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Some Properties and Integral Representations of the one variable New Generalized Gauss Hypergeometric Polynomials

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Abstract

In the present paper we have obtained finite difference formula, Simple Generating relation, Contour Integral Representation, Real Integral Representation, Single Infinite Integral Representation, Finite Single Integral Representation, Infinite Single Integral Representation, Finite Double Integral Representation and Infinite Double Integral Representation of the one variable Generalized Gauss Hypergeometric Polynomials.

Keywords: New Generalized Gauss Hypergeometric Polynomials, Integral Representation, Generating functions.

1. Introduction

Bajpai and Aroara [1] studied the Gauss Hypergeometric polynomials which give rise to generalization of the some classical polynomials like Jacobi polynomial, Legendre polynomial Gegenbauer polynomial and Chebyshev polynomial [2-6]. Therefore, it is important to study their properties like orthogonality, recurrence relations, integral representations and other aspects [5].

In the present paper, we derived finite difference formula, Simple Generating relation and Additional Generating relation, Contour Integral Representation, Real Integral Representation, Single Infinite Integral Representation, Finite Single Integral Representation, Infinite Single Integral Representation, Finite Double Integral Representation and Infinite Double Integral Representation of the one variable New Generalized Gauss Hypergeometric Polynomials.

1.1 Definition

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The one variable New Generalized Gauss Hypergeometric Polynomial $R_n^{(a,b)}(x,\alpha), n = 0,1,2,3,\ldots$ has been defined and stated as follows

$$R_n^{(a,b)}(x;\alpha) = x^n \,{}_2F_1\left[-n, a; b; \frac{\alpha}{x}\right] \tag{1}$$

where, i) n a nonnegative integer $n = 0,1,2,\ldots$

ii) a is a real number

iii) b is a positive integer i.e., $b \neq 0,-1,-2,-3,\ldots$.

iv) a and b are independent n

v) α is a non zero real number.

For the sake of conciseness, the class defined by (1) will be denoted by $R_n^{(a,b)}(x,\alpha)$.

$$R_n^{(a,b)}(x,\alpha) = x^n \sum_{k=0}^{\infty} \frac{(-n)_k\,(a)_k}{(b)_k\,k!}\left(\frac{\alpha}{x}\right)^k$$

$$= \sum_{k=0}^{\infty} \frac{(-n)_k\,(a)_k\;\alpha^k\,x^{n-k}}{(b)_k\,k!}.$$

Using $$(-n)_k = \frac{(-1)^k\,n!}{(n-k)!}, 0 \le k \le n$$

therefore, $$= \sum_{k=0}^{n} \frac{(-1)^k\,n!\,(a)_k\;\alpha^k\,x^{n-k}}{(n-k)!\,(b)_k\,k!}$$

$$R_n^{(a,b)}(x,\alpha) = \sum_{k=0}^{n} \frac{n!\,(a)_k\,(-\alpha)^k\,x^{n-k}}{(n-k)!\,(b)_k\,k!} \tag{2}$$

For $\alpha = 1$, in Eq. (1) , we get the class of Gauss' Hypergeometric polynomials [1] which is Semi-Orthogonal and is defined as follows :

$$A_n^{(a,b)}(x) = x^n \,{}_2F_1\left[-n, a; b; \frac{1}{x}\right]$$

2. Finite Difference Formula

From (2),

$$R_n^{(a,b)}(x,\alpha) = \sum_{k=0}^{n} \frac{n!\,(a)_k\,(-\alpha)^k\,x^{n-k}}{(n-k)!\,(b)_k\,k!}$$

$$= \frac{\Gamma(b)}{\Gamma(a)} \sum_{k=0}^{n} \frac{n!\,\Gamma(a+k)\,(-\alpha)^k\,x^{n-k}}{(n-k)!\;\Gamma(b+k)k!}$$

$$= (-1)^n \frac{\Gamma(b)}{\Gamma(a)} \sum_{k=0}^{n} C_k^n (-1)^{n-k}\,\alpha^k\,x^{n-k}\,\frac{\Gamma(a+k)}{\Gamma(b+k)}$$

replacing a and b by $a+\lambda$ and $b+\lambda$ respectively,

$$R_n^{(a+\lambda,b+\lambda)}(x,\alpha) = (-1)^n \frac{\Gamma(b+\lambda)}{\Gamma(a+\lambda)} \sum_{k=0}^{n} C_k^n (-1)^{n-k}\,\alpha^k\,x^{n-k}\,\frac{\Gamma(a+\lambda+k)}{\Gamma(b+\lambda+k)}$$

$$= (-1)^n \frac{\Gamma(b+\lambda)}{\Gamma(a+\lambda)} \sum_{k=0}^{n} C_k^n (-1)^{n-k} \alpha^{\lambda+k} x^{n+\lambda} \frac{\Gamma(a+\lambda+k)}{\Gamma(b+\lambda+k)} \frac{1}{\alpha^{\lambda} x^{\lambda+k}}$$

