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Laplace transform of Bessel functions with its applications

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Abstract:

Many academics are fascinated by integral transforms, which leads to the gradual formation of many sorts. In the realm of special functions, integral transformations are crucial. The Laplace Transforms, an important class of Integral Transform, of a number of notable Special Functions are examined in this work. Bessel's functions are mathematically significant for solving differential equations, particularly in real-world applications. Numerous application equations and their solutions using these functions are the subject of extensive research. Because integral transforms are so effective at obtaining precise solutions to a wide range of differential and integral equations, this study applies the "Laplace Transforms method for Bessel functions."

The two-dimensional chlorine transport model in pipes is examined in this work. The model under study takes the shape of a partial differential equation of second order with a set of boundary conditions. Because of the nature of the involved boundary, it is difficult to provide a precise solution for the current model. circumstances, particularly when using the Laplace transform. Nevertheless, these issues are resolved by applying the residues approach. The Bessel functions are used to express the precise solution. Additionally, the expression for the average concentration of a dimensionless cup-mixing is determined analytically. Numerical examples are used to validate the suggested method and compare the outcomes with those found in the literature. The current method is clear-cut and efficient, and it may be used on other comparable models with other boundary conditions.

Keywords: Special functions, Bessel's Functions, Laplace Transform.

1. Introduction

This article defines some renowned Special Functions (Debnath [1], Snedon [2]). and derives their Laplace transforms. Initially, we define the Laplace transform of F(t) as

$$L\{F(t)\} = \int_0^\infty e^{-st} F(t)dt, where\ s > 0 \tag{1}$$

The subsequent renowned Special Functions are defined as per Rainville [3], and Srivastava and Manocha [4]. Define the Poch hammer symbol $(\alpha)_n$ by the equation

$$(\alpha)_n = \alpha(\alpha+1)(\alpha+2)(\alpha+3)\ldots\ldots\ldots..(\alpha-n+1)$$

$$= \prod_{m=1}^{n}(\alpha-n+1), n \geq 0\ and\ (\alpha)_0 = 1, \alpha \neq 0 \tag{2}$$

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The function $(\alpha)_n$ is referred to as the fractional function and it is clear that $(1)_n = n!$ The data presented here are highly beneficial for the study of Special Functions.

A. If n$\in N, then$

$$\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = (\alpha)_n\,, \quad \alpha \neq 0$$

B. If α $is\ not\ an\ integer\ , then$ $\frac{\Gamma(1-\alpha+n)}{\Gamma(1-\alpha)} = \frac{(-1)^n}{(\alpha)_n}$

C. $(1-z)^{-\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n (z)^n}{n!}$

D. $(\alpha)_{n-k} = \frac{(\alpha)_n (-1)^n}{(1-\alpha-n)_n}$ $,0 \leq k \leq n$

$$particularly\ \alpha = 1, then\ \frac{n!\ (-1)^k}{(-n)_k}\ ,0 \leq k \leq n$$

E. $(\alpha)_{2n} = 2^{2n} \left(\frac{\alpha}{2}\right)_n \left(\frac{\alpha+1}{2}\right)_n$

Bessel functions are increasingly employed alongside other equations, including those in physics and mathematics, to address a broad spectrum of scientific and engineering challenges. Bessel functions can be employed to address many issues in mathematical and physical applications [5].

Functions of Bessel of order is given by [5]

$$J_n(t) = \sum_{n=0}^{\infty} \frac{(-1)^k (t)^{2k+1}}{(2)^{2k+1} k!\,\Gamma(1+n+k)}$$

$$= \frac{t^n}{2^n n!}\left[1 - \frac{t^2}{4\Gamma(n+1)} + \frac{(t^2)^2}{4.4\Gamma(n^2+3n+2)} - \frac{(t^2)^3}{4.4.6\Gamma\,(n^2+3n+2)\Gamma(n+2)} + \cdots \ldots..\right] \qquad (3)$$

If n=0, The infinite power series yields functions of Bessel of zero order, denoted by $J_0(t)$.

$$J_0(t) = \left[1 - \frac{t^2}{4} + \frac{(t^2)^2}{4^3} - \frac{(t^2)^3}{4^3.6^2} + \cdots \ldots..\right] \qquad (4)$$

If n=1, The infinite power series yields functions of Bessel of first order, denoted by $J_1(t)$.

$$J_1(t) = \frac{t}{2} + \frac{t^3}{4^2} + \frac{t^5}{4^3.6} - \frac{t^7}{4.4^2.6^2 6.8} + \cdots \ldots .... \qquad (5)$$

Now eq. (5) can be written as

$$J_1(t) = \frac{t}{2} + \frac{t^3}{2^3.2!} + \frac{t^5}{2^5.2!.3!} - \frac{t^7}{2^7.3!.4!} + \cdots \ldots .... \qquad (6)$$

And if n=2, The infinite power series yields functions of Bessel of second order, denoted by $J_2(t)$.

$$J_2(t) = \frac{t^2}{2.4} - \frac{t^4}{4^2.6} + \frac{t^6}{4^3.6^2.8.10} + \cdots \ldots .... \qquad (7)$$

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1.1.Relationships Between $J_0(t)$ and $J_1(t)$

$$\frac{d}{dt}J_0(t) = -J_1(t) \tag{8}$$

1.2. Relationships Between $J_0(t)$ and $J_2(t)$

$$J_2(t) = J_0(t) + 2J_0''(t) \tag{9}$$

1.3. Laplace Transform of Bessel Functions

From equations (1) and (4), we derive

$$L\{J_n(t)\} = L\left\{\sum_{k=0}^{\infty}\frac{(-1)^k(t)^{2k+1}}{(2)^{2k+1}k!\Gamma(1+n+k)}\right\}$$

$$= \int_0^{\infty} e^{-st}\sum_{k=0}^{\infty}\frac{(-1)^k(t)^{2k+1}}{(2)^{2k+1}k!\Gamma(1+n+k)}dt$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k}{(2)^{2k+1}k!\Gamma(1+n+k)}\int_0^{\infty} e^{-st}\sum_{k=0}^{\infty}(t)^{2k+1}\,dt$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k}{(2)^{2k+1}k!\Gamma(1+n+k)}\frac{\Gamma(2k+n+1)}{s^{2k+n+1}}$$

$$= \frac{1}{2^n}\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k}k!}\frac{\Gamma(n+1)}{\Gamma(n+k+1)}\frac{\Gamma(2k+n+1)}{\Gamma(n+1)}.\frac{1}{s^{2k}}$$

