containing T. The forcing subset of S of minimum cardinality is the minimum forcing subset of S. The forcing signal number of S denoted by $f_s(S)$ is the cardinality of the minimum forcing subset of S and is given by $f_s(G) = \min \{f_s(S)$, where the minimum is taken over all minimum signal sets of G.

Example 2.2

For the graph G in Figure 1, $S = \{x_1, x_4\}$ is the unique minimum signal set of G and so $f_s(G) = 0$.

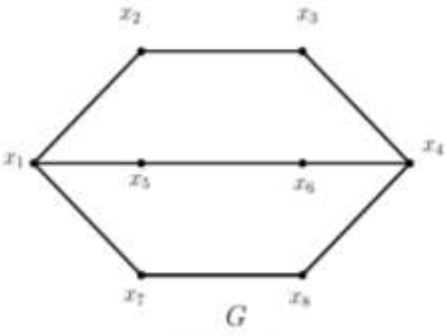

Figure 1

Also the graph G in Figure 2, $S_1 = \{x_1, x_3, x_6\}, S_2 = \{x_1, x_4, x_5\}, S_3 = \{x_1, x_3, x_5\}, S_4 = \{x_1, x_4, x_6\}, S_5 = \{x_1, x_4, x_7\}$ and $S_6 = \{x_1, x_2, x_5\}$ are the minimum signal sets of this G.

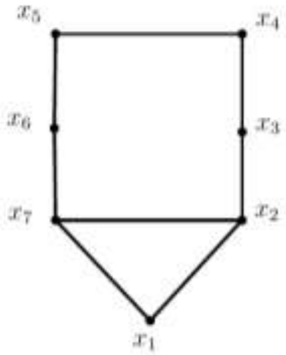

Figure 2

Since S_5 and S_6 are the only minimum sign set containing v_7 and v_5 respectively. Thus$f_s(S_5) = f_s(S_6) = 1$. And no other possible vertices in G belongs to only one minimum signal set and thus, $f_s(S_i) > 2$ for $1 \le i \le 4$. Hence $f_s(G) = 1$.

Theorem 2.3

For any connected graph $G, 0 \le f_s(G) \le s_n(G) \le n$.

Proof:

From the definition of forcing signal set, $f_s(G) \ge 0$. Clearly, $f_s(G) \le S_n(G)$ and $f_s(G) = \min\{f_s(G)\}$, for every minimum signal set $S, f_s(G) \le s_n(G)$. Also, the set of all vertices form a signal set of G and hence, $s_n(G) \le n$. Thus, $0 \le f_s(G) \le s_n(G) \le n$.

Remark: 2.4

The bounds in Theorem 2.3 are strict. For the graph G given in Figure 1, $f_s(G) = 0$. For the graph G given in Figure 2, $V(G) = 7, s_n(G) = 3$ and $f_s(G) = 1$. Thus, $0 < f_s(G) < s_n(G) < n$.

Theorem: 2.5

Let G be any connected graph. Then,

(i) $f_s(G) = 0$ if and only if G has a unique minimum signal set.

(ii) $f_s(G) = 1$ if and only if G has at least two minimum signal sets, one of which is the unique minimum signal set containing one of its elements.

(iii) $f_s(G) = s_n(G)$ if and only if no minimum signal set of G is the unique minimum signal set containing any of its proper subsets.

Proof:

(i) Assume $f_s(G) = 0$. Then by the definition 2.1, $f_s(G) = 0$ for some signal set S of G and so the empty set ϕ is the minimum forcing subset for S. Since the empty set ϕ is a subset for every set, it follows S is the unique minimum signal set of G. Conversely, assume that S is the unique signal set. It is so clear that ϕ is the forcing subset of S. Hence, $f_s(G) = 0$.

(ii) Assume $f_s(G) = 1$. Then by Theorem 2.5 (i), G has at least two minimum signal sets. Also, since $f_s(G) = 1$, there is a singleton subset $S_1$of a minimum signal set S of G such that $S_1$is not a subset of any other minimum signal set of G. Thus that S is the unique minimum signal set containing one of its elements.

Conversely, assume that G has at least two signal sets, there exists an element in one of the signal sets which is not in any other signal set. Hence $f_s(G) = 1$.

(iii) Let $f_s(G) = s_n(G)$. Then $f_s(G) = s_n(G)$ for every minimum signal set S in G. also, by Theorem 1.3, $s_n(G) \geq 2$ and so $f_s(G) \geq 2$. Then by Theorem 2.5 (i), G has at least two minimum signal set of G. Since $f_s(G) = s_n(G)$, no proper subset of S is a forcing subset for S. Hence no signal set of G is the unique signal set containing any of its proper subsets.

Conversely assume that there is no minimum signal set of G is the unique minimum signal set of G is the unique minimum signal set containing any of its proper subsets we prove that $f_s(G) = s_n(G)$. By our assumption G contains more than one minimum signal set and no subset of any minimum signal set S other than S is a forcing subset for S. Hence $f_s(G) = s_n(G)$.

Definition: 2.6

A vertex x of a connected graph G is said to be a signal vertex of G of x belongs to every minimum signal set of G.

Theorem: 2.7

Let G be a connected graph and let S be a minimum signal set of G. Then no signal vertex of G belongs to any minimum forcing set of G.