$$= (-1)^n \frac{\Gamma(b+\lambda)}{\Gamma(a+\lambda)} \alpha^{-\lambda} x^{n+\lambda} \sum_{k=0}^{n} C_k^n (-1)^{n-k} \left(\frac{\alpha}{x}\right)^{\lambda+k} \frac{\Gamma(a+\lambda+k)}{\Gamma(b+\lambda+k)}$$

using, $\Delta_{\lambda} g(\lambda) = g(\lambda+1) - g(\lambda)$ and $\Delta_{\lambda}^{n} g(\lambda) = \sum_{k=0}^{n} (-1)^k C_k^n g(\lambda+k)$,

$$R_n^{(a+\lambda,b+\lambda)}(x,\alpha) = (-1)^n \frac{\Gamma(b+\lambda)}{\Gamma(a+\lambda)} \alpha^{-\lambda} x^{n+\lambda} \Delta_{\lambda}^{n} \left[\frac{\Gamma(a+\lambda)}{\Gamma(b+\lambda)} \left(\frac{\alpha}{x}\right)^{\lambda} \right] \tag{3}$$

3. Generating Relation

3.1 Simple Generating Relation

From (1), we write

$$\sum_{n=0}^{\infty} \frac{R_n^{(a,b)}(x,\alpha)\, t^n}{n!} = \sum_{n=0}^{\infty} \frac{t^n}{n!} R_n^{(a,b)}(x,\alpha)$$

From (2),

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{n} \frac{n!\,(a)_k\,(-\alpha)^k\, x^{n-k}}{(n-k)!\,(b)_k\, k!}$$

$$= \sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} \frac{(a)_k\,(-\alpha)^k\, x^{n-k}}{(n-k)!\,(b)_k\, k!}$$

using , $$\sum_{n=0}^{\infty} \sum_{k=0}^{n} A(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k,n+k)$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(a)_k\,(-\alpha)^k\, x^{n}}{n!\,(b)_k\, k!} t^{n+k}$$

$$= \sum_{n=0}^{\infty} \frac{(xt)^n}{n!} \sum_{k=0}^{\infty} \frac{(a)_k\,(-\alpha t)^k}{(b)_k\, k!}$$

$$= e^{xt}\ {}_1F_1[a;b;-\alpha t] \tag{4}$$

3.2 Additional Generating Relation

By using definition (1),

$$\sum_{n=0}^{\infty} \frac{(\delta)_n\, R_n^{(a,b)}(x,\alpha)\, t^n}{n!} = \sum_{n=0}^{\infty} \frac{(\delta)_n\, t^n}{n!} R_n^{(a,b)}(x,\alpha)$$

From (2),

$$= \sum_{n=0}^{\infty} \frac{(\delta)_n t^n}{n!} \sum_{k=0}^{n} \frac{n!\,(a)_k\,(-\alpha)^k\, x^{n-k}}{(n-k)!\,(b)_k\, k!}$$

$$= \sum_{n=0}^{\infty} (\delta)_n t^n \sum_{k=0}^{n} \frac{(a)_k\,(-\alpha)^k\, x^{n-k}}{(n-k)!\,(b)_k\, k!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(\delta)_n (a)_k \ (-\alpha)^k \ x^{n-k} t^n}{(n-k)! \ (b)_k \ k!}$$

using,

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} A(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n+k)$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\delta)_{n+k} (a)_k \ (-\alpha)^k \ x^n}{n! \ (b)_k \ k!} t^{n+k}$$

by using the result, $(\delta)_{n+k} = (\delta)_k (\delta+k)_n$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\delta)_k (\delta+k)_n (a)_k \ (-\alpha t)^k \ (xt)^n}{n! \ (b)_k \ k!}$$

$$= \sum_{k=0}^{\infty} \left[\sum_{n=0}^{\infty} \frac{(\delta+k)_n \ (xt)^n}{n!} \right] \frac{(\delta)_k (a)_k \ (-\alpha t)^k}{(b)_k \ k!}$$

$$= \sum_{k=0}^{\infty} \left[\frac{1}{(1-xt)^{\delta+k}} \right] \frac{(\delta)_k (a)_k \ (-\alpha t)^k}{(b)_k \ k!}$$

$$= \frac{1}{(1-xt)^{\delta}} \sum_{k=0}^{\infty} \frac{(\delta)_k (a)_k}{(b)_k \ k!} \left[\frac{-\alpha t}{(1-xt)} \right]^k$$

$$= \frac{1}{(1-xt)^{\delta}} \ {}_2F_1\left[\delta, a; b; \frac{-\alpha t}{1-xt} \right] \qquad (5)$$