Based on outcomes A and E, this may be expressed as

$$L\{J_n(t)\} = \frac{1}{2^n}\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k}k!}\frac{(1+n)_{2k}}{(1+n)_k}.\frac{1}{s^{2k}}$$

$$= \frac{1}{2^n}\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k}k!}\frac{2^{2k}(\frac{1+n}{2})_k(1+\frac{n}{2})_k}{(1+n)_k}.\frac{1}{s^{2k}} \tag{10}$$

2. MAIN RESULTS

Water quality research is expanding because of its significance in engineering sciences and industry. Chlorine is utilized in most parts of the world as a guarantee for the provision of clean drinking water because of its significance in drinking water quality. [6, 7]. To maintain the water's purity, some chlorine must be left in the water to stop any microbes from surviving and growing while the water moves through the networks. Thus, keeping the chlorine level at a specific level guarantee that no hazardous byproducts are created inside the distribution networks.

According to Ref. [8,] {0.2} mgl $^{\{-1\}}$ is the proportion of chlorine concentration that guarantees the avoidance of public health hazards. As a result, maintaining the previously indicated chlorine content and making sure that this percentage stays below the designated threshold are essential for the effective management of drinking water quality. Biswas et al. [9] developed the fundamental model of chlorine transport, which looks like this:

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$$\frac{\partial f}{\partial x} = \frac{A_0}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial f}{\partial r}\right) - A_1 f, \tag{11}$$

and

$$\begin{aligned} f(0,r) &= 1, && 0 \le r \le 1, \\ \frac{\partial}{\partial r} f(x,0) &= 0, && 0 \le x \le 1, \\ \frac{\partial}{\partial r} f(x,1) + A_2 f(x,1) &= 0, && 0 \le x \le 1, \end{aligned} \tag{12}$$

where $f(x,r)$ is the chlorine concentration, and A_0, A_1 and A_2 are dimensionless parameters related to the chlorine decay. Further, details about the dimensionless parameters and derivation of the Eqs. (11)-(12) can be found in Ref. [9]. The purpose of the present work is to solve the system (11)-(12) through applying the Laplace transform (LT). The LT is a well-known and effective approach to solving various scientific models in physics and engineering $11-18$. Ebaid and Al sharif [11] applied the LT on the ODE governing the heat transfer of nanofluids suspended with carbon-nanotubes. Ebaid et. al [12] provided the analytic solution for a class of singular boundary value problems (SBVPs) via the LT. Khaled [13] obtained the exact solution of the model describing the radiation effect on MHD Marangoni convection over a flat plate. Ebaid et. al [14] solved a general class of SBVPs with applications in nanofulids via the LT. Bakodah and Ebaid [15] addressed the Ambartsumian delay equation using the LT. A variety of other of LT applications in addition to other transforms can be found in Refs. $[16-35]$. In this paper we consider the application of the LT for the solution of the system (11)-(12). The paper is structured as follows. The LT approach is applied in "Application of LT" on Eqs. (11)-(4). Section "Analysis and exact solution of the chlorine decay model" is devoted to obtaining exact solution. Section "Discussion of results" analyses the results and discusses their physical meaning. In addition, the results are to be compared with those in Ref. [9]. Finally, Section "Conclusion" outlines the main conclusions.

Now Applying LT on Eq. (10) with respect to the variable x, we can write

$$L\left(\frac{\partial f(x,r)}{\partial x}\right) = L\left(\frac{A_0}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial f(x,r)}{\partial r}\right)\right) - L\big(A_1 f(x,r)\big), \tag{13}$$

which gives

$$sf(s,r) - f(0,r) = \frac{A_0}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial f(s,r)}{\partial r}\right) - A_1 f(s,r), \tag{14}$$

where $L(\cdot)$ and s denote the Laplace operator and variable, respectively. After introducing the BC given in (2), Eq. (14) becomes

$$(s + A_1) f(s,r) - 1 = \frac{A_0}{r}\frac{d}{dr}\left(\frac{1}{r}\frac{df(s,r)}{dr}\right). \tag{15}$$

Re-arranging Eq. (15), yields

$$\frac{d^2 f(s,r)}{dr^2} + \frac{1}{r}\frac{df(s,r)}{dr} - \left(\frac{s+A_1}{A_0}\right) f(s,r) = -\frac{1}{A_0}, \tag{16}$$

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which is the well-known Bessel differential equation with inhomogeneous part ($-\frac{1}{A_0}$). The solution of Eq. (16) is given by

$$f(s,r) = c_1 J_0\left(i\sqrt{\frac{s+A_1}{A_0}}r\right) + c_2 Y_0\left(i\sqrt{\frac{s+A_1}{A_0}}r\right) + \frac{1}{s+A_1}, \tag{17}$$

where $J_0(\cdot)$ and $Y_0(\cdot)$ are Bessel functions and c_1 and c_2 denote unknown constants, $i = \sqrt{-1}$. Since $f(x,r)$ and its LT, $f(s,r)$, must be bounded at $r = 0$, the value of c_2 must be zero since $Y_0\left(i\sqrt{\frac{s+A_1}{A_0}}r\right) \to \infty$ as $r \to 0$. Thus, we can write

$$f(s,r) = c_1 J_0\left(i\sqrt{\frac{s+A_1}{A_0}}r\right) + \frac{1}{s+A_1} \tag{18}$$

and

$$\frac{df(s,r)}{dr} = -c_1 i\sqrt{\frac{s+A_1}{A_0}} J_1\left(i\sqrt{\frac{s+A_1}{A_0}}r\right), \tag{19}$$

where $J_0'(\lambda r) = -\lambda J_1(\lambda r)$. From Eqs. (18) and (19), we obtain

$$\frac{df(s,r)}{dr} + A_2 f(s,r) = c_1\left[A_2 J_0\left(i\sqrt{\frac{s+A_1}{A_0}}r\right) - i\sqrt{\frac{s+A_1}{A_0}} J_1\left(i\sqrt{\frac{s+A_1}{A_0}}r\right)\right] + \frac{A_2}{(s+A_1)}. \tag{20}$$

