

Total Geo Chromatic Number for Splitting Graphs of Certain graphs

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Abstract

A total geo chromatic set of a graph G is a geo chromatic set S_c such that the subgraph induced by S_c has no isolated vertices. The minimum cardinality of a total geo chromatic set of G is the total geo chromatic number of G and is denoted by $\chi_{tg}(G)$. A total geo chromatic set of cardinality $\chi_{tg}(G)$ is called a χ_{tg} -set of G . The total geo chromatic number of some standard graphs are determined and some general properties satisfied by this concept are studied. In this paper, we investigate the total geo chromatic number of splitting graphs of certain graphs.

Keywords: Geodetic number, chromatic number, geo chromatic number, total geo chromatic number, Splitting graphs.

AMS: 05C12

1. Introduction

Let $G = (V, E)$ be a finite undirected connected graph without multiple edges or loops. The order and size of G are denoted by k and l respectively. For basic graph theoretic terminology we refer to Harary [6]. For vertices p and q in a connected graph G , the distance $d(p, q)$ is the length of a shortest $p - q$ path in G . An $p - q$ path of length $d(p, q)$ is called an $p - q$ geodesic. A vertex x is said to lie on an $p - q$ geodesic P' if x is a vertex of P' including the vertices of p and q . The neighborhood of a vertex x is the set $N(x)$ consisting of all vertices y which are adjacent with x . A vertex x is an extreme vertex of G if the subgraph induced by its neighbors is complete.

The closed interval $I[p, q]$ consists of all vertices lying on some $p - q$ geodesic of G , while for $S \subseteq V, I[S] = \cup_{p, q \in S} I[p, q]$. If $I[S] = V$, then a set S of vertices is a geodetic set and the minimum cardinality of a geodetic set is the geodetic number $g(G)$. A geodetic number of a graph was introduced in [3,4] and further studied in [5,7].

The concept of geo chromatic number was introduced by S. B. Samli and S. R. Chellathurai in [1] and further studied in [2,9]. A geodetic set S is said to be a geo chromatic set S_c of G , if S is both a geodetic set and a chromatic set of G . The minimum cardinality of a geo chromatic set of G is the geo chromatic number of G and is denoted by $\chi_{gc}(G)$.

The concept of geo chromatic set of G has motivated us to introduce the new geo chromatic set conception of totoal geo chromatic set. We call the minimum cardinality of a totoal geo chromatic set of G , the total geo chromatic number of G .

Definition 1.1[10]

A subset $S_c \subseteq V(G)$ is said to be total geo chromatic set of G if S_c is both a total geodesic and a chromatic set of G . The minimum cardinality of a total geo chromatic set is the total geo chromatic number of G and is denoted by $\chi_{tg}(G)$. A total geo chromatic set of minimum cardinality is called a χ_{tg} - set of G .

2. Preliminaries

Theorem 2.1. [10] Every extreme vertex of a connected graph G belongs to every total geo chromatic set of G .

Theorem 2.2. [10] Every support vertex of a connected graph G belongs to every total geo chromatic set of G .

3 Splitting graphs of Some known graphs and their total Geo Chromatic Number

Definition 3.1 [11]: The splitting graph $S'(G)$ of G is obtained by adding a new vertex x' corresponding to each vertex x of G such that x' is adjacent to every vertex adjacent to x in G . If n is the order of G , then $2n$ is the order of $S'(G)$. Also, we say the vertices x_1, x_2, \dots, x_n are duplicated by x'_1, x'_2, \dots, x'_n .

Example 3.2

Consider the graph G and the corresponding its splitting graph $S'(G)$ is given in Figure 3.1

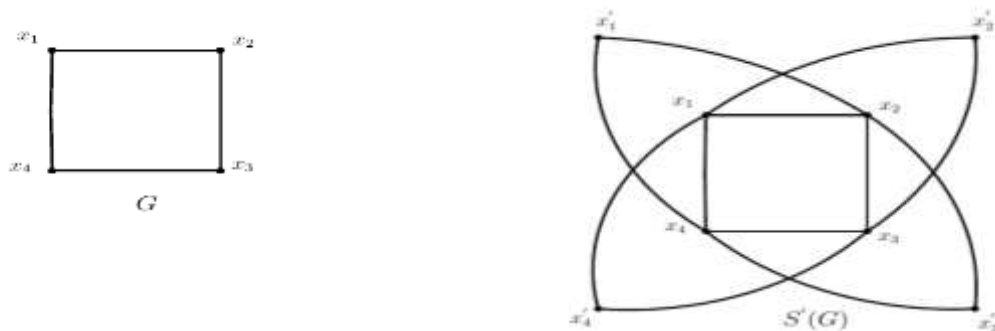


Figure 3.1

Here $S = \{x_1, x_2, x_3\}$ is a minimum total geo chromatic set of G and so $\chi_{tg}(G) = 3$.