Substituting $\delta = b$, we get

$$\sum_{n=0}^{\infty} \frac{(b)_n \ R_n^{(a,b)}(x,\alpha) \ t^n}{n!} = \frac{1}{(1-xt)^b} \sum_{k=0}^{\infty} \frac{(b)_k (a)_k}{(b)_k \ k!} \left[\frac{-\alpha t}{(1-xt)} \right]^k$$

$$= \frac{1}{(1-xt)^b} \sum_{k=0}^{\infty} \frac{(a)_k}{k!} \left[\frac{-\alpha t}{(1-xt)} \right]^k$$

$$= \frac{1}{(1-xt)^b} \frac{1}{\left[1 - \left(\frac{-\alpha t}{(1-xt)} \right) \right]^a}$$

$$= \frac{1}{(1-xt)^b} \frac{1}{\left[1 + \frac{\alpha t}{(1-xt)} \right]^a}$$

$$=\frac{1}{(1-xt)^{b}}\frac{1}{\left[\frac{1+(\alpha-x)t}{(1-xt)}\right]^{a}}$$

$$=(1-xt)^{a-b}\frac{1}{(1+(\alpha-x)t)^{a}}$$

$$=(1-xt)^{a-b}(1+(\alpha-x)t)^{-a} \quad (6)$$

4. Integral Representation

4.1 Contour Integral Representation

From (4), we have, $\sum_{n=0}^{\infty}\frac{R_n^{(a,b)}(x,\alpha)\,t^n}{n!}= e^{xt}\ {}_1F_1[a;b;-\alpha t]$

let, $f(t)=\sum_{n=0}^{\infty}\frac{R_n^{(a,b)}(x,\alpha)\,t^n}{n!}= e^{xt}\ {}_1F_1[a;b,-\alpha t]$

using Maclaurin's theorem, $f(t)=\sum_{n=0}^{\infty}\frac{f^n(0)\,t^n}{n!}$

where, $f^n(0), n=0,1,2,3,\ldots$ is given by formula,

$$f^n(0)=\frac{n!}{2\pi i}\int^{(0+)}\frac{f(t)}{t^{n+1}}dt, \forall n=0,1,2,3,\ldots$$

$$R_n^{(a,b)}(x,\alpha)=\frac{n!}{2\pi i}\int^{(0+)}t^{-n-1}e^{xt}\ {}_1F_1[a;b;-\alpha t]dt \quad (7)$$

and from (5), if we take $f(t)=\sum_{n=0}^{\infty}\frac{(\delta)_n R_n^{(a,b)}(x,\alpha)t^n}{n!}==\frac{1}{(1-xt)^{\delta}}\ {}_2F_1\left[\delta,a;b;\frac{-\alpha t}{1-xt}\right]$

again by using Maclaurin's theorem, we have

$$R_n^{(a,b)}(x,\alpha)=\frac{n!}{2\pi i(\delta)_n}\int^{(0+)}t^{-n-1}\frac{1}{(1-xt)^{\delta}}\ {}_2F_1\left[\delta,a;b;\frac{-\alpha t}{1-xt}\right]dt \quad (8)$$

where ,the origin of the t plane lies inside contour in positive direction.

4.2 Real Integral Representation

Substituting $t=e^{i\theta}\,(0\le\theta\le 2\pi)$ and $dt=ie^{i\theta}\,d\theta$ in (7), we get

$$R_n^{(a,b)}(x,\alpha)=\frac{n!}{2\pi i}\int_0^{2\pi}\left(e^{i\theta}\right)^{-n-1}e^{xe^{i\theta}}\ {}_1F_1\left[a;b;-\alpha e^{i\theta}\right]ie^{i\theta}d\theta$$

$$=\frac{n!}{2\pi}\int_0^{2\pi} e^{-in\theta}\exp\left(xe^{i\theta}\right)\left[\sum_{k=0}^{\infty}\frac{(a)_k}{(b)_k k!}\left(-\alpha e^{i\theta}\right)^k\right]d\theta$$

$$=\frac{n!}{2\pi}\sum_{k=0}^{\infty}\left[\int_0^{2\pi}\frac{(a)_k}{(b)_k k!}e^{i(k-n)\theta}\exp\left(xe^{i\theta}\right)(-\alpha)^k d\theta\right]$$

$$=\frac{n!}{2\pi}\sum_{k=0}^{\infty}\frac{(a)_k(-\alpha)^k}{(b)_k k!}\left[\int_0^{2\pi}e^{i(k-n)\theta}\exp\left(xe^{i\theta}\right)d\theta\right]$$

$$=\frac{n!}{2\pi}\sum_{k=0}^{\infty}\frac{(a)_k(-\alpha)^k}{(b)_k k!}\left[\int_0^{2\pi}e^{i(k-n)\theta}\left(\sum_{m=0}^{\infty}\frac{\left(xe^{i\theta}\right)^m}{m!}\right)d\theta\right]$$

$$=\frac{n!}{2\pi}\sum_{k=0}^{\infty}\frac{(a)_k(-\alpha)^k}{(b)_k k!}\left[\int_0^{2\pi}e^{i(k-n)\theta}\left(\sum_{m=0}^{\infty}\frac{x^m e^{im\theta}}{m!}\right)d\theta\right]$$

$$=\frac{n!}{2\pi}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(a)_k(-\alpha)^k x^m}{(b)_k k!m!}\left[\int_0^{2\pi}e^{i(k-n)\theta}e^{im\theta}d\theta\right]$$

$$=\frac{n!}{2\pi}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(a)_k(-\alpha)^k x^m}{(b)_k k!m!}\left[\int_0^{2\pi}e^{i(k-n+m)\theta}d\theta\right]$$

$$=\frac{n!}{2\pi}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(a)_k(-\alpha)^k x^m}{(b)_k k!m!}\left[\int_0^{2\pi}Cis(k-n+m)\theta\, d\theta\right]$$

where, $Cis\varphi=\cos\varphi+i\sin\varphi$.

therefore, we get

$$R_n^{(a,b)}(x,\alpha)=\frac{n!}{2\pi}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(a)_k(-\alpha)^k x^m}{(b)_k k!m!}\left[\int_0^{2\pi}(\cos\varphi+i\sin\varphi)d\theta\right]\tag{9}$$

where, $\varphi=(k-n+m)\theta$.