Applying LT on (12) yields

$$\left[\frac{df(s,r)}{dr} + A_2 f(s,r)\right]_{r=1} = 0. \tag{21}$$

From Eqs. (20) and (21), we obtain

$$c_1 = -\frac{A_2}{(s+A_1)\left[A_2 J_0\left(i\sqrt{\frac{s+A_1}{A_0}}\right) - i\sqrt{\frac{s+A_1}{A_0}} J_1\left(i\sqrt{\frac{s+A_1}{A_0}}\right)\right]}. \tag{22}$$

Substituting (22) into (18) leads to

$$f(s,r) = -\frac{A_2 J_0\left(i\sqrt{\frac{s+A_1}{A_0}}r\right)}{(s+A_1)\left[A_2 J_0\left(i\sqrt{\frac{s+A_1}{A_0}}\right) - i\sqrt{\frac{s+A_1}{A_0}} J_1\left(i\sqrt{\frac{s+A_1}{A_0}}\right)\right]} + \frac{1}{s+A_1} \tag{23}$$

However, Eq. (23) can be written as

$$f(s,r) = -A_2 F(s,r) + \frac{1}{s+A_1}, \tag{24}$$

were

$$F(s,r) = \frac{J_0\left(i\sqrt{\frac{s+A_1}{A_0}}r\right)}{(s+A_1)\left[A_2 J_0\left(i\sqrt{\frac{s+A_1}{A_0}}\right) - i\sqrt{\frac{s+A_1}{A_0}} J_1\left(i\sqrt{\frac{s+A_1}{A_0}}\right)\right]}. \tag{25}$$

Applying the inverse LT on Eq. (24), yields

$$L^{-1}\big(f(s,r)\big) = -A_2 L^{-1}\big(F(s,r)\big) + L^{-1}\left(\frac{1}{s+A_1}\right), \tag{26}$$

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or

$$f(x,r) = -A_2 f(x,r) + e^{-A_1 x}, \tag{27}$$

where $f(x,r)$ is the inverse LT of $F(s,r)$ so that

$$f(x,r) = L^{-1}\left(\frac{J_0\left(i\sqrt{\frac{s+A_1}{A_0}}r\right)}{(s+A_1)\left[A_2 J_0\left(i\sqrt{\frac{s+A_1}{A_0}}\right) - i\sqrt{\frac{s+A_1}{A_0}}J_1\left(i\sqrt{\frac{s+A_1}{A_0}}\right)\right]}\right). \tag{28}$$

Analysis and exact solution of the chlorine decay model

Analysis. The below theorem introduces the method of residues when applied to calculating the inverse LT.
Theorem 1 (Method of residues 31) Let s_i are the poles of $F(s,r)$, then $f(x,r)$ (inverse LT of $F(s,r)$) is $f(x,r) = \sum_{i=1}^{n} Res\left(e^{s_i x}F(s_i,r)\right)$ at all poles s_i.

It will be shown later that the inverse LT of the function $F(s,r)$, defined in (25), using the residues, can be obtained in terms of Bessel functions with the help of their properties. In this regard, the Bessel functions $J_0(y)$, $J_1(y)$ and $J_2(y)$ are defined by the expressions:

$$\begin{aligned} J_0(y) &= \sum_{k=0}^{\infty}\frac{(-1)^k}{(k!)^2}\left(\frac{y}{2}\right)^{2k}, \\ J_1(y) &= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+1)!}\left(\frac{y}{2}\right)^{2k+1}, \\ J_2(y) &= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+2)!}\left(\frac{y}{2}\right)^{2k+2}, \end{aligned} \tag{29}$$

that satisfy the following properties:

$$\begin{aligned} \frac{d}{dy}\left(J_0(\lambda y)\right) &= -\lambda J_1(\lambda y) \\ \frac{d}{dy}\left(J_1(\lambda y)\right) &= \frac{\lambda}{2}\left(J_0(\lambda y) - J_2(\lambda y)\right), \\ yJ_2(y) + yJ_0(y) &= 2J_1(y). \end{aligned} \tag{30}$$

Exact solution of the chlorine decay model.

The main challenge of this paper is to obtain the inverse LT of the expression in Eq. (28). The expression (28) is really complex due to the nature of the boundary conditions (12). This is because the denominator in expression (28) involves Bessel functions of first and second kind which leads to actual difficulties when deriving the inverse LT of the expression in Eq. (28). However, such difficulties are overcome through applying the method of residues as indicated below.

