10.48047/jocaaa.2024.33.08.211

Co-secure Semitotal Domination in Graph

A. Punitha Tharani[1], D. Joslin Amala[2*]

[1]Associate Professor, Department of Mathematics, St. Mary's College (Autonomous), Thoothukudi-628001, Tamil Nadu, India, Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012, Tamil Nadu, India.

[2]Research Scholar, (Register No.22212212092001), Department of Mathematics, St. Mary's College (Autonomous), Thoothukudi-628001, Tamil Nadu, India, Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012, Tamil Nadu, India.

E-mail: [1]punitha_tharani@yahoo.co.in , [2]joslinamala123@gmail.com

ABSTRACT: In this paper, we introduce a new domination parameter called the *co-secure semitotal domination number*. A set D of vertices in a graph G with no isolated vertices is a co-secure semitotal dominating set of G if D is a semitotal dominating set of G and for each $u \in D$ there exists a vertex $v \in V \setminus D$ such that $uv \in E$ and $(D \setminus \{u\}) \cup \{v\}$ is a semitotal dominating set of G. The minimum cardinality of a co-secure semitotal dominating set of G is called the co-secure semitotal domination number and is denoted by $\gamma_{cst2}(G)$. We investigate the basic properties of this parameter and establish relationships with known domination parameters. Exact values and bounds for $\gamma_{cst2}(G)$ are determined for various standard graphs. This study opens a new directions in the ongoing development of domination theory in graphs.

Keywords: graphs, co-secure semitotal dominating set, co-secure semitotal domination number

INTRODUCTION: The graphs considered here are simple, finite, nontrivial, undirected and without isolated vertices. The graph $G = (V, E)$ considered here have $n = |V|$ vertices and $m = |E|$ edges. A set D of vertices in a graph G is a dominating set if every vertex not in D is adjacent to at least one vertex in D. The domination number $\gamma(G)$ of G is the order of a smallest dominating set in G. Any dominating set with $\gamma(G)$ vertices is called a $\gamma(G)$ – set of G. The concept of co-secure domination was introduced by S. Arumugam, Karam Ebadi, Martin Manrique[1]. A dominating set D of a graph G is called a co-secure dominating set, if for each $u \in D$ there exists a vertex $v \in V \setminus D$ such that $v \in N(u)$ and $(D \setminus \{u\}) \cup \{v\}$ is a dominating set of G. The co-secure domination number $\gamma_{cs}(G)$ is the minimum cardinality of a co-secure dominating set of G. Any co-secure dominating set with $\gamma_{cs}(G)$ vertices is called a $\gamma_{cs}(G)$ – set of G.The concept of semitotal domination was introduced by Wayne Goddard, Michael A. Henning and Charles A. McPillan[2]. A set D of vertices in a graph G with no isolated vertices to be a semitotal dominating set of G if it is a dominating set of G and every vertex in D is within distance 2 of another vertex of D. The semitotal domination number, denoted by $\gamma_{t2}(G)$, is the minimum cardinality of a semitotal dominating set of G. Any semitotal dominating set with $\gamma_{t2}(G)$ vertices is called a $\gamma_{t2}(G)$ – set of G. Motivated by these two domination parameters, the co-secure semitotal domination number is devised. In this paper we determine the co-secure semitotal domination number of some standard graphs and basic properties of this parameter are also discussed.

METHODS:

Definition: A set D of vertices in a graph G with no isolated vertices is a *co-secure semitotal dominating set* of G if D is a semitotal dominating set of G and for each $u \in D$ there exists a

10.48047/jocaaa.2024.33.08.211

vertex $v \in V \setminus D$ such that $uv \in E$ and $(D \setminus \{u\}) \cup \{v\}$ is a semitotal dominating set of G. The minimum cardinality of a co-secure semitotal dominating set of G is called the *co-secure semitotal domination number* of G and is denoted by $\gamma_{cst2}(G)$. Any co-secure semitotal dominating set with $\gamma_{cst2}(G)$ vertices is called a $\gamma_{cst2}(G)$ – set of G.

Example: For the graph G_1 in Figure 1, $D = \{v_4, v_5, v_6\}$ forms a $\gamma_{cst2}(G_1)$ – set. Hence $\gamma_{cst2}(G_1) = 3$.

