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AN EFFICIENT COMPUTATIONAL TECHNIQUE FOR MULTI-DIMENSIONAL DIFFUSION MODELS INVOLVING ATANGANA-BALEANU FRACTIONAL OPERATORS

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Abstract. In this paper, we propose a precise and analytical approach known as, the Shehu Transform Decomposition Method (STDM) to solve multi-dimensional fractional diffusion equations that model density dynamics in diffusive materials. These equations incorporate time-fractional derivatives, formulated using the Atangana-Balenu (AB) derivative. Our method integrates the Shehu Transform (ST) with the Adomian Decomposition Method (ADM) and utilizes Adomian polynomials to effectively handle nonlinear terms. To demonstrate the accuracy and efficiency of the proposed method, we apply it to two illustrative examples of multidimensional fractional diffusion problems. Analytical solutions are presented alongside graphical and numerical simulations, performed in MATLAB, to validate the close agreement between the derived solutions and known exact solutions. The results indicate that STDM is a reliable and efficient tool for solving multi-dimensional diffusion equations with fractional-order derivatives.

1. Introduction

The theory of fractional differential equations (FDEs) has greatly advanced mathematical modeming by enabling more precise and realistic representations compared to classical integer-order differential equations. The utilisation of FC in applied studies has progressively broadened over time. Contemporary investigations have not only demonstrated but also anticipated the existence of fractional operators exhibiting non-locality even without a unique kernel. Furthermore, amidst the various definitions of fractional derivatives categorised as local and non-local, certain derivatives stand out for their ability to aptly elucidate models. The remarkable attributes of memory and heritability exhibited by numerous materials and processes are among the most noteworthy advantages conferred by fractional derivatives. Non-local fractional derivatives offer a straightforward means of describing these characteristics

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across diverse processes. Consequently, from both a physical and practical perspective, non-local fractional derivatives are distinctly favored over their local counterparts.

Among the various fractional operators, the Caputo and R–L derivatives have been the most frequently used in complex fractional models ([1], [2], [3], [20], [21]). To further enhance the understanding of model dynamics, several fractional operators have been introduced over the years. These include the Riemann–Liouville (RL) and Caputo derivatives ([6], [7], [8], [9], [10]), as well as the Caputo–Fabrizio (CF) ([11], [12], [13], [14]) and Atangana–Baleanu [15] operators. In 2016, Atangana and Baleanu [16] proposed a novel non-linear fractional derivative, now known as the Atangana–Baleanu derivative operator. One of the key advantages of the Atangana–Baleanu–Caputo (ABC) fractional derivative operator with a non-singular kernel is its ability to transform a stretched exponential waiting time distribution into a power-law form, and to convert a Gaussian density distribution into a non-Gaussian one.

Integral transformations undeniably stand out as one of the most beneficial and efficient techniques in theoretical and applied mathematics, finding numerous applications in fields such as biology ([18], [20], [21], [22], [25], [26]), electrodynamics [6], mechanics ([7], [8], [29]), nanotechnology [9], biotechnology [10], chaos theory ([11]), and many others ([27], [28]). In recent years, various integral transforms, such as Laplace transform (LT) ([12], [13]), Elzaki transform (ET)

([14], [15]), Aboodh transform (AT) [16], Sumudu transform (SMT) ([17], [19]), etc., have been employed for solving different physical models.

The ST, a novel transformation [23], has recently captured the interest of many mathematicians. The key advantages of the ST include:

(i) Serving as an extension of the LT and SMT [16].

(ii) When considering $v = 1$, it transforms into the LT, and for $\rho = 1$, it corresponds to the Yang integral transform.

(iii) Facilitating the efficient and rapid solution of both exact and numerical solutionsfor fractional-order differential equations.

The diffusion equation [24] is a partial differential equation which describes density dynamics in a material undergoing diffusion. It is also used to describe processes exhibiting diffusive-like behaviour, for instance the 'diffusion' of alleles in a population in population genetics. The equation can be written as,

$$\frac{\partial \varpi (r,\tau)}{\partial \lambda} = \nabla . \left(D \left(\varpi (r,\tau), r \right) \nabla \varpi (r,\tau) \right), \tag{1.1}$$

where, ϖ (r,τ) is the density of the diffusing material at location $r = (x,y,z)$ and time τ. D $(\varpi$ $(r,\tau),r)$ denotes the collective diffusion coefficient for density ϖ at location r.