At first sight, the expression $(s+A_1)\left[A_2 J_0\left(i\sqrt{\frac{s+A_1}{A_0}}\right) - i\sqrt{\frac{s+A_1}{A_0}}J_1\left(i\sqrt{\frac{s+A_1}{A_0}}\right)\right]$ has simple zeros at $s = -A_1$ and $i\sqrt{\frac{s+A_1}{A_0}} = \lambda_1, \lambda_2, \ldots \lambda_n, \ldots$, and thus we find simple poles at $s_1 = -A_1$ and $s_2 = -A_1 - A_0\lambda_n^2, n = 1,2,3, \ldots$. Therefore, the inverse LT of $F(s,r)$, i.e., $f(x,r)$ can be obtained from Theorem 1 by calculating the residues (Res) of $e^{sx}F(s,r)$ at $s_1 = -A_1$ and $s_2 = -A_1 - A_0\lambda_n^2$, and then by taking their sum. At $s_1 = -A_1$, we have

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$$(\text{Res})_{s_1} = \lim_{s\to s_1}(s - s1)e^{sx}F(s,r) \tag{31}$$

$$= e^{-A1x}\lim_{s\to -A1}\left(\frac{J_0\left(i\sqrt{\frac{s+A_1}{A_0}}\right)r}{\left[A_2J_0\left(i\sqrt{\frac{s+A_1}{A_0}}\right)-i\sqrt{\frac{s+A_1}{A_0}}J_1\left(i\sqrt{\frac{s+A_1}{A_0}}\right)\right]}\right)$$

$$= e^{-A1x}\lim_{s\to -A1}\left(\frac{J_0(0)}{[A_2J_0(0) - 0]}\right)$$

$$= \frac{e^{-A1x}}{A2} \tag{32}$$

where J$_0$(0) =1.

At $s_2 = -A_1 - A_0\lambda_n^2$, we have

$$(\text{Res})_{s2} = \lim_{s\to s_1}\left(\frac{(s-s2)e^{sx}J_0\left(i\sqrt{\frac{s+A_1}{A_0}}\right)r}{(s+A_1)\left[A_2J_0\left(i\sqrt{\frac{s+A_1}{A_0}}\right)-i\sqrt{\frac{s+A_1}{A_0}}J_1\left(i\sqrt{\frac{s+A_1}{A_0}}\right)\right]}\right) \tag{33}$$

$$= \lim_{s\to s_2}\left(\frac{(s+A_1+A_0\lambda_n^2\ )}{\left[A_2J_0\left(i\sqrt{\frac{s+A_1}{A_0}}\right)-i\sqrt{\frac{s+A_1}{A_0}}J_1\left(i\sqrt{\frac{s+A_1}{A_0}}\right)\right]}\right)\times\lim_{s\to s_1}\left(\frac{e^{sx}J_0\left(i\sqrt{\frac{s+A_1}{A_0}}\right)r}{(s+A1)}\right) \tag{34}$$

$$= \lim_{s\to s_2}\left(\ (G(s,r))\times\frac{e^{-(A_1+A_0\lambda_n^2)x}J_0(-\lambda_n r)}{-A_0\lambda_n^2}\right) \tag{35}$$

The limit of $G(s,r)$ as $s \to s_2$ can be calculated using the L'Hospital's rule as follows

$$\lim_{s\to s_2}\ (G(s,r)) = \frac{lim_{s\to s_2}(s-s_2)}{lim_{s\to s_2}\left[A_2J_0\left(i\sqrt{\frac{s+A_1}{A_0}}\right)-i\sqrt{\frac{s+A_1}{A_0}}J_1\left(i\sqrt{\frac{s+A_1}{A_0}}\right)\right]} = \frac{0}{0} \tag{36}$$

$$= \frac{lim_{s\to s_2}\frac{d}{ds}(s+A_1+A_0\lambda_n^2)}{lim_{s\to s_2}\frac{d}{ds}\left[A_2J_0\left(i\sqrt{\frac{s+A_1}{A_0}}\right)-i\sqrt{\frac{s+A_1}{A_0}}J_1\left(i\sqrt{\frac{s+A_1}{A_0}}\right)\right]} \tag{37}$$

$$= \frac{1}{d} \tag{38}$$

were

$$d = \lim_{s\to s_2}\frac{d}{ds}\left[A_2J_0\left(i\sqrt{\frac{s+A_1}{A_0}}\right) - i\sqrt{\frac{s+A_1}{A_0}}J_1\left(i\sqrt{\frac{s+A_1}{A_0}}\right)\right] \tag{39}$$

$$= \frac{1}{4iA_0\lambda_n}[-2i(1 + A_2)J_1(-\lambda_n) + i\,\lambda_n(J_0(-\lambda_n) - J_2(-\lambda_n))]. \tag{40}$$

Since the functions J_0 and J_2 are even and J_1 is odd, we obtain

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$$d=\frac{1}{4A_0\lambda_n}\left[2(1+A_2)J_1(\lambda_n)+\lambda_n\left(J_0(\lambda_n)-J_2(\lambda_n)\right)\right]$$

$$=\frac{1}{4A_0\lambda_n}\left[2(1+A_2)J_1(\lambda_n)+\lambda_n\left(J_0(\lambda_n)-J_2(\lambda_n)\right)\right]$$

$$=\frac{1}{4A_0\lambda_n}\left[2(1+A_2)J_1(\lambda_n)+\lambda_nJ_0(\lambda_n)-2J_1(\lambda_n)+\lambda_nJ_0(\lambda_n)\right]$$

$$=\frac{1}{2A_0\lambda_n}\left[A_2J_1(\lambda_n)+\ \lambda_nJ_0(\lambda_n)\right]. \tag{41}$$

From (38) and (41), we get

$$\lim_{s\to s_2}(G(s,r))=\frac{2A_0\lambda_n}{[A_2J_1(\lambda_n)+\lambda_nJ_0(\lambda_n)]},$$

Substituting above equation into (35), yields

$$(\text{Res})_{s_2}=\left(\frac{e^{-(A_1+A_0\lambda_n^2)x}J_0(-\lambda_nr)}{-A_0\lambda_n^2}\right)\times\frac{2A_0\lambda_n}{[A_2J_1(\lambda_n)+\lambda_nJ_0(\lambda_n)]} \tag{42}$$

$$=-\frac{2e^{-(A_1+A_0\lambda_n^2)x}J_0(-\lambda_nr)}{\lambda_n[A_2J_1(\lambda_n)+\lambda_nJ_0(\lambda_n)]}$$