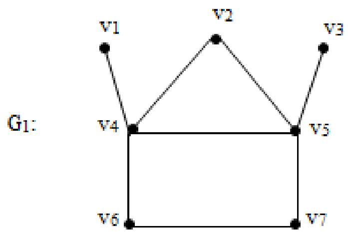

Figure 1: A graph G_1 with $\gamma_{cst2}(G_1) = 3$

RESULTS AND DISCUSSION

Observation: If a co-secure semitotal dominating set exists for the graph G, then $\gamma_{cst2}(G) \geq 2$.

Observation: Every co-secure semitotal dominating set is a co-secure dominating set, but not conversely.

Observation: Every co-secure semitotal dominating set is a semitotal dominating set, but not conversely.

Observation: Let G be a connected graph that has a $\gamma_{cst2}(G)$ – set but the complement of the graph G need not contain a $\gamma_{cst2}(\bar{G})$ – set.

Observation: If a spanning subgraph H of a graph G has a co-secure semitotal dominating set, then G also has a co-secure semitotal dominating set.

Observation: Let G be a connected graph and H be a spanning subgraph of G. If H has a co-secure semitotal dominating set, then $\gamma_{cst2}(G) \leq \gamma_{cst2}(H)$.

Example: For the graph G_2 in Figure 2, $D = \{v_1, v_2\}$ is a $\gamma_{cst2}(G_2)$ – set and so $\gamma_{cst2}(G_2) = 2$. For the spanning subgraph H_1 of G_2, $D = \{v_1, v_2\}$ is a $\gamma_{cst2}(H_1)$ – set and so $\gamma_{cst2}(H_1) =$

10.48047/jocaaa.2024.33.08.211

2. Hence $\gamma_{cst2}(G_2) = \gamma_{cst2}(H_1)$. For the spanning subgraph H_2 of G_2, $D = \{v_1, v_2, v_4\}$ is a $\gamma_{cst2}(H_2)$ – set and so $\gamma_{cst2}(H_2) = 3$. Hence $\gamma_{cst2}(G_2) < \gamma_{cst2}(H_2)$.

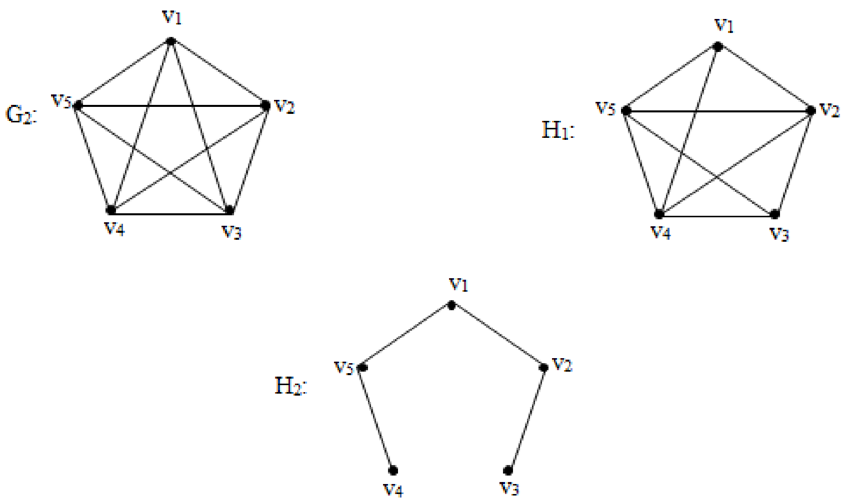

Figure 2: Shows that $\boldsymbol{\gamma_{cst2}(G_2) = \gamma_{cst2}(H_1)}$ and $\boldsymbol{\gamma_{cst2}(G_2) < \gamma_{cst2}(H_2)}$

Exact values for some standard graphs:

1. For any path P_n of order $n \geq 3$, $\gamma_{cst2}(P_n) =$

$$\begin{cases} \left\lceil \frac{n}{2} \right\rceil, \; n = 3,4,5,7,8,9,10 \\ 4, \; n = 6 \\ 4r+1, \; n = 9r \\ 4r+2, \; n = 9r+s; \; where \; s = 1,2,3 \; and \; n \geq 11 \\ 4r+3, \; n = 9r+s; \; where \; s = 4,5 \\ 4r+4, \; n = 9r+s; \; where \; s = 6,7 \\ 4r+5, \; n = 9r+8 \end{cases}$$