Now, substituting $t=e^{i\theta}\ (0\le\theta\le 2\pi)$ and $dt=ie^{i\theta}\,d\theta$ in (8), we get

$$R_n^{(a,b)}(x,\alpha)=\frac{n!}{2\pi i(\delta)_n}\int_0^{2\pi}\left(e^{i\theta}\right)^{-n-1}\frac{1}{\left(1-xe^{i\theta}\right)^{\delta}}\ {}_2F_1\left[\delta,a;b;\frac{-\alpha e^{i\theta}}{1-xe^{i\theta}}\right]ie^{i\theta}d\theta$$

$$=\frac{n!}{2\pi(\delta)_n}\int_0^{2\pi}\left(e^{i\theta}\right)^{-n-1}\frac{1}{\left(1-xe^{i\theta}\right)^{\delta}}\left[\sum_{k=0}^{\infty}\frac{(\delta)_k(a)_k}{(b)_k k!}\left(\frac{-\alpha e^{i\theta}}{1-xe^{i\theta}}\right)^k\right]e^{i\theta}d\theta$$

$$=\frac{n!}{2\pi(\delta)_n}\sum_{k=0}^{\infty}\frac{(\delta)_k(a)_k(-\alpha)^k}{(b)_k k!}\int_0^{2\pi}e^{i(k-n)\theta}\left(1-xe^{i\theta}\right)^{-k-\delta}d\theta$$

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$$= \frac{n!}{2\pi(\delta)_n} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\delta)_k (a)_k (-\alpha)^k (\delta+k)_m}{(b)_k \, k! m!} \int_0^{2\pi} e^{i(k-n)\theta} \left(xe^{i\theta}\right)^m d\theta$$

$$= \frac{n!}{2\pi(\delta)_n} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\delta)_k (a)_k (-\alpha)^k (\delta+k)_m \, x^m}{(b)_k \, k! m!} \int_0^{2\pi} e^{i(k-n+m)\theta} \, d\theta$$

$$= \frac{n!}{2\pi(\delta)_n} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\delta)_k (a)_k (-\alpha)^k (\delta+k)_m \, x^m}{(b)_k \, k! m!} \int_0^{2\pi} e^{i(k-n+m)\theta} \, d\theta$$

$$= \frac{n!}{2\pi(\delta)_n} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\delta)_{k+m} (a)_k (-\alpha)^k \, x^m}{(b)_k \, k! m!} \int_0^{2\pi} Cis(k-n+m)\theta \, d\theta$$

where, $Cis\varphi = \cos\varphi + i\sin\varphi$

therefore, we get

$$R_n^{(a,b)}(x,\alpha) = \frac{n!}{2\pi(\delta)_n} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\delta)_{k+m} (a)_k (-\alpha)^k \, x^m}{(b)_k \, k! m!} \int_0^{2\pi} (\cos\varphi + i\sin\varphi) d\theta \tag{10}$$

where, $\varphi = (k-n+m)\theta$.

4.3 Single Infinite Integral Representation

By using definition (1), $R_n^{(a,b)}(x,\alpha) = x^n \; {}_2F_1\left[-n, a; b; \frac{\alpha}{x}\right]$

$$= x^n \sum_{k=0}^{\infty} \frac{(-n)_k \, (a)_k}{(b)_k \, k!} \left(\frac{\alpha}{x}\right)^k,$$

by using,

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)},$$

we get

$$= \sum_{k=0}^{\infty} \frac{(-n)_k \, \Gamma(a+k) \alpha^k \, x^{n-k}}{\Gamma(a)(b)_k \, k!}$$

$$= \frac{1}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{(-n)_k \, \alpha^k \, x^{n-k}}{(b)_k \, k!} \int_{-\infty}^{\infty} e^{-t^2} t^{2\left(a+k-\frac{1}{2}\right)} dt$$

$$= \frac{1}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{(-n)_k \, \alpha^k \, x^{n-k}}{(b)_k \, k!} \int_{-\infty}^{\infty} e^{-t^2} t^{2a+2k-1} dt$$

$$=\frac{1}{\Gamma(a)}\int_{-\infty}^{\infty}e^{-t^2}t^{2a-1}\left[\sum_{k=0}^{\infty}\frac{(-n)_k\ \alpha^k\ x^{n-k}}{(b)_k\ k!}\left(t^2\right)^k\right]dt$$