Hence,

$$f(x,r)=\frac{e^{-A_1x}}{A_2}-2\sum_{n=1}^{\infty}\frac{e^{-(A_1+A_0\lambda_n^2)x}J_0(\lambda_nr)}{\lambda_n[A_2J_1(\lambda_n)+\lambda_nJ_0(\lambda_n)]}. \tag{43}$$

Inserting (43) into (24) leads to

$$f(x,r)=2\sum_{n=1}^{\infty}\frac{A_2J_0(\lambda_nr)e^{-(A_1+A_0\lambda_n^2)x}}{\lambda_n[A_2J_1(\lambda_n)+\lambda_nJ_0(\lambda_n)]}, \tag{44}$$

where the symbols λ_n denote the roots of the equation:

$$A_2J_0(\lambda_n)-\lambda_nJ_1(\lambda_n)=0. \tag{45}$$

Using (45), the solution (44) can be written as

$$f(x,r)=2\sum_{n=1}^{\infty}\frac{\lambda_nJ_1(\lambda_n)J_0(\lambda_nr)}{(A_2^2+\lambda_n^2)J_0^2(\lambda_n)}e^{-(A_1+A_0\lambda_n^2)x}, \tag{46}$$

which agrees with solution in Ref. 4 derived by the separation of variables technique.

Discussion of results

The dimensionless cup-mixing average concentration is defined by

$$f_{av}=2\int_0^1 f\,(x,r)rdr. \tag{47}$$

Substituting (46) into (47), yields

$$f_{av}=4\sum_{n=1}^{\infty}\frac{\lambda_nJ_1(\lambda_n)}{(A_2^2+\lambda_n^2)J_0^2(\lambda_n)}e^{-(A_1+A_0\lambda_n^2)x}\int_0^1 r\,J_0(\lambda_nr)dr \tag{48}$$

or

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$$f_{av} = 4\sum_{n=1}^{\infty}\frac{J_1^2(\lambda_n)}{(A_2^2+\lambda_n^2)J_0^2(\lambda_n)}e^{-(A_1+A_0\lambda_n^2)x}. \tag{49}$$

After including the relation (45) we obtain

$$f_{av} = 4\sum_{n=1}^{\infty}\frac{A_2^2}{\lambda_n^2(A_2^2+\lambda_n^2)}e^{-(A_1+A_0\lambda_n^2)x}. \tag{50}$$

If $A_2 \rightarrow \infty$, then

$$f_{av} = \lim_{A_2\rightarrow\infty}\left(\sum_{n=1}^{\infty}\frac{A_2^2}{\lambda_n^2(A_2^2+\lambda_n^2)}e^{-(A_1+A_0\lambda_n^2)x}\right) \tag{51}$$

which gives

$$f_{av} = \sum_{n=1}^{\infty}\frac{4}{\lambda_n^2}e^{-(A_1+A_0\lambda_n^2)x}. \tag{52}$$

Moreover, if $A_2 \rightarrow 0$ then Eq. (27) implies

$$f(x,r) = e^{-A_1 x}, \tag{53}$$

and the corresponding u_{av} is obtained from Eq. (47) as

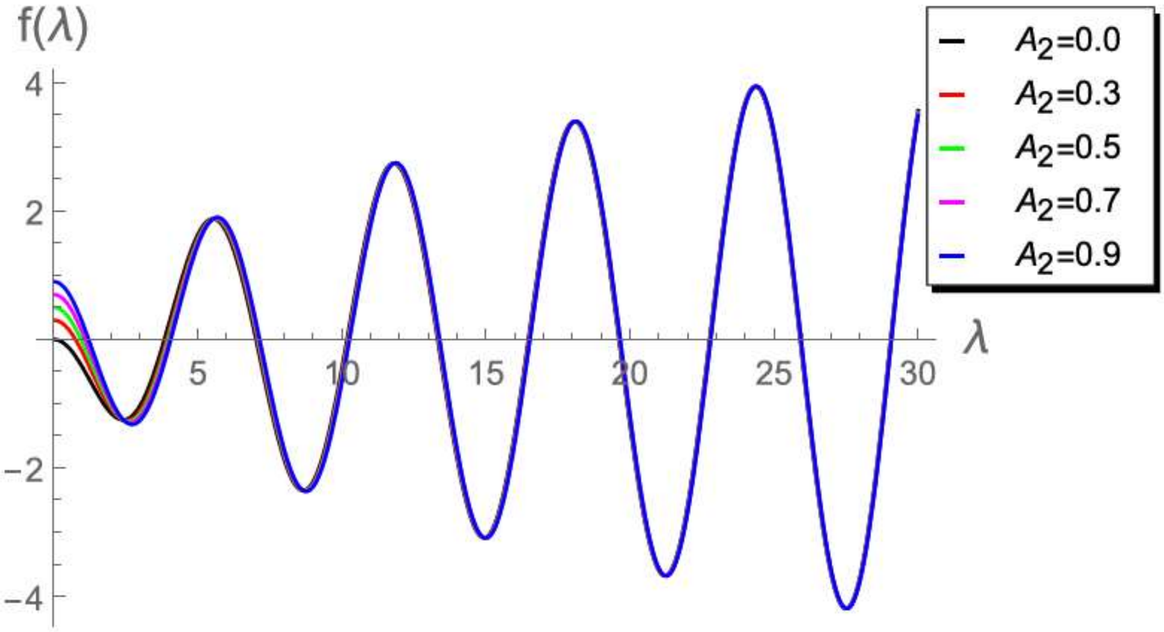

Figure 1. Plot of $f(\lambda)$ vs λ for $A_2 = \{0.0,0.3,0.5,0.7,0.9\}$.