2. For any cycle C_n of order $n \geq 3$, $\gamma_{cst2}(C_n) = \begin{cases} 2, \; if \; n = 3,4,5 \\ 3, \; if \; n = 6 \\ 4, \; if \; n = 7,8 \\ 4r, \; if \; n = 9r \\ 4r+1, \; if \; n = 9r+s; \; where \; s = 1,2 \\ 4r+2, \; if \; n = 9r+s; \; where \; s = 3,4 \\ 4r+3, \; if \; n = 9r+s; \; where \; s = 5,6 \\ 4r+4, \; if \; n = 9r+s; \; where \; s = 7,8 \end{cases}$

3. For any complete graph K_n of order $n \geq 3$, $\gamma_{cst2}(K_n) = 2$

4. For any complete bipartite graph $K_{m,n}$, where $m \leq n, m \geq 1 \; and \; n > 1$,

10.48047/jocaaa.2024.33.08.211

$$\gamma_{cst2}(K_{m,n}) = \begin{cases} n, & m = 1 \\ 2, & m = 2 \\ 3, & m = 3 \\ 4, & m \geq 4 \end{cases}$$

Proof:

1. Let $G = P_n$ be a path. Let $V = \{v_i : 1 \leq i \leq n\}$ be the vertex set of G. For $3 \leq n \leq 10$, the result is obvious. Now, for $n \geq 11$, we consider the following cases:

Case (i): For $n \equiv 0\ (mod\ 9)$,

The set, $D = \left\{v_{9i-8}, v_{9i-6}, v_{9i-4}, v_{9i-2} : 1 \leq i \leq \frac{n}{9}\right\} \cup \{v_n\}$, forms a $\gamma_{cst2}(G)$ – set.

Case (ii): For $n \equiv 1\ (mod\ 9)$,

The set, $D = \left\{v_{9i-8}, v_{9i-6}, v_{9i-4}, v_{9i-2} : 1 \leq i \leq \frac{n-1}{9}\right\} \cup \{v_{n-2}, v_n\}$, forms a $\gamma_{cst2}(G)$ – set.

Case (iii): For $n \equiv 2\ (mod\ 9)$,

The set, $D = \left\{v_{9i-8}, v_{9i-6}, v_{9i-4}, v_{9i-2} : 1 \leq i \leq \frac{n-2}{9}\right\} \cup \{v_{n-2}, v_{n-1}\}$, forms a $\gamma_{cst2}(G)$ – set.

Case (iv): For $n \equiv 3\ (mod\ 9)$,

The set, $D = \left\{v_{9i-8}, v_{9i-6}, v_{9i-4}, v_{9i-2} : 1 \leq i \leq \frac{n-3}{9}\right\} \cup \{v_{n-2}, v_{n-1}\}$, forms a $\gamma_{cst2}(G)$ – set.

Case (v): For $n \equiv 4\ (mod\ 9)$,

The set, $D = \left\{v_{9i-8}, v_{9i-6}, v_{9i-4}, v_{9i-2} : 1 \leq i \leq \frac{n-4}{9}\right\} \cup \{v_{n-3}, v_{n-2}, v_n\}$, forms a $\gamma_{cst2}(G)$ – set.

Case (vi): For $n \equiv 5\ (mod\ 9)$,

The set, $D = \left\{v_{9i-8}, v_{9i-6}, v_{9i-4}, v_{9i-2} : 1 \leq i \leq \frac{n-5}{9}\right\} \cup \{v_{n-4}, v_{n-2}, v_{n-1}\}$, forms a $\gamma_{cst2}(G)$ – set.

Case (vii): For $n \equiv 6\ (mod\ 9)$,

The set, $D = \left\{v_{9i-8}, v_{9i-6}, v_{9i-4}, v_{9i-2} : 1 \leq i \leq \frac{n-6}{9}\right\} \cup \{v_{n-5}, v_{n-3}, v_{n-2}, v_n\}$, forms a $\gamma_{cst2}(G)$ – set.

Case (viii): For $n \equiv 7\ (mod\ 9)$,

The set, $D = \left\{v_{9i-8}, v_{9i-6}, v_{9i-4}, v_{9i-2} : 1 \leq i \leq \frac{n+2}{9}\right\}$, forms a $\gamma_{cst2}(G)$ – set.

Case (ix): For $n \equiv 8\ (mod\ 9)$,

The set, $D = \left\{v_{9i-8}, v_{9i-6}, v_{9i-4}, v_{9i-2} : 1 \leq i \leq \frac{n-8}{9}\right\} \cup \{v_{n-7}, v_{n-5}, v_{n-3}, v_{n-2}, v_n\}$ forms a $\gamma_{cst2}(G)$ – set.