$$=\frac{1}{\Gamma(a)}\int_{-\infty}^{\infty}e^{-t^2}t^{2a-1}\left[x^n\sum_{k=0}^{\infty}\frac{(-n)_k}{(b)_k\ k!}\left(\frac{\alpha t^2}{x}\right)^k\right]dt$$

$$=\frac{x^n}{\Gamma(a)}\int_{-\infty}^{\infty}e^{-t^2}t^{2a-1}\ {}_1F_1\left[-n;b;\frac{\alpha t^2}{x}\right]dt$$

$$R_n^{(a,b)}(x,\alpha)=\frac{x^n}{\Gamma(a)}\int_{-\infty}^{\infty}e^{-t^2}t^{2a-1}\ {}_1F_1\left[-n;b;\frac{\alpha t^2}{x}\right]dt\text{, if }\operatorname{Re}(a)>\frac{1}{2}\text{.}\qquad(11)$$

4.4 Finite Single Integral Representation

By (1)

$$R_n^{(a,b)}(x,\alpha)=x^n\ {}_2F_1\left[-n,a;b;\frac{\alpha}{x}\right]$$

$$=x^n\sum_{k=0}^{\infty}\frac{(-n)_k\,(a)_k}{(b)_k\ k!}\left(\frac{\alpha}{x}\right)^k$$

$$=\sum_{k=0}^{\infty}\frac{(-n)_k\,(a)_k\,\alpha^k}{(b)_k\ k!}x^{n-k}$$

by using, $(v)_k=\frac{\Gamma(v+k)}{\Gamma(v)}$, we write

$$=\frac{\Gamma(b)}{\Gamma(a)}\sum_{k=0}^{\infty}\frac{(-n)_k\ \Gamma(a+k)\alpha^k\ x^{n-k}}{\Gamma(b+k)k!}$$

$$=\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\sum_{k=0}^{\infty}\frac{(-n)_k\ \alpha^k\ x^{n-k}\Gamma(a+k)}{k!\ \Gamma(b+k)}\Gamma(b-a)$$

By using definition of Beta Function

$$=\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\sum_{k=0}^{\infty}\frac{(-n)_k\ \alpha^k\ x^{n-k}}{k!}\int_0^1 t^{a+k-1}(1-t)^{b-a-1}dt$$

$$=\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 t^{a-1}(1-t)^{b-a-1}\left[\sum_{k=0}^{\infty}\frac{(-n)_k}{k!}\left(\frac{\alpha t}{x}\right)^k\right]dt$$

$$=\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 t^{a-1}(1-t)^{b-a-1}\left(1-\frac{\alpha t}{x}\right)^n dt$$

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$$= \frac{\Gamma(b)x^{-n}}{\Gamma(a)\Gamma(b-a)}\int_0^1 t^{a-1}(1-t)^{b-a-1}(x-\alpha t)^n\,dt$$

$$R_n^{(a,b)}(x,\alpha) = \frac{\Gamma(b)x^{-n}}{\Gamma(a)\Gamma(b-a)}\int_0^1 t^{a-1}(1-t)^{b-a-1}(x-\alpha t)^n\,dt \qquad (12)$$

if $\mathrm{Re}(a) > 0$ and $\mathrm{Re}(b-a) > 1$

4.5 Finite Double Integral Representation

Srivastava and Karlsson [3,p.275,(2)] stated the result that if $\mathrm{Re}(a) > 0, \mathrm{Re}(b) > 0$ and $\mathrm{Re}(c) > 0$

$$\iint_D u^{a-1}v^{b-1}(1-u-v)^{c-1}\,du\,dv = \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b+c)}$$

where D is bounded by the lines $u \geq 0, v \geq 0$ and $u+v \leq 1$.

From (1) $$R_n^{(a,b)}(x,\alpha) = x^n\ {}_2F_1\left[-n, a; b; \frac{\alpha}{x}\right]$$

$$= x^n \sum_{k=0}^{\infty} \frac{(-n)_k\,(a)_k}{(b)_k\,k!}\left(\frac{\alpha}{x}\right)^k$$

$$= \sum_{k=0}^{\infty} \frac{(-n)_k\,(a)_k\,\alpha^k}{(b)_k\,k!}x^{n-k}$$

by using, $(v)_k = \frac{\Gamma(v+k)}{\Gamma(v)}$, we write

$$= \frac{\Gamma(b)}{\Gamma(a)}\sum_{k=0}^{\infty}\frac{(-n)_k\,\Gamma(a+k)\alpha^k\,x^{n-k}}{\Gamma(b+k)k!}$$

$$= \frac{\Gamma(b)\Gamma(\gamma)}{\Gamma(a)\Gamma(\gamma-b-a)}\sum_{k=0}^{\infty}\frac{(-n)_k\,\Gamma(a+k)(\gamma)_k}{k!\,\Gamma(b+k)\Gamma(\gamma+k)}\Gamma(\gamma-b-a)\alpha^k\,x^{n-k}$$