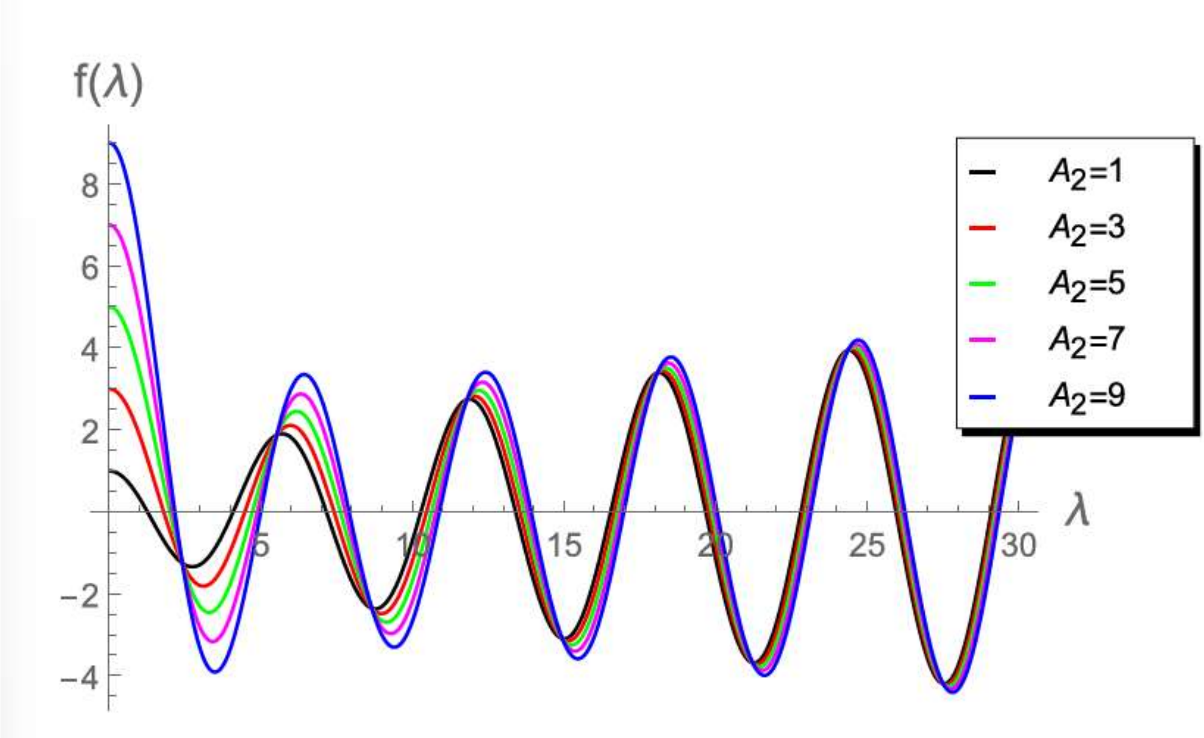

Figure 2. Plot of $f(\lambda)$ vs λ for $A_2 = \{1,3,5,7,9\}$.

$$f_{av} = 2\int_0^1 e^{-A_1 x}\, r dr = e^{-A_1 x} \tag{54}$$

Therefore, Eq. (50) gives the general expression for the f_{av} while Eqs. (52) and (54) are limiting cases. According to Biswas et al. 4, three roots λ_1, λ_2, and λ_3 of Eq. (45) were used. In addition, the following fitting functions were used to reproduce $(\lambda_1, \lambda_2, \lambda_3)$ in terms of A_2 at several ranges.

$$(i)\ \text{For} 0.01 \le A_2 < 1$$
$$\lambda_1 = 1.29861(A_2)^{0.477433}, \lambda_2 = 4.00946(A_2)^{0.0119894}, \lambda_3 = 7.11555(A_2)^{0.00376107}$$
$$\text{(ii) For} 1 \le A_2 < 10$$
$$\lambda_1 = 1.30427(A_2)^{0.239289}, \lambda_2 = 4.05693(A_2)^{0.0927629}, \lambda_3 = 7.10846(A_2)^{0.0463785}$$
$$\text{(iii) For} 10 \le A_2 < 1000$$
$$\lambda_1 = 2.10218(A_2)^{0.021361}, \lambda_2 = 4.86441(A_2)^{0.0200514}, \lambda_3 = 7.71165(A_2)^{0.0182292}$$

In order to have a numerical comparison between the current calculations of the first three roots λ_1, λ_2, and λ_3 of Eq. (45) and the corresponding results in 4 (using $(\lambda_1, \lambda_2, \lambda_3)$ in terms of A_2), we may write Eq. (45) as a function of λ in the form:

$$f(\lambda) = A_2 J_0(\lambda) - \lambda J_1(\lambda) \tag{55}$$

Figures 1,2,3,4 and 5) highlight details about the roots of $f(\lambda)$ for different values of A_2. We verify that we have an infinite number of roots. In addition, all the roots of $f(\lambda)$, excepting the first one, are nearly identical for small $A_2 \in$, as shown by Fig. 1. However, for the range $1 \le A_2 < 10$, Fig. 2 reveals that the first seven roots are considerably different, while the others have approximately the same values. Figure 3 indicates that the first two roots are nearly identical, while the rest of roots are considerably different for the range $10 \le A_2 < 45$. For higher values of A_2, namely in the ranges $50 \le A_2 < 90$ and $100 \le A_2 < 900$, Figs. 4 and 5 reveal that the roots of $f(\lambda)$ are nearly identical. However, the results introduced in^4 were mainly depend on obtaining the first three roots λ_1, λ_2 and λ_3 and, therefore, the proposed approach is more accurate.