10.48047/jocaaa.2024.33.08.211

2. Let $G = C_n$ be a cycle. Let $V = \{v_i : 1 \leq i \leq n\}$ be the vertex set of G. For $3 \leq n \leq 8$, the result is obvious. Now, for $n \geq 9$, we consider the following cases:

Case (i): For $n \equiv 0 \ (mod\ 9)$,

The set, $D = \left\{v_{9i-8}, v_{9i-6}, v_{9i-4}, v_{9i-2} : 1 \leq i \leq \frac{n}{9}\right\}$, forms a $\gamma_{cst2}(G)$ – set.

Case (ii): For $n \equiv 1 \ (mod\ 9)$,

The set, $D = \left\{v_{9i-8}, v_{9i-6}, v_{9i-4}, v_{9i-2} : 1 \leq i \leq \frac{n-1}{9}\right\} \cup \{v_{n-1}\}$, forms a $\gamma_{cst2}(G)$ – set.

Case (iii): For $n \equiv 2 \ (mod\ 9)$,

The set, $D = \left\{v_{9i-8}, v_{9i-6}, v_{9i-4}, v_{9i-2} : 1 \leq i \leq \frac{n-2}{9}\right\} \cup \{v_{n-1}\}$, forms a $\gamma_{cst2}(G)$ – set.

Case (iv): For $n \equiv 3 \ (mod\ 9)$,

The set, $D = \left\{v_{9i-8}, v_{9i-6}, v_{9i-4}, v_{9i-2} : 1 \leq i \leq \frac{n-3}{9}\right\} \cup \{v_{n-2}, v_n\}$, forms a $\gamma_{cst2}(G)$ – set.

Case (v): For $n \equiv 4 \ (mod\ 9)$,

The set, $D = \left\{v_{9i-8}, v_{9i-6}, v_{9i-4}, v_{9i-2} : 1 \leq i \leq \frac{n-4}{9}\right\} \cup \{v_{n-3}, v_{n-1}\}$, forms a $\gamma_{cst2}(G)$ – set.

Case (vi): For $n \equiv 5 \ (mod\ 9)$,

The set, $D = \left\{v_{9i-8}, v_{9i-6}, v_{9i-4}, v_{9i-2} : 1 \leq i \leq \frac{n-5}{9}\right\} \cup \{v_{n-4}, v_{n-2}, v_n\}$, forms a $\gamma_{cst2}(G)$ – set.

Case (vii): For $n \equiv 6 \ (mod\ 9)$,

The set, $D = \left\{v_{9i-8}, v_{9i-6}, v_{9i-4}, v_{9i-2} : 1 \leq i \leq \frac{n-6}{9}\right\} \cup \{v_{n-5}, v_{n-3}, v_{n-1}\}$, forms a $\gamma_{cst2}(G)$ – set.

Case (viii): For $n \equiv 7 \ (mod\ 9)$,

The set, $D = \left\{v_{9i-8}, v_{9i-6}, v_{9i-4}, v_{9i-2} : 1 \leq i \leq \frac{n+2}{9}\right\}$, forms a $\gamma_{cst2}(G)$ – set.

Case (ix): For $n \equiv 8 \ (mod\ 9)$,

The set, $D = \left\{v_{9i-8}, v_{9i-6}, v_{9i-4}, v_{9i-2} : 1 \leq i \leq \frac{n+1}{9}\right\}$, forms a $\gamma_{cst2}(G)$ – set.

3. Let $G = K_n$ be a complete graph. Let $V = \{v_i : 1 \leq i \leq n\}$ be the vertex set of G. Then the γ_{cst2} – set is $D = \{v_k, v_l\}, \forall k \neq l \ and \ 1 \leq k, l \leq n$. Since $(D \setminus \{v_k\}) \cup \{v_i\}$ and $(D \setminus \{v_l\}) \cup \{v_i\}$, where $v_i \in V \setminus D$ form semitotal dominating sets. Therefore, $\gamma_{cst2}(K_n) = 2$.