Clearly S is a chromatic set of $S'(G)$. But vertices x'_1 and x'_3 does not lie on any geodesic path. So that S is not a total geo chromatic set of $S'(G)$. Now, $S_1 = S \cup \{x_4\}$ is a minimum total geo chromatic set of $S'(G)$ and hence $\chi_{tg}(S'(G)) = 4$.

Thus, the total geo chromatic number of G and $S'(G)$ are different.

Remark 3.3

Every extreme vertex of G need not belongs to every total geo chromatic set of $S'(G)$.

For example, we consider the graph G which is given in Figure 3.2.



Figure 3.2

Here $S = \{x_1, x_2, x_1', x_2', x_3\}$ is a total geo chromatic set of $S'(G)$. But x_3 is an extreme vertex of G which does not belong to S .

It is to be noted that $S_1 = \{x_1, x_2, x_3, x_2', x_3'\}$ is also a total geo chromatic set of $S'(G)$. This S_1 contains all the extreme vertices of G .

Observation 3.4

For any connected graph G of order $n \geq 2$, $\chi_{tg}(G) \leq \chi_{tg}(S'(G))$.

Example 3.2 shows that $\chi_{tg}(G) < \chi_{tg}(S'(G))$.

Example for $\chi_{tg}(G) = \chi_{tg}(S'(G))$.

Consider the graph G given in Figure 3.3

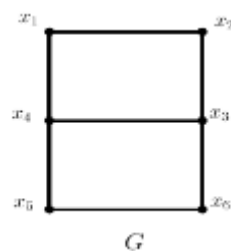


Figure 3.3

Here, $S = \{x_1, x_4, x_3, x_6\}$ is a minimum total geo chromatic set of G and so, $\chi_{tg}(G) = 4$.

Now, we can split each edge of G to obtained a new connected graph $S'(G)$ and is given in Figure 3.4.

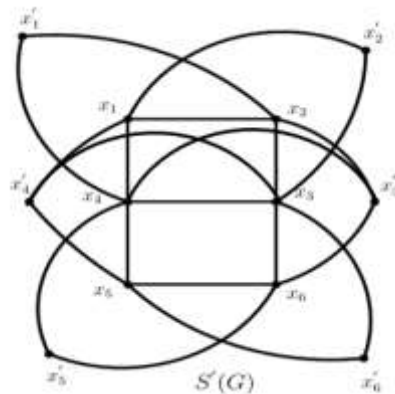


Figure 3.4

Here $S_1 = \{x_1, x_2, x_3, x_4\}$ is the unique minimum total geo chromatic set of $S'(G)$ and so, $\chi_{tg}(S'(G)) = 4$.

Thus, in this case, $\chi_{tg}(G) = \chi_{tg}(S'(G))$.

Theorem 3.5

Every support vertex of G belongs to every total geo chromatic set of $S'(G)$.

Proof:

Let G be any connected graph and v be a support vertex of G . Then there must be at least one end vertex of G which is adjacent to v in G . Let them be u . By the definition of $S'(G)$, u have a duplicated vertex u' , which is adjacent to v in $S'(G)$.

Let S be a total geo chromatic set of $S'(G)$. To prove $v \in S$. Suppose $v \notin S$. Then u' does not lie on any geodesic path of vertices S in $S'(G)$. This shows that $u' \in S$ in $S'(G)$. Since $\deg(u') = 1$ in $S'(G)$, that u' has exactly one neighbor v in $S'(G)$ and also u' has no adjacent

vertex in S . This implies that the subgraph induced by S has isolated vertices in $S'(G)$. It follows that S is not a total geo chromatic set of $S'(G)$, which is a contradiction. Hence $v \in S$ in $S'(G)$.

Theorem 3.6

Every duplicated vertex of an extreme vertex of G belong to every total geo chromatic set of $S'(G)$.

Proof:

Let v be an extreme vertex of G . Then the subgraph that induced by v is complete. That is, $N(v)$ is complete. Let S be a total geo chromatic set of $S'(G)$. Let v' be the duplicated vertex of v in $S'(G)$. Then, by the definition of $S'(G)$, that v' is adjacent to every vertices of the neighbours of v in G . It follows that the neighbours of v' must be complete in $S'(G)$ and so that v' is an extreme vertex of $S'(G)$. Therefore, by Theorem 2.1, $v' \in S$.