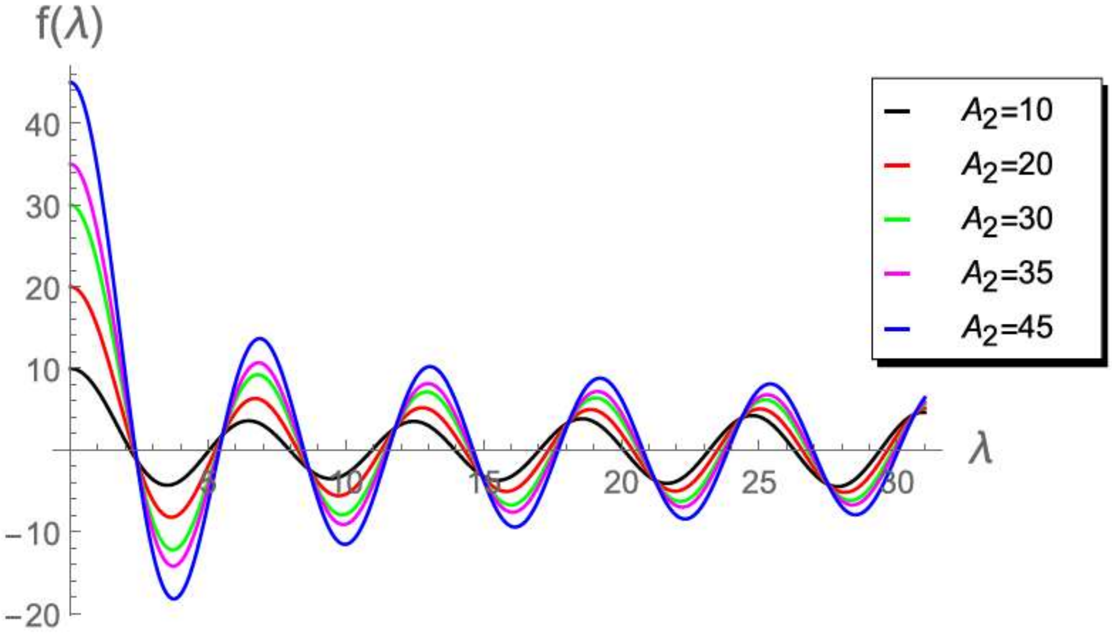

Figure 3. Plot of $f(\lambda)$ vs λ for $A_2 = \{10,20,30,35,45\}$.

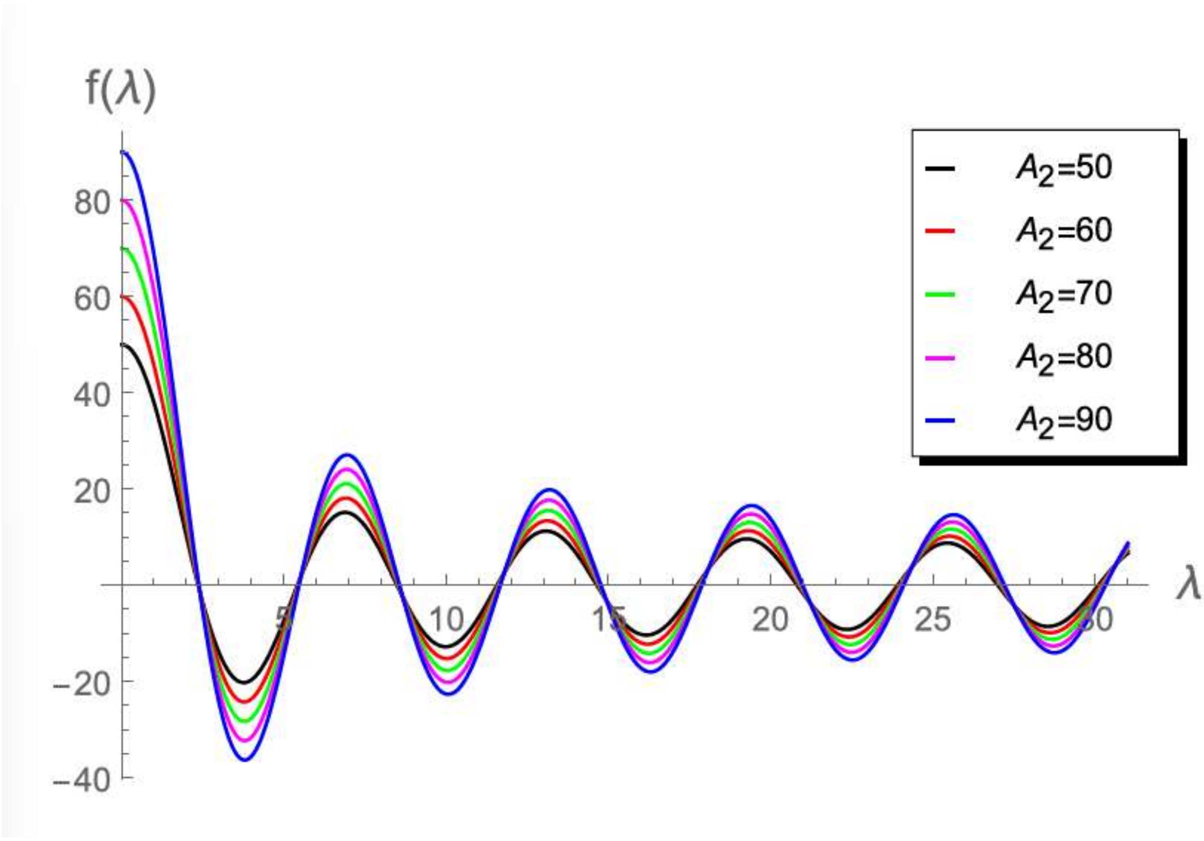

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Figure 4. Plot of $f(\lambda)$ vs λ for $A_2 = \{50,60,70,80,90\}$.