$$= \frac{\Gamma(b)\Gamma(\gamma)}{\Gamma(a)\Gamma(\gamma-b-a)}\sum_{k=0}^{\infty}\frac{(-n)_k\,\Gamma(a+k)(\gamma)_k}{k!\,\Gamma(b+k)\Gamma(\gamma+k)}\Gamma(\gamma-b-a)\alpha^k\,x^{n-k}$$

$$= \frac{\Gamma(\gamma)}{\Gamma(a)\Gamma(\gamma-b-a)}\sum_{k=0}^{\infty}\frac{(-n)_k\,(\gamma)_k\,\alpha^k\,x^{n-k}}{k!\,\Gamma(b+k)}\left[\frac{\Gamma(a+k)\Gamma(\gamma-b-a)\Gamma(b)}{\Gamma(\gamma+k)}\right]$$

$$= \frac{\Gamma(\gamma)}{\Gamma(a)\Gamma(\gamma-b-a)}\sum_{k=0}^{\infty}\frac{(-n)_k\,(\gamma)_k\,\alpha^k\,x^{n-k}}{k!\,\Gamma(b+k)}\iint_D u^{a+k-1}v^{b-1}(1-u-v)^{\gamma-a-b-1}\,du\,dv$$

$$= \frac{\Gamma(\gamma)x^n}{\Gamma(a)\Gamma(b)\Gamma(\gamma-b-a)} \iint_D u^{a-1} v^{b-1} (1-u-v)^{\gamma-a-b-1} \left[\sum_{k=0}^{\infty} \frac{(-n)_k\ \Gamma(b)(\gamma)_k}{k!\ \Gamma(b+k)} \left(\frac{\alpha u}{x} \right)^k \right] du\, dv$$

$$= \frac{\Gamma(\gamma)x^n}{\Gamma(a)\Gamma(b)\Gamma(\gamma-b-a)} \iint_D u^{a-1} v^{b-1} (1-u-v)^{\gamma-a-b-1} \left[\sum_{k=0}^{\infty} \frac{(-n)_k\ (\gamma)_k}{k!\ (b)_k} \left(\frac{\alpha u}{x} \right)^k \right] du\, dv$$

$$= \frac{\Gamma(\gamma)}{\Gamma(a)\Gamma(b)\Gamma(\gamma-b-a)} \iint_D u^{a-1} v^{b-1} (1-u-v)^{\gamma-a-b-1}\ x^n\ {}_2F_1\left[-n, \gamma; b; \frac{\alpha u}{x} \right] du\, dv$$

$$= \frac{\Gamma(\gamma)}{\Gamma(a)\Gamma(b)\Gamma(\gamma-b-a)} \iint_D u^{a-1} v^{b-1} (1-u-v)^{\gamma-a-b-1}\ R_n^{(a,b)}\left(\frac{\alpha u}{x}; \alpha \right) du\, dv$$

$$R_n^{(a,b)}(x;\alpha) = \frac{\Gamma(\gamma)}{\Gamma(a)\Gamma(b)\Gamma(\gamma-b-a)} \iint_D u^{a-1} v^{b-1} (1-u-v)^{\gamma-a-b-1}\ R_n^{(a,b)}\left(\frac{\alpha u}{x}; \alpha \right) du\, dv, \tag{13}$$

if $\mathrm{Re}(a) > 0, \mathrm{Re}(b) > 0$ and $\mathrm{Re}(\gamma - a - b) > 0$.

4.6 Infinite Double Integral Representation

J. Edwards [2,p.177,(16)] stated the result that

$$\int_0^\infty \int_0^\infty \varphi(x+y) x^\alpha y^\beta\, dx\, dy = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \int_0^\infty \varphi(z) z^{\alpha+\beta+1} dz, \tag{14}$$

From (1),

$$R_n^{(a,b)}(x;\alpha) = x^n\ {}_2F_1\left[-n, a; b; \frac{\alpha}{x} \right]$$

$$= x^n \sum_{k=0}^{\infty} \frac{(-n)_k\ (a)_k}{(b)_k\ k!} \left(\frac{\alpha}{x} \right)^k$$

$$\int_0^\infty \int_0^\infty u^{\mu+\frac{1}{2}} v^\mu (1-u-v)^{-\frac{3}{2}}\ R_n^{(a,b)}\left(\frac{x}{4uvc}; \alpha \right) du\, dv = \int_0^\infty \int_0^\infty u^{\mu+\frac{1}{2}} v^\mu (1-u-v)^{-\frac{3}{2}} \left[\sum_{k=0}^{\infty} \frac{(-n)_k\ (a)_k\ \alpha^k}{(b)_k\ k!} \left(\frac{4uvc}{x} \right)^{n-k} \right] du\, dv$$

$$= \int_0^\infty \int_0^\infty u^{\mu+\frac{1}{2}} v^\mu (1-u-v)^{-\frac{3}{2}} \left[\sum_{k=0}^{\infty} \frac{(-n)_k\ (a)_k\ \alpha^k}{(b)_k\ k!} \left(\frac{4c}{x} \right)^{n-k} u^{n-k} v^{n-k} \right] du\, dv$$

$$= \sum_{k=0}^{\infty} \frac{(-n)_k\ (a)_k\ \alpha^k}{(b)_k\ k!} \left(\frac{4c}{x} \right)^{n-k} \int_0^\infty \int_0^\infty u^{\mu+n-k+\frac{1}{2}} v^{\mu+n-k} (1-u-v)^{-\frac{3}{2}}\, du\, dv$$