Proof:

Let G be a connected graph and S be a minimum signal set of G. Let x be a signal vertex of G. Then by definition x belongs to every minimum signal set S of G. Let $T \subseteq S$ be any minimum forcing subset for any minimum signal set S of G. We claim that $x \notin T$. Suppose $x \in T$. Then $T' = T - \{x\}$ is a proper subset of T such that T' is the unique minimum signal set containing T' so that T' is a forcing subset for S with $|T'| < |T|$, which is a contradiction to T is a minimum forcing subset for S. Thus, $x \notin T$. Therefore no signal vertex of G belongs to any minimum forcing set for S.

Theorem: 2.8

Let G be a connected graph and T be the set of all signal vertices of G. Then $f_s(G) \leq s_n(G) - T$.

Proof:

Let S be any minimum signal set of G. Then $s_n(G) = |S|, T \subseteq S$ and S is the unique minimum signal set containing $S - T$. Hence $f_s(G) \leq |S - T| = |S| - |T| = s_n(G) - T$.

Corollary: 2.9

If G is a connected graph with k extreme points, then $f_s(G) \leq s_n(G) - k$.

Theorem: 2.10

For any complete graph $G = K_n$ $(n \geq 2)$ or any non-trivial tree G, $f_s(G) = 0$.

Proof:

For $G = K_n$, it is clear that the set of all vertices of G is the unique minimum signal set. Therefore, by Theorem 2.5 (i), it follows that $f_s(G) = 0$.

If G is a non-trivial tree, then the set of all end vertices of G is the unique minimum signal set of G and so by Theorem 2.5 (i), $f_s(G) = 0$.

Theorem: 2.11

For any even cycle $G = C_n(n \geq 4)$, a set $S \subseteq V(G)$ is a minimum signal set of G if and only if S consists of two antipodal vertices.

Proof:

If S contains only the two antipodal vertices, then it is clear that S is a minimum signal set of G. Conversely, let S be any minimum signal set of G. Then $s_n(G) = |S|$. Let S' be any set of two antipodal vertices of G. Then as in the first part of this theorem, S' is a minimum signal set of G. Thus $|S| = |S'|$. So S contains two vertices, say $S = \{x, y\}$. If x and y are not antipodal, then any part of vertices that is not on the $x - y$ geosig. Hence S is not a minimum signal set, which is a contradiction. Thus S consists of two antipodal vertices.

Theorem: 2.12

For any cycle $C_n(n \geq 4), f_s(C_n) = \begin{cases} 1 & if\ n\ is\ even \\ 2 & otherwise \end{cases}$.

Proof:

If n is even, then $s_n(C_n) = 2$ and by Theorem 2.11, every minimum signal set of C_n consists of pair of antipodal vertices. But C_n does not have a unique minimum signal set because C_n has $\frac{n}{2}$ minimum signal sets of C_n. Moreover, every vertex of C_n has a unique vertex which is antipodal to it and so by Theorem2.5 (ii), $f_s(C_n) = 1$.

If n is odd, then $s_n(C_n) = 3$. Also it is clear that C_n contains more than one minimum signal set. Again every vertex of C_n belongs to at least two distinct minimum signal sets and so $f_s(G) \geq 2$. other hand, for every pair x, y of adjacent vertices in C_n, there is a unique vertex z in C_n such that $d(x,z) = d(y,x)$. Therefore, it follows that $\{x, y, z\}$ is the unique minimum signal set of G containing $\{x, y\}$. This shows that $f_s(C_n) = 2$.

Theorem: 2.13

If G is a connected graph with $s_n(G) = 2$, then $f_s(G) < 2$.

Proof:

Let $S = \{x, y\}$ be a signal set of G. Then $d(x,y) = diam(G)$ and every vertex of G lies on some $x - y$ geosig of G. To prove $f_s(G) < 2$. By Theorem 2.3, we have $f_s(G) \leq 2$. Suppose $f_s(G) = 2$. Then there exists a vertex $z \neq y$ such that $\{x, z\}$ is also a minimum signal set of G. it follows that z lies on $x - y$ – geosig of G. This shows that $d(x,z) < d(x,y) = diam(G)$, which is a contradiction. Hence $f_s(G) < 2$.

3. Conclusion

In this paper, we present an in depth of forcing signal number of a graph focusing on their mathematical properties and applications.

References

[1] F. Buckley and F. Harary, Distance in Graphs, Addison-Wesley, Redwood City, CA (1990)

[2] S. Balamurgan and R. Antony Dass, The Edge Signal Number of a graph, Discrete Mathematics Algorithms and Applications, Vol. 13(3), Article umber 2150024 (2021)

[3] G. Chartrand and F. Harary, On the geodetic number of a graph, Networks, Vol. 39(1) (2002)

[4] K. Kathiresan and R. Sumathi, A study on Signal Distance in graphs, Algebra, graph Theory and its applications, Narosa Publishing house Pvt. Ltd. (2009)

[5] X. Lenin Xaviour and S. Ancy May, On Double Signal number of a graph, Ural Mathematical Journal, Vol. 8(1) (2022)

[6] S. Sethu Ramalingam and S. Balamurugan, On the Signal Distance in Grpahs, Ars Combinatoria (2018)