In this paper, we implement the Shehu transform decomposition method (STDM) with non-singular kernel, for obtaining analytical and numerical solutions of the fractional multi-dimensional diffusion equations. The advantage of this technique is its capability of combining two powerful methods for obtaining exact and approximate analytical solutions for nonlinear equations. It is worth mentioning that the proposed method is capable of reducing the volume of the computational work as compared to the classical methods while still maintaining the high accuracy of the numerical result; the size reduction amounts to an improvement of the performance of the approach.

2. Preliminaries

This section provides key definitions, which are essential for understanding the subsequent results.

Definition 2.1. The usual definition of the Atangana-Baleanu-Caputo derivative (ABC) of order $0 < \delta < 1$ is defined [16] as follows:

$$ {}_{0}^{ABC}D_{\zeta}^{\lambda}\left[\varpi(\eta,\zeta)\right]=\frac{\mathcal{A}(\lambda)}{1-\lambda}\int_{0}^{\eta}E_{\lambda}\left(-\frac{\lambda(\eta-\vartheta)^{\lambda}}{1-\lambda}\right)\varpi'\left((\zeta,\vartheta)\right)d\vartheta,\quad \zeta>0 \quad , \tag{2.1} $$

where A(λ) is a normalization constant, E_λ is the Mittag-Leffler function, and $\Gamma(\lambda)$ is the Gamma function.

Definition 2.2. S. Maitama and W. Zaho [4] developed a noval transform of exponential order function ϖ (ζ) over the set of B,

$$ \mathcal{B}=\left\{\varpi(\zeta):\exists\ \kappa_1,\zeta_1,\ \zeta_2>0,\ |\varpi(\zeta)|<\zeta_1 e^{\frac{|\eta|}{\zeta_j}}\ ,\ if\ \eta\in(-1)^j\times[0,\infty)\right\} $$

by to integral

$$ S\left[\varpi(\zeta)\right]=\mathrm{T}(\nu,\rho)=\int_{0}^{\infty}\varpi(\zeta)\ e^{\frac{-\nu\eta}{\rho}}d\eta\ ,\ \nu>0,\ \eta>0 \quad . \tag{2.2} $$

Remark 2.1. If ρ = 1, then ST becomes Laplace's transform, and also for ν = 1, this transform converts into Yang's integral transform [5].

Definition 2.3. The Shehu transform (ST) for fractional ABC derivative is given as

$$ S\left({}_{0}^{ABC}D_{\zeta}^{\lambda}\left[\varpi(\eta,\zeta)\right]\right)=\frac{\mathcal{A}(\lambda)}{1-\lambda+\lambda\left(\frac{\rho}{\nu}\right)^{\lambda}}\left(V(\nu,\rho)-\frac{\rho}{\nu}\varpi(0)\right) , \tag{2.3} $$

here, $V(\nu,\rho)$ is ST of $\varpi(\zeta,\eta)$.

3. Proposed Methodology

Consider a nonlinear fractional partial differential equation as

$$ {}^{ABC}D_{\zeta}^{\lambda}\varpi(\eta,\varrho,\zeta)=\Re\varpi(\eta,\varrho,\zeta)+\aleph\varpi(\eta,\varrho,\zeta)+P(\eta,\varrho,\zeta)\ ,\ m-1<\lambda\leq m, \tag{3.1} $$

with initial condition

$$ \varpi(\eta,\varrho,\zeta)=\varpi(\eta,\varrho), \tag{3.2} $$

where ${}^{ABC}D_{\zeta}^{\lambda}=\frac{\partial^{\lambda}}{\partial\zeta^{\lambda}}$ represents the fractional AB derivative of order λ, $\Re$ is a linear function of (η,ϱ,ζ), $\aleph$ denotes the nonlinear function, and P is the source term.