From (14), we write

$$=\sum_{k=0}^{\infty}\frac{(-n)_k\,(a)_k\,\alpha^k}{(b)_k\,k!}\left(\frac{4c}{x}\right)^{n-k}\frac{\Gamma\left(\mu+n-k+\frac{3}{2}\right)\Gamma(\mu+n-k+1)}{\Gamma\left(2\mu+2n-2k+\frac{5}{2}\right)}\int_0^{\infty} z^{2\mu+2n-2k+\frac{3}{2}}(1-z)^{-\frac{3}{2}}dz$$

$$=\sum_{k=0}^{\infty}\frac{(-n)_k\,(a)_k\,\alpha^k}{(b)_k\,k!}\left(\frac{4c}{x}\right)^{n-k}\frac{\Gamma\left(\mu+n-k+\frac{3}{2}\right)\Gamma(\mu+n-k+1)}{\Gamma\left(2\mu+2n-2k+\frac{5}{2}\right)}\int_0^{\infty} z^{2\mu+2n-2k+\frac{3}{2}}(1-z)^{-\frac{3}{2}}dz$$

A. Erdelyi [6,p.10,(12)] stated the result $\int_0^{\infty} t^{x-1}(1+bt)^{-x-y}\,dt=b^{-x}\,B(x,y)$, we have

$$=\sum_{k=0}^{\infty}\frac{(-n)_k\,(a)_k\,\alpha^k}{(b)_k\,k!}\left(\frac{4c}{x}\right)^{n-k}\frac{\Gamma\left(\mu+n-k+\frac{3}{2}\right)\Gamma(\mu+n-k+1)}{\Gamma\left(2\mu+2n-2k+\frac{5}{2}\right)}B\left(2\mu+2n-2k+\frac{5}{2},2k-2\mu-2n-1\right)$$

$$=\sum_{k=0}^{\infty}\frac{(-n)_k\,(a)_k\,\alpha^k}{(b)_k\,k!}\left(\frac{4c}{x}\right)^{n-k}\frac{\Gamma\left(\mu+n-k+\frac{3}{2}\right)\Gamma(\mu+n-k+1)}{\Gamma\left(2\mu+2n-2k+\frac{5}{2}\right)}\frac{\Gamma\left(2\mu+2n-2k+\frac{5}{2}\right)\Gamma(2k-2\mu-2n-1)}{\Gamma\left(\frac{3}{2}\right)}$$

$$=\sum_{k=0}^{\infty}\frac{(-n)_k\,(a)_k\,\alpha^k}{(b)_k\,k!}\left(\frac{4c}{x}\right)^{n-k}\Gamma\left(\mu+n-k+\frac{3}{2}\right)\Gamma(\mu+n-k+1)\,\frac{\Gamma(2k-2\mu-2n-1)}{\frac{1}{2}\sqrt{\pi}}$$

$$=\sum_{k=0}^{\infty}\frac{(-n)_k\,(a)_k\,\alpha^k}{(b)_k\,k!}\left(\frac{4c}{x}\right)^{n-k}\Gamma\left(\mu+n-k+1+\frac{1}{2}\right)\Gamma(\mu+n-k+1)\frac{\Gamma(2k-2\mu-2n-1)}{\frac{1}{2}\sqrt{\pi}}$$

by using the result, $\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)=\frac{\sqrt{\pi}\,\Gamma(2z)}{2^{2z-1}}$

$$=\sum_{k=0}^{\infty}\frac{(-n)_k\,(a)_k\,\alpha^k}{(b)_k\,k!}\left(\frac{4c}{x}\right)^{n-k}\frac{\sqrt{\pi}\,\Gamma(2\mu+2n-2k+2)}{2^{2\mu+2n-2k+1}}\;\frac{\Gamma(2k-2\mu-2n-1)}{\frac{1}{2}\sqrt{\pi}}$$

$$=\sum_{k=0}^{\infty}\frac{(-n)_k\,(a)_k\,\alpha^k}{(b)_k\,k!}\left(\frac{4c}{x}\right)^{n-k}\frac{\Gamma(2\mu+2n-2k+2)}{2^{2\mu+2n-2k}}\;\Gamma(2k-2\mu-2n-1)$$

$$=\frac{1}{2^{2\mu}}\sum_{k=0}^{\infty}\frac{(-n)_k\,(a)_k\,\alpha^k}{(b)_k\,k!}\left(\frac{c}{x}\right)^{n-k}\ \Gamma(2\mu+2n-2k+2)\Gamma(2k-2\mu-2n-1)$$

$$=\frac{1}{2^{2\mu}}\sum_{k=0}^{\infty}\frac{(-n)_k\,(a)_k\,\alpha^k}{(b)_k\,k!}\left(\frac{c}{x}\right)^{n-k}\ \Gamma(2\mu+2n-2k+2)\Gamma(2k-2\mu-2n-1)$$