Theorem 3.7

Every full vertex of G or every duplicated vertex of a full vertex in G belong to every total geo chromatic set of $S'(G)$.

Proof:

Let G be any connected graph. Let v be a full degree vertex in G . Clearly v is adjacent to every other vertices in G . By the definition of $S'(G)$, that v is adjacent to all the vertices in $S'(G)$ except v' , where v' is the duplicated vertex of v in $S'(G)$. By a proper coloring in $S'(G)$, v and v' received same colour, say (c_1) . Clearly no vertices of $S'(G) - \{v, v'\}$ receive colour c_1 . This shows that every total geo chromatic set must contain either v or v' or both. Therefore, full vertex of G or duplicated vertex of full vertex must be in every total geo chromatic set of $S'(G)$.

Theorem 3.8

For integer $n \geq 2$, $\chi_{tg}(S'(K_{1,n-1})) = n + 2$.

Proof:

Let x_1, x_2, \dots, x_{n-1} are the pendant vertices and x be the full vertex of the star graph $K_{1,n-1}$. Let the corresponding duplicated vertices of $x, x_1, x_2, \dots, x_{n-1}$ be $x', x'_1, x'_2, \dots, x'_{n-1}$, respectively added to obtain $S'(K_{1,n-1})$. This graph is given in Figure 3.5.

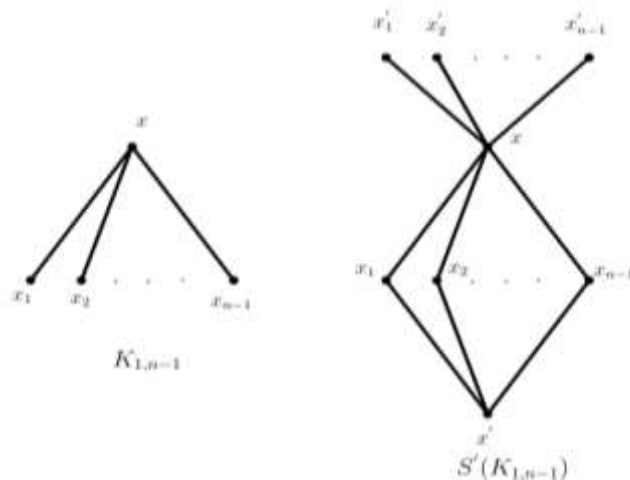


Figure 3.5

Clearly, $V(S'(K_{1,n-1})) = \{x, x', x_1, x_2, \dots, x_{n-1}, x'_1, x'_2, \dots, x'_{n-1}\}$. Thus, $|V(S'(K_{1,n-1}))| = 2n$. Since each $x'_1, x'_2, \dots, x'_{n-1}$ is the extreme vertex of $S'(K_{1,n-1})$ by Theorem 2.1, $S = \{x'_1, x'_2, \dots, x'_{n-1}\}$ is a subset of a minimum total geo chromatic set of $S'(G)$. Also x is a support vertex of $S'(G)$ and hence by Theorem 2.2, $S_1 = S \cup \{x\}$ is a subset of a minimum total geo chromatic set of $S'(G)$. Clearly S_1 is a chromatic set of $S'(G)$. But $x_1, x_2, \dots, x_{n-1}, x'$ does not lie a any geodesic path between vertices from S_1 . Now, it is clear that $S_2 = S_1 \cup \{x'\}$ is a geo chromatic set of $S'(G)$. But the subgraph induced by S_2 has isolated vertices. Therefore that S_2 is not a total geo chromatic set of $S'(G)$ and so $\chi_{tg}(S'(G)) > |S_2| = n + 1$. This implies, $\chi_{tg}(S'(G)) \geq n + 2$. Now, if we take, $S_3 = S_2 \cup \{x_1\}$. Then clearly S_3 is a total geo chromatic set of $S'(G)$ and so $\chi_{tg}(S'(G)) \leq |S_3| = n + 2$. Hence, $\chi_{tg}(S'(G)) = n + 2$.

Example 3.9

For the star graph $S'(K_{1,6})$ given in Figure 3.6, by Theorem 3.8, $S = \{x, x', x'_1, x'_2, x'_3, x'_4, x'_5, x'_6, x_1\}$ is a minimum total geo chromatic set of $S'(K_{1,6})$ and hence $\chi_{tg}(S'(K_{1,6})) = |S| = 8$.