Applying the Shehu transform (ST) to both sides of equation (3.1), we obtain

$$ S\left[{}^{ABC}D_{\zeta}^{\lambda}\varpi(\eta,\varrho,\zeta)\right]=S\left[\Re\varpi(\eta,\varrho,\zeta)\right]+S\left[\aleph\varpi(\eta,\varrho,\zeta)\right]+\ S\left[P(\eta,\varrho,\zeta)\right]. \tag{3.3} $$

Utilizing the differentiation property of the Shehu transform (ST), we obtain

$$S\left[\varpi\left(\eta,\varrho,\zeta\right)\right]=\left(\frac{\varrho}{\upsilon}\right)\left[\varpi\left(\eta,\varrho,0\right)\right]+\frac{\left(1-\lambda+\lambda\left(\frac{\varrho}{\upsilon}\right)^{\lambda}\right)}{\mathcal{A}(\lambda)}S\left[P\left(\eta,\varrho,\zeta\right)\right]$$
$$+\frac{\left(1-\lambda+\lambda\left(\frac{\varrho}{\upsilon}\right)^{\lambda}\right)}{\mathcal{A}(\lambda)}S\left[\Re\left[\varpi\left(\eta,\varrho,\zeta\right)\right]+\aleph\left[\varpi\left(\eta,\varrho,\zeta\right)\right]\right]. \quad (3.4)$$

Now, applying the inverse ST to both sides of the equation, we get (3.4), we get

$$\varpi\left(\eta,\varrho,\zeta\right)=S^{-1}\left[\left(\frac{\varrho}{\upsilon}\right)\left[\varpi\left(\eta,\varrho,0\right)\right]\right]+S^{-1}\left[\frac{\left(1-\lambda+\lambda\left(\frac{\varrho}{\upsilon}\right)^{\lambda}\right)}{\mathcal{A}(\lambda)}S\left[P\left(\eta,\varrho,\zeta\right)\right]\right]$$
$$+S^{-1}\left[\frac{\left(1-\lambda+\lambda\left(\frac{\varrho}{\upsilon}\right)^{\lambda}\right)}{\mathcal{A}(\lambda)}S\left[\Re\left[\varpi\left(\eta,\varrho,\zeta\right)\right]+\aleph\left[\varpi\left(\eta,\varrho,\zeta\right)\right]\right]\right]. \quad (3.5)$$

Let, $\varpi(\eta,\varrho)$ has infinite series solution as

$$\varpi\left(\eta,\varrho,\zeta\right)=\sum_{\tau=0}^{\infty}\varpi_{\tau}\left(\eta,\varrho\right), \quad (3.6)$$

and the nonlinear term $\aleph\varpi(\eta,\varrho)$ is expressed as

$$\aleph\varpi\left(\eta,\varrho,\zeta\right)=\sum_{\tau=0}^{\infty}A_{\tau}, \quad (3.7)$$

where A_τ is the Adomian polynomial, given by

$$A_{\tau}=\frac{1}{\Gamma\left(\tau+1\right)}\left[\frac{d^{\tau}}{d\Pi^{\tau}}\left\{\aleph\left(\sum_{\iota=0}^{\infty}\Pi^{\iota}\eta_{\iota},\sum_{\iota=0}^{\infty}\Pi^{\iota}\varrho_{\iota},\sum_{\iota=0}^{\infty}\Pi^{\iota}\zeta_{\iota}\right)\right\}\right]_{\Pi=0}. \quad (3.8)$$

Using equations (3.6) and (3.7) in equation (3.5), we get

$$\sum_{\ell=0}^{\infty}\varpi\left(\eta,\varrho,\zeta\right)=\varpi\left(\eta,\varrho,0\right)+S^{-1}\left[\frac{\left(1-\lambda+\lambda\left(\frac{\varrho}{\upsilon}\right)^{\lambda}\right)}{\mathcal{A}(\lambda)}S\left[P\left(\eta,\varrho,\zeta\right)\right]\right]$$
$$+S^{-1}\left[\frac{\left(1-\lambda+\lambda\left(\frac{\varrho}{\upsilon}\right)^{\lambda}\right)}{\mathcal{A}(\lambda)}S\left[\Re\left[\varpi\left(\sum_{\tau=0}^{\infty}\eta_{\tau},\sum_{\tau=0}^{\infty}\varrho_{\tau},\sum_{\tau=0}^{\infty}\zeta_{\tau}\right)\right]+\sum_{\tau=0}^{\infty}A_{\ell}\right]\right]. \quad (3.9)$$