$$=\frac{1}{2^{2\mu}}\sum_{k=0}^{\infty}\frac{(-n)_k\,(a)_k\,\alpha^k}{(b)_k\,k!}\left(\frac{c}{x}\right)^{n-k}\ \Gamma(2\mu+2n-2k+2)\Gamma(2k-2\mu-2n-1)$$

$$=\left(\frac{1}{2}\right)^{2\mu}\sum_{k=0}^{\infty}\frac{(-n)_k\,(a)_k\,\alpha^k}{(b)_k\,k!}\left(\frac{c}{x}\right)^{n-k}\ \Gamma(2\mu+2n-2k+2)\Gamma(1-(2\mu+2n-2k+2))$$

by using result, $\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z}$, we write

$$=\left(\frac{1}{2}\right)^{2\mu}\sum_{k=0}^{\infty}\frac{(-n)_k\,(a)_k\,\alpha^k}{(b)_k\,k!}\left(\frac{c}{x}\right)^{n-k}\ \frac{\pi}{\sin[\pi(2\mu+2n-2k+2)]}$$

$$=\left(\frac{1}{2}\right)^{2\mu}\sum_{k=0}^{\infty}\frac{(-n)_k\,(a)_k\,\alpha^k}{(b)_k\,k!}\left(\frac{c}{x}\right)^{n-k}\ \frac{\pi}{\sin(2\mu\pi)}$$

$$=\left(\frac{1}{2}\right)^{2\mu}\ \frac{\pi}{\sin(2\mu\pi)}\sum_{k=0}^{\infty}\frac{(-n)_k\,(a)_k\,\alpha^k}{(b)_k\,k!}\left(\frac{c}{x}\right)^{n-k}$$

$$\int_0^{\infty}\int_0^{\infty}u^{\mu+\frac{1}{2}}v^{\mu}\,(1-u-v)^{-\frac{3}{2}}\,R_n^{(a,b)}\left(\frac{x}{4uvc};\alpha\right)du\,dv=\left(\frac{1}{2}\right)^{2\mu}\ \frac{\pi}{\sin(2\mu\pi)}R_n^{(a,b)}\left(\frac{x}{c},\alpha\right)$$

$$R_n^{(a,b)}\left(\frac{x}{c};\alpha\right)\ =\frac{(2)^{2\mu}\sin(2\mu\pi)}{\pi}\int_0^{\infty}\int_0^{\infty}u^{\mu+\frac{1}{2}}v^{\mu}\,(1-u-v)^{-\frac{3}{2}}\,R_n^{(a,b)}\left(\frac{x}{4uvc};\alpha\right)du\,dv$$

$$R_n^{(a,b)}\left(\frac{x}{c};\alpha\right)\ =\frac{(4)^{\mu}\sin(2\mu\pi)}{\pi}\int_0^{\infty}\int_0^{\infty}u^{\mu+\frac{1}{2}}v^{\mu}\,(1-u-v)^{-\frac{3}{2}}\,R_n^{(a,b)}\left(\frac{x}{4uvc};\alpha\right)du\,dv \qquad (15)$$

The equations (3) to (13) and (15) are not in literature.

5. Conclusion

In the present paper, we obtained finite difference formula, Contour Integral Representation, Real Integral Representation, Single Infinite Integral Representation, Finite Single Integral Representation, Infinite Single Integral Representation, Finite Double Integral Representation and Infinite Double Integral Representation of the one variable New Generalized Gauss Hypergeometric Polynomials.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

1. Bajpai, S. D., and M. S. Arora. "Semi-Orthogonality of a Class of the Gauss' Hypergeometric Polynomials." *Annales Mathématiques Blaise Pascal* 1, no. 1 (1994): 75–83. https://www.numdam.org/item/AMBP_1994__1_1_75_0.pdf.
2. Edwards, J. *A Treatise on the Integral Calculus*. Vol. 2. London: Macmillan and Co. Ltd., 1922. https://archive.org/details/in.ernet.dli.2015.501480.
3. Srivastava, H. M., and P. W. Karlsson. *Multiple Gaussian Hypergeometric Series*. New York: Halsted Press, John Wiley & Sons, 1985. https://ia803401.us.archive.org/1/items/in.ernet.dli.2015.134480/2015.134480.Multiple-Gaussian-Hypergeometric-Series_text.pdf.
4. Rainville, E. D. *Special Functions*. New York: Chelsea Publishing Company, 1971.
5. Andhare, P. G., and R. R. Jagtap. "Integral Representation of Polynomial." *International Journal for Research in Applied Science & Engineering Technology* 9, no. 8 (2021): 2452–2457. https://doi.org/10.22214/ijraset.2021.37788.
6. Erdélyi, A. *Higher Transcendental Functions*. Vol. 1. New York: McGraw-Hill Book Company, Inc., 1953. https://s3.amazonaws.com/apps.nrbook.com/bateman/Vol1.pdf.