Figure 3.6

Theorem 3.10

For the bistar $B_{m,n}$, ($m, n \geq 2$), $\chi_{tg}(S'(B_{m,n})) = m + n + 4$.

Proof:

Consider the graph $B_{m,n}$ with vertex set $V(B_{m,n}) = \{x, y, x_i, y_j; 1 \leq i \leq m, 1 \leq j \leq n\}$. Here x_i and y_j are the adjacent vertices with x and y , respectively. So clearly x and y are the support vertices and x_i, y_j are the end vertices of $B_{m,n}$. Let $x', y', x'_1, x'_2, \dots, x'_m, y_1, y_2, \dots, y'_n$ be the corresponding duplicated vertices of $x, y, x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n$, respectively which has been added to obtain $S'(B_{m,n})$ and is shown in Figure 3.7. So $V(S'(B_{m,n})) = \{x, x_i, y, y_j, x', x'_i, y', y'_j; 1 \leq i \leq m, 1 \leq j \leq n\}$ and hence, $|V(S'(B_{m,n}))| = 2m + 2n + 2 = 2(m + n + 1)$.

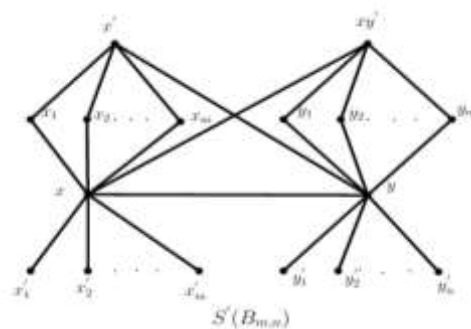
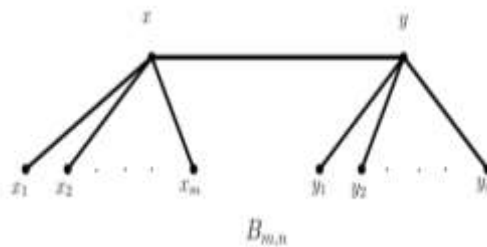


Figure 3.7

Since x and y are the support vertices of $B_{m,n}$, by Theorem 3.5, $S = \{x, y\}$ is a subset of a minimum total geo chromatic set of $S'(B_{m,n})$. Since $x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n$ are the extreme vertices of $B_{m,n}$, by Theorem 3.6, $S_1 = \{x'_1, x'_2, \dots, x'_m, y'_1, y'_2, \dots, y'_n\}$ must be in every total geo chromatic set of $S'(B_{m,n})$. Since $m, n \geq 2$, there is a geodesic between x' and y' , which includes the remaining vertices $x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n$. So x' and y' also in every minimum total geo chromatic set of $S'(B_{m,n})$. If we consider $S_2 = S \cup S_1 \cup \{x', y'\}$, then clearly $I[S_2] = V(S'(B_{m,n}))$ and S_2 is a chromatic set of $S'(B_{m,n})$. Since x' is adjacent with y and y' is adjacent with x , that S_2 itself form a minimum total geo chromatic set of $S'(B_{m,n})$ and hence, $\chi_{tg}(S'(B_{m,n})) = |S_2| = m + n + 4$.

Example 3.11

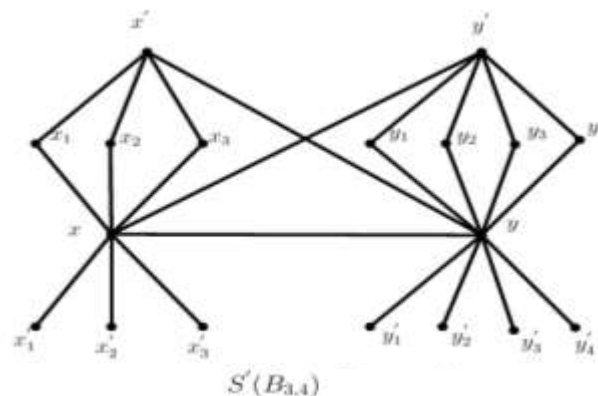


Figure 3.8

Consider the graph $S'(B_{3,4})$ shown in Figure 3.8. By Theorem 3.10, $S = \{x', y', x, y, x'_1, x'_2, x'_3, y'_1, y'_2, y'_3, y'_4\}$ is a minimum total geo chromatic set of $S'(B_{m,n})$ and so $\chi_{tg}(S'(B_{m,n})) = |S| = 4 + 3 + 4 = 11$.

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