From equation (3.9), we get

$$\varpi_{0}\left(\eta,\varrho,\zeta\right)=\varpi\left(\eta,\varrho,0\right)+S^{-1}\left[\frac{\left(1-\lambda+\lambda\left(\frac{\varrho}{\upsilon}\right)^{\lambda}\right)}{A(\lambda)}S\left[P\left(\eta,\varrho,\zeta\right)\right]\right],$$
$$\varpi_{1}\left(\eta,\varrho,\zeta\right)=S^{-1}\left[\frac{\left(1-\lambda+\lambda\left(\frac{\varrho}{\upsilon}\right)^{\lambda}\right)}{A(\lambda)}S\left[\Re\left[\varpi\left(\eta_{0},\varrho_{0},\zeta_{0},\right)\right]+A_{0}\right]\right],$$
$$\varpi_{\tau+1}\left(\eta,\varrho,\zeta\right)=S^{-1}\left[\frac{\left(1-\lambda+\lambda\left(\frac{\varrho}{\upsilon}\right)^{\lambda}\right)}{A(\lambda)}S\left[\Re\left[\varpi\left(\eta_{\tau},\varrho_{\tau},\zeta_{\tau}\right)\right]+A_{\tau}\right]\right],\qquad\tau\geq1. \quad (3.10)$$

4. applications

Application 1. Consider the two-dimensional fractional-order diffusion equation given by

$$^{ABC}D_{\zeta}^{\lambda}\varpi\left(\eta,\varrho,\zeta\right)=\frac{\partial^2\varpi}{\partial\eta^2}+\frac{\partial^2\varpi}{\partial\varrho^2}\,,\ 0<\lambda\leq 1, \tag{4.1}$$

accompanied by the initial condition:

$$\varpi\left(\eta,\varrho,0\right)=\sin(\eta)\cos(\varrho). \tag{4.2}$$

applying the ST to both sides of the equation (4.1) and after simplification, the resultant expression is

$$S\left[\varpi\left(\eta,\varrho,\zeta\right)\right]=\left(\frac{\rho}{\nu}\right)\varpi\left(\eta,\varrho,0\right)+\frac{\left(1-\lambda+\lambda\left(\frac{\varrho}{\nu}\right)^{\zeta}\right)}{\mathcal{A}\left(\lambda\right)}S\left[\frac{\partial^2\varpi}{\partial\eta^2}+\frac{\partial^2\varpi}{\partial\varrho^2}\right], \tag{4.3}$$

Subsequently, applying the inverse ST to both sides of the equation (4.3), we get

$$\varpi\left(\eta,\varrho,\zeta\right)=\varpi\left(\eta,\varrho,0\right)+S^{-1}\left[\frac{\left(1-\lambda+\lambda\left(\frac{\varrho}{\nu}\right)^{\zeta}\right)}{\mathcal{A}\left(\lambda\right)}S\left[\frac{\partial^2\varpi}{\partial\eta^2}+\frac{\partial^2\varpi}{\partial\varrho^2}\right]\right], \tag{4.4}$$

As a result, we obtain the following recursive scheme

$$\varpi_0\left(\eta,\varrho,\zeta\right)=\sin(\eta)\cos(\varrho), \tag{4.5}$$

$$\varpi_{\ell+1}\left(\eta,\varrho,\zeta\right)=S^{-1}\left[\frac{\left(1-\lambda+\lambda\left(\frac{\varrho}{\nu}\right)^{\zeta}\right)}{\mathcal{A}\left(\lambda\right)}S\left[\frac{\partial^2\varpi_{\ell}}{\partial\eta^2}+\frac{\partial^2\varpi_{\ell}}{\partial\varrho^2}\right]\right],\ \ell\geq 0 . \tag{4.6}$$

$\ell=0,1,2,\ldots$

Substituting $\left(\ \lambda\ \right)$ into equation (4.6), we derive the following values

$$\varpi_1\left(\eta,\varrho,\zeta\right)=-2\sin\left(\eta\right)\cos\left(\varrho\right)\left(\frac{1-\lambda+\lambda\ \frac{\zeta}{\Gamma(\lambda+1)}}{\mathcal{A}(\lambda)}\right),$$