10.48047/jocaaa.2024.33.08.211

4. Let $G = K_{m,n}$ be the complete bipartite graph, where $m \leq n$. Let $V = V_1 \cup V_2 = \{u_i : 1 \leq i \leq m\} \cup \{v_j : 1 \leq j \leq n\}$ be the vertex set of G. To find the $\gamma_{cst2}(G)$ – set, consider the following cases:

Case (i): Suppose $m = 1$ and $n > 1$.

This represents a star graph $K_{1,n}$. Suppose $D = \{u_1, v_j\}$ is the co-secure semitotal dominating set. The vertex v_j has no adjacent vertex in $V \setminus D$ and the universal vertex u_1 has no vertex v_k in $V \setminus D$ such that $(D \setminus \{u_1\}) \cup v_k$ is a semitotal dominating set. Hence, the universal vertex u_1 of $K_{1,n}$ should not be included in the $\gamma_{cst2}(G)$ – set. Thus, the set of all pendant vertices of $K_{1,n}$ forms a $\gamma_{cst2}(G)$ – set. Therefore, $\gamma_{cst2}(K_{1,n}) = n$.

Case (ii): Suppose $m = 2$ and $n > 1$.

Then the set, $D = \{u_1, u_2\}$ forms a $\gamma_{cst2}(G)$ – set. Since $(D \setminus \{u_1\}) \cup \{v_j\}$ and $(D \setminus \{u_2\}) \cup \{v_j\}$, where $v_j \in V \setminus D$ form semitotal dominating sets.

Case (iii): Suppose $m = 3$ and $n > 1$.

Then the set, $D = \{u_1, u_2, u_3\}$ forms a $\gamma_{cst2}(G)$ – set. Since $(D \setminus \{u_1\}) \cup \{v_j\}$, $(D \setminus \{u_2\}) \cup \{v_j\}$ *and* $(D \setminus \{u_3\}) \cup \{v_j\}$, where $v_j \in V \setminus D$ form semitotal dominating sets.

Case (iv): Suppose $m \geq 4$ and $n > 1$.

Then, $D = \{u_k, u_l, v_k, v_l\}$, where $k \neq l$, forms a $\gamma_{cst2}(G)$ – set. Since $(D \setminus \{v_k\}) \cup \{u_i\}$, $(D \setminus \{v_l\}) \cup \{u_i\}$, $(D \setminus \{u_k\}) \cup \{v_j\}$ and $(D \setminus \{u_l\}) \cup \{v_j\}$, where $u_i, v_j \in V \setminus D$, form semitotal dominating sets.

Theorem: For any connected graph G, $\gamma_{cs}(G) \leq \gamma_{cst2}(G)$.

Proof: The result follows directly from the observation, which states that every co-secure semitotal dominating set is also a co-secure dominating set, but not necessarily the converse.

Theorem: For any connected graph G, $\gamma_{t2}(G) \leq \gamma_{cst2}(G)$.

Proof: The result follows directly from the observation, which states that every co-secure semitotal dominating set is also a semitotal dominating set, but not necessarily the converse.

Theorem: For any connected graph G, with order $n \geq 3$, $2 \leq \gamma_{cst2}(G) \leq n - 1$.

Proof: It is known that the minimum cardinality of a co-secure semitotal dominating set is at least two. Hence, $\gamma_{cst2}(G) \geq 2$. Moreover, in any connected graph G, each vertex in a co-secure semitotal dominating set requires a neighbor in the complement of the set to satisfy the co-secure semitotal domination condition. If the set included all vertices of G, then no vertex

10.48047/jocaaa.2024.33.08.211

can find outside the set, which violates the co-secure semitotal domination condition. Thus, at most $n-1$ vertices of G forms a $\gamma_{cst2}(G)$ – set. Therefore, $\gamma_{cst2}(G) \le |V(G)| - 1 = n - 1$.

Theorem: If a connected graph G has n vertices and maximum degree Δ, then $\gamma_{cst2}(G) \ge \frac{2n}{2\Delta+1}$.

Proof: It is know that for any connected graph G of order n and maximum degree Δ,

$$\gamma_{t2}(G) \ge \frac{2n}{2\Delta+1}.$$

Since $\gamma_{t2}(G) \le \gamma_{cst2}(G)$, it follows that

$$\gamma_{cst2}(G) \ge \frac{2n}{2\Delta+1}.$$

Theorem: For the complete t – partite graph $G = K_{p_{1,}p_{2,\dots,}p_t}$, where $p_1 \le p_2 \le \cdots \le p_t$ and $t \ge 3$, $\gamma_{cst2}(G) = 2$.