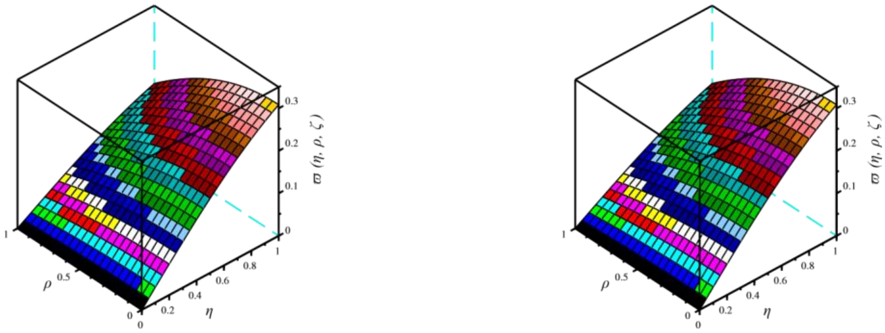
$$\varpi_2\left(\eta,\varrho,\zeta\right)=4\sin\left(\eta\right)\cos\left(\varrho\right)\left(\frac{1}{\mathcal{A}(\lambda)}\right)^2\left(1+2\lambda\left(1+\frac{\zeta^{\lambda}}{\Gamma(\lambda+1)}\right)+\lambda^2\left(1+\frac{\zeta^{2\lambda}}{\Gamma(2\lambda+1)}-2\frac{\zeta^{\lambda}}{\Gamma(\lambda+1)}\right)\right),$$

Similarly, we iteratively derive subsequent terms in the same manner. Thus, the approximate solution to equation (4.1) is expressed as

$$\begin{aligned}\varpi\left(\eta,\varrho,\zeta\right)&=\sin\left(\eta\right)\cos\left(\varrho\right)-2\sin\left(\eta\right)\cos\left(\varrho\right)\left(\frac{1-\lambda+\lambda\left(\frac{\zeta^{\lambda}}{\Gamma(\lambda+1)}\right)}{\mathcal{A}(\lambda)}\right)\\&+4\sin\left(\eta\right)\cos\left(\varrho\right)\left(\frac{1}{\mathcal{A}(\lambda)}\right)^2\left(1+2\lambda\left(1+\frac{\zeta^{\lambda}}{\Gamma(\lambda+1)}\right)+\lambda^2\left(1+\frac{\zeta^{2\lambda}}{\Gamma(2\lambda+1)}-2\frac{\zeta^{\lambda}}{\Gamma(\lambda+1)}\right)\right)+\ldots..\end{aligned} \tag{4.7}$$

Especially noteworthy is the case where $\lambda = 1$, whereby equation (4.1) exhibits rapid convergence towards the exact solution, which is denoted as

$$\varpi(\eta,\varrho,\zeta) = \sin(\eta)\cos(\varrho)e^{-2\zeta}. \tag{4.8}$$



(a) Approx. solution (b) Exact solution

Figure 1. $3D$ surface of the approximate solutions of application 1 and the exact solution at $\zeta = 0.50$.

Application 2. Consider the three-dimensional fractional-order diffusion equation given by

$$^{ABC}D_{\zeta}^{\lambda}\varpi(\eta,\varrho,\beta,\zeta) = \frac{\partial^2\varpi}{\partial\eta^2} + \frac{\partial^2\varpi}{\partial\varrho^2} + \frac{\partial^2\varpi}{\partial\beta^2},\ 0 < \lambda \leq 1, \tag{4.9}$$

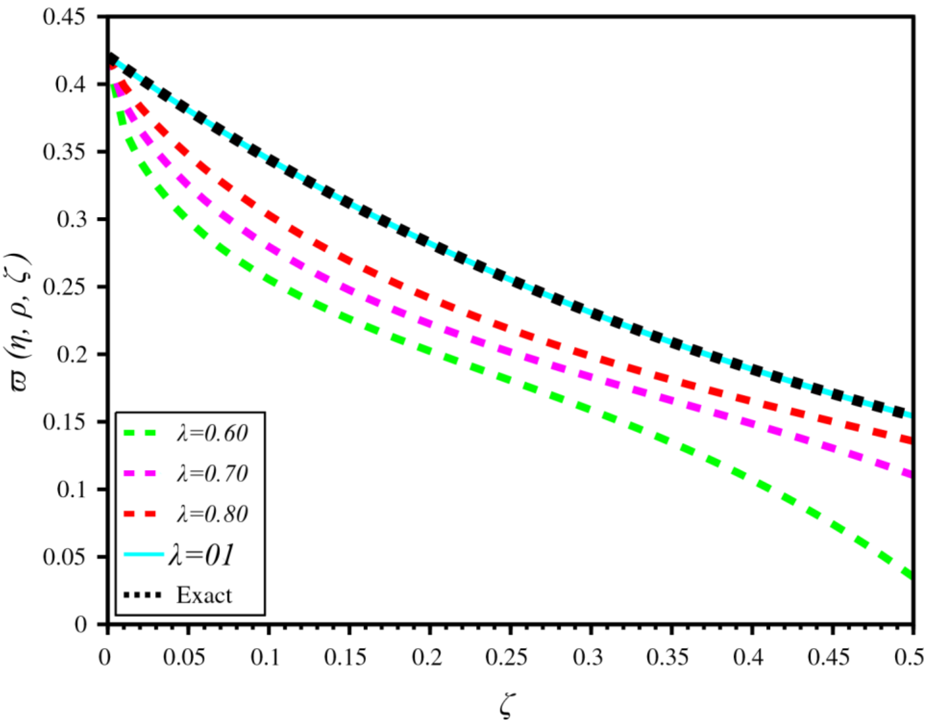


Figure 2. comparison of $2D$ plots for distinct λ and exact solution for application 1 at $\eta = \varrho = 0.50$.

accompanied by the initial condition:

$$\varpi(\eta,\varrho,\beta,0) = \sin(\eta)\sin(\varrho)\sin(\beta). \tag{4.10}$$

applying the ST to both sides of the equation (4.9) and after simplification, the resultant expression is

$$S\left[\varpi(\eta,\varrho,\beta,\zeta)\right] = \left(\frac{\mu}{v}\right)\varpi(\eta,\varrho,\beta,0) + \frac{\left(1-\lambda+\lambda\left(\frac{\varrho}{\nu}\right)^{\zeta}\right)}{\mathcal{A}(\lambda)} S\left[\frac{\partial^2\varpi}{\partial\eta^2} + \frac{\partial^2\varpi}{\partial\varrho^2} + \frac{\partial^2\varpi}{\partial\beta^2}\right], \tag{4.11}$$

Subsequently, applying the inverse ST to both sides of the equation (4.11), we get

$$\varpi(\eta,\varrho,\beta,\zeta) = \varpi(\eta,\varrho,\beta,0) + S^{-1}\left[\frac{\left(1-\lambda+\lambda\left(\frac{\varrho}{\nu}\right)^{\zeta}\right)}{\mathcal{A}(\lambda)} S\left[\frac{\partial^2\varpi}{\partial\eta^2} + \frac{\partial^2\varpi}{\partial\varrho^2} + \frac{\partial^2\varpi}{\partial\beta^2}\right]\right], \tag{4.12}$$

As a result, we obtain the following recursive scheme

$$\varpi_0(\eta,\varrho,\zeta) = \sin(\eta)\sin(\varrho)\sin(\beta), \tag{4.13}$$

Table 1: The SDTM results for fifth order $\varpi(\eta,\varrho,\zeta)$ with $\eta = 0.5$, $\varrho = 0.5$, for application 1.

ζ	Exact Sol.	STDM Sol. ($\lambda = 1$)	$\lvert\varpi_{Exact} - \varpi_{STDM}\rvert$ ($\lambda = 1$)
0.05	0.38069721	0.38069721	$5.7611E-10$
0.10	0.34446908	0.34446905	$3.6356E-08$
0.15	0.31168851	0.31168811	$4.0840E-07$
0.20	0.28202743	0.28202517	$2.2633E-06$
0.25	0.25518897	0.25518045	$8.5170E-06$
0.30	0.23090453	0.23087944	$2.5019E-05$
0.35	0.20893106	0.20886863	$6.2432E-05$
0.40	0.18904864	0.18891135	$1.3728E-04$
0.45	0.17105828	0.17078358	$2.7470E-04$
0.50	0.15477993	0.15426968	$5.1026E-04$

$$\varpi_{\ell+1}(\eta,\varrho,\beta,\zeta) = S^{-1}\left[\frac{\left(1-\lambda+\lambda\left(\frac{\varrho}{\nu}\right)^{\zeta}\right)}{\mathcal{A}(\lambda)} S\left[\frac{\partial^2\varpi_\ell}{\partial\eta^2} + \frac{\partial^2\varpi_\ell}{\partial\varrho^2} + \frac{\partial^2\varpi_\ell}{\partial\beta^2}\right]\right],\ \ell \geq 0. \tag{4.14}$$

Substituting into equation (4.14), we derive the following values.

$$\varpi_1(\eta,\varrho,\beta,\zeta) = -3\sin(\eta)\sin(\varrho)\sin(\beta)\left(\frac{1-\lambda+\lambda\ \frac{\zeta^\lambda}{\Gamma(\lambda+1)}}{\mathcal{A}(\lambda)}\right),$$