Proof: Let $G = K_{p_{1,}p_{2,\dots,}p_t}$ be a complete $t-$ partite graph with partite sets $V_1, V_2, \dots, V_t$, where $|V_r| = p_r$ for each r. Choosing any vertices $u \in V_i$ and $v \in V_j$ where $i \ne j$ and $1 \le i, j \le t$. Then the set, $D = \{u, v\}$ forms a $\gamma_{cst2}(G)$ – set. Since $(D \setminus \{u\}) \cup \{w\}$ and $(D \setminus \{v\}) \cup \{w\}$, where $w \in V_r$, and $r \ne i, j$, form semitotal dominating sets. Therefore, $\gamma_{cst2}(G) = 2$.

Theorem: For any connected graph G with $n \ge 3$ vertices, $\gamma_{cst2}(G) = n - 1$ iff G is isomorphic to $K_{1,n-1}$ or K_3.

Proof: Suppose G is isomorphic to $K_{1,n-1}$ *or* K_3, then clearly $\gamma_{cst2}(G) = n - 1$. Conversely, suppose $\gamma_{cst2}(G) = n - 1$. Let D be a co-secure semitotal dominating set with $|D| = n - 1$ and $|V \setminus D| = 1$. If $uv \in E$ where $u, v \in D$ and $D \setminus \{u\}$ forms a co-secure semitotal dominating set, then the cardinality of $D \setminus \{u\}$ is $n - 2$, which is a contradiction. Hence, D is an independent set and therefore G is isomorphic to $K_{1,n-1}$ or K_3.

Theorem: For any connected graph G with at least two vertices of degree $n-2$, then $\gamma_{cst2}(G) = 2$.

Proof: Let G be a connected graph with at least two vertices of degree $n - 2$. Let $u, v \in V(G)$, where $d(u) = d(v) = n - 2$. To find the co-secure semitotal dominating set D, consider the following cases:

Case (i): Suppose u and v are not adjacent.

Then, $D = \{u, v\}$ forms a $\gamma_{cst2}(G)$ – set, since $(D \setminus \{u\}) \cup \{w\}$ and $(D \setminus \{v\}) \cup \{w\}$ both form semitotal dominating sets, where $w \in V \setminus D$. Therefore, $\gamma_{cst2}(G) = 2$.

10.48047/jocaaa.2024.33.08.211

Case (ii): Suppose u and v are adjacent.

There are two possible cases: (i) there exists a vertex adjacent to a vertex other than u and v. (ii) a vertex adjacent to u is not adjacent to v and a vertex adjacent to v is not adjacent to u.

Subcase (i): Since u and v are adjacent, there exists a vertex z such that z is adjacent to a vertex other than u and v. Then $D = \{u, z\}$ forms a $\gamma_{cst2}(G)$ – set. Since $(D \setminus \{u\}) \cup \{v\}$ and $(D \setminus \{z\}) \cup \{w\}$ form semitotal dominating sets, where $wz \in E$. Therefore $\gamma_{cst2}(G) = 2$.

Subcase (ii): Since u and v are adjacent, there exists two vertices x and y such that u is adjacent to x not to y and v is adjacent to y not to x. Then $D = \{u, v\}$ forms a $\gamma_{cst2}(G)$ – set. Since $(D \setminus \{u\}) \cup \{x\}$ and $(D \setminus \{v\}) \cup \{y\}$ both form semitotal dominating sets. Therefore $\gamma_{cst2}(G) = 2$.

CONCLUSION

In this paper we have determined the co-secure semitotal domination number of some standard graphs and also discussed its fundamental properties. Further, the characterization of the graphs with this parameter and its relationship with other graph theoretical parameters can be investigated.

REFERENCES

1. S. Arumugam, Karam Ebadi, Martin Manrique, Co-secure and Secure Domination in Graphs, *Utilitas Mathematica,* Vol.94(2014), 167-182.
2. Wayne Goddard, Michael A. Henning, Charles A. McPillan, Semitotal Domination in Graphs, *Utilitas Mathematica*, Vol.94(2014), 67-81.
3. Aleena Joseph, V. Sangeetha, Bounds on Co-secure Domination in Graphs, *International Journal of Mathematics Trends and Technology,* Vol.55(2018), 158-164.
4. Teresa W. Haynes, Stephen Hedetniemi, Peter Slater, Fundamentals of Domination in Graphs, *CRC Press*, (1998).