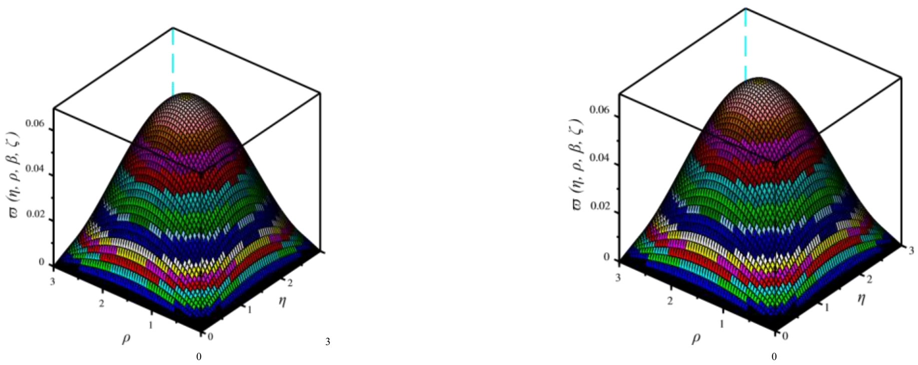
$$\varpi_2(\eta,\varrho,\beta,\zeta) = 9\sin(\eta)\sin(\varrho)\sin(\beta)\left(\frac{1}{\mathcal{A}(\lambda)}\right)^2\left(1+2\lambda\left(1+\frac{\zeta^\lambda}{\Gamma(\lambda+1)}\right)+\lambda^2\left(1+\frac{\zeta^{2\lambda}}{\Gamma(2\lambda+1)}-2\frac{\zeta^\lambda}{\Gamma(\lambda+1)}\right)\right)$$

Similarly, we iteratively derive subsequent terms in the same manner. Thus, the approximate solution to equation (4.9) is expressed as

$$\begin{aligned}\varpi(\eta,\varrho,\beta,\zeta) &= \sin(\eta)\sin(\varrho)\sin(\beta) - 3\sin(\eta)\sin(\varrho)\sin(\beta)\left(\frac{1-\lambda+\lambda\left(\frac{\zeta^\lambda}{\Gamma(\lambda+1)}\right)}{\mathcal{A}(\lambda)}\right)\\ &+9\sin(\eta)\sin(\varrho)\sin(\beta)\left(\frac{1}{\mathcal{A}(\lambda)}\right)^2\left(1+2\lambda\left(1+\frac{\zeta^\lambda}{\Gamma(\lambda+1)}\right)+\lambda^2\left(1+\frac{\zeta^{2\lambda}}{\Gamma(2\lambda+1)}-2\frac{\zeta^\lambda}{\Gamma(\lambda+1)}\right)\right)+\ldots..\end{aligned} \tag{4.15}$$

Especially noteworthy is the case where $\lambda = 1$, whereby equation (4.9) exhibits rapid convergence towards the exact solution, which is denoted as

$$\varpi(\eta,\varrho,\beta,\zeta) = \sin(\eta)\sin(\varrho)\sin(\beta)e^{-3\zeta}. \tag{4.16}$$



(a) λ = 01 (b) Exact solution

Figure 3. comparison of the approximate solutions of application 2 w.r. to the exact solution at $\beta = 0.10$ and $\zeta = 0.10$.

5. Results and Discussions

In this article, we examine Table-1, revealing that the simulation outcomes obtained by the proposed technique closely approximate the exact solution for fractional-order diffusion equations. Fig. 1 illustrates the nature of the $3D$ plots of the approximate and exact solution of application 1. Fig. 2 explores $2D$ nature of the approximate solution for distinct values of λ and the exact solution of application 1 at $\varrho = \eta = 0.5$. Fig. 5 depict the dynamic behaviour of the approximate and exact solutions of application 2 at $\beta = \zeta = 0.1$. We observe from the graphical solutions that the approximate solutions closely match the exact solutions at ð = 1.

6. Conclusions

In this article, we effectively applied a hybrid approach STDM to examine the multi-dimensional fractional-order diffusion equation involving the CF derivative. The uniqueness and existence of proposed method are proven by a fixed-point theorem. We solved three types of fractional-order diffusion equations to demonstrate the effectiveness and accuracy of the proposed scheme. The obtained outcomes indicate that the proposed technique is computationally accurate and reliable.

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