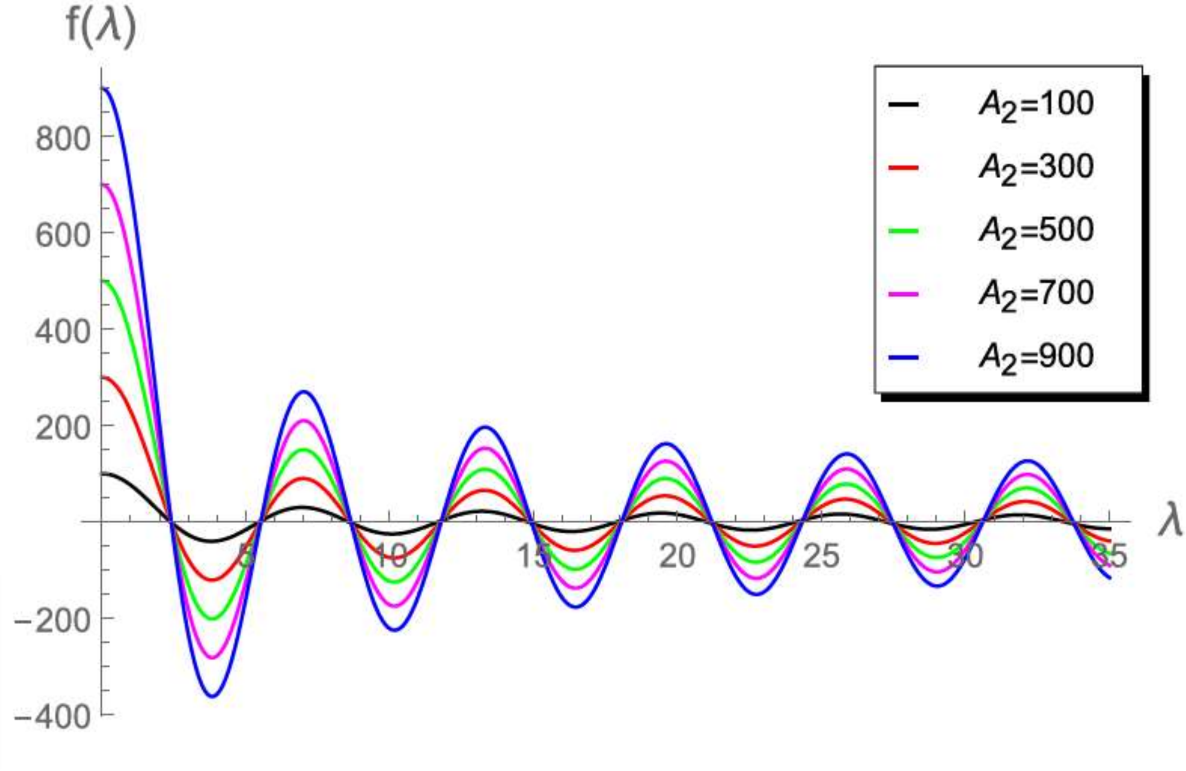

Figure 5. Plot of $f(\lambda)$ vs λ for $A_2 = \{100,300,500,700,900\}$.

Table 1 presents the three roots λ_1, λ_2 and λ_3 of Eq. (45) and the corresponding results from 4 (using Eq. (46)) in the interval $A_2 \in$. The calculations of the present roots are accomplished through MATHEMATICA. The absolute errors listed in Table 1 show that the results presented in 4 agree with the obtained ones only up to two/three digits at most. This means that the new approach leads to better results than those of 4 after 3 decimal places. A similar conclusion is also obtained in Table 2 regarding λ_1, λ_2 and λ_3 for the range $1 \le A_2 < 10$.

Table 3 shows that the absolute errors increase in the range $10 \le A_2 < 1000$. Such differences in the values may lead to differences when calculating the chlorine concentration or the cup-mixing average concentration. The behavior of the cup-mixing average concentration u_{av}, at the outlet $x = 1$ of a pipe, versus A_1 are displayed in Figs. 6, 7, 8 and 9 for several values of A_0 and A_2. These figures indicate that the f_{av} is always a decreasing function in the parameter A_1. This means that the cup-mixing average concentration decays with increasing the parameter A_1. In conclusion, the proposed approach gives a clear and precise solution of the mathematical model.

A_2	Present			Ref. 4			Absolute error		
	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
0.01	0.141 245	3.834 31	7.017 01	0.144 083	3.794 09	6.993 37	0.002 838	0.040 230	0.023 645

A_2	Present			Ref. 4			Absolute error		
0.10	0.441682	3.85771	7.02983	0.432559	3.90029	7.05419	0.009122	0.042576	0.024369
0.20	0.616975	3.88351	7.04403	0.602237	3.93283	7.07261	0.014738	0.049329	0.028579
0.50	0.940771	3.95937	7.08638	0.932732	3.97628	7.09702	0.008038	0.016907	0.010643

Table 1. Comparisons of present roots λ_1, λ_2, and λ_3 of Eq. (45) and the corresponding results in Ref. 4 using$(\lambda_1, \lambda_2, \lambda_3)$ in terms of A_2),) $(0.01 \le A_2 < 1)$.

A_2	Present			Ref. 4			Absolute error		
	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
1	1.25578	4.07948	7.15580	1.30427	4.05693	7.10846	0.04849	0.02255	0.04734
2	1.59945	4.29096	7.28839	1.53957	4.32635	7.34069	0.05987	0.03539	0.05230
5	1.98981	4.71314	7.61771	1.91701	4.71016	7.65936	0.07281	0.00298	0.04166

Table 2. Comparisons of present roots λ_1, λ_2, and λ_3 of Eq. (45) and the corresponding results in Ref. 4 using $(\lambda_1, \lambda_2, \lambda_3)$ in terms of A_2)) $(1 \le A_2 < 10)$.

A_2	Present			Ref. 4			Absolute error		
	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
10	2.17950	5.03321	7.95688	2.20816	5.09427	8.04223	0.02867	0.06105	0.08535
50	2.35724	5.41120	8.48399	2.28540	5.26135	8.28167	0.07185	0.14985	0.20231
100	2.38090	5.46521	8.56783	2.31949	5.33498	8.38698	0.06141	0.13023	0.18085

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Table 3. Comparisons of present roots λ_1, λ_2, and λ_3 of Eq. (45) and the corresponding results in Ref. 4 using $(\lambda_1, \lambda_2, \lambda_3)$ in terms of A_2)) ($10 \leq A_2 < 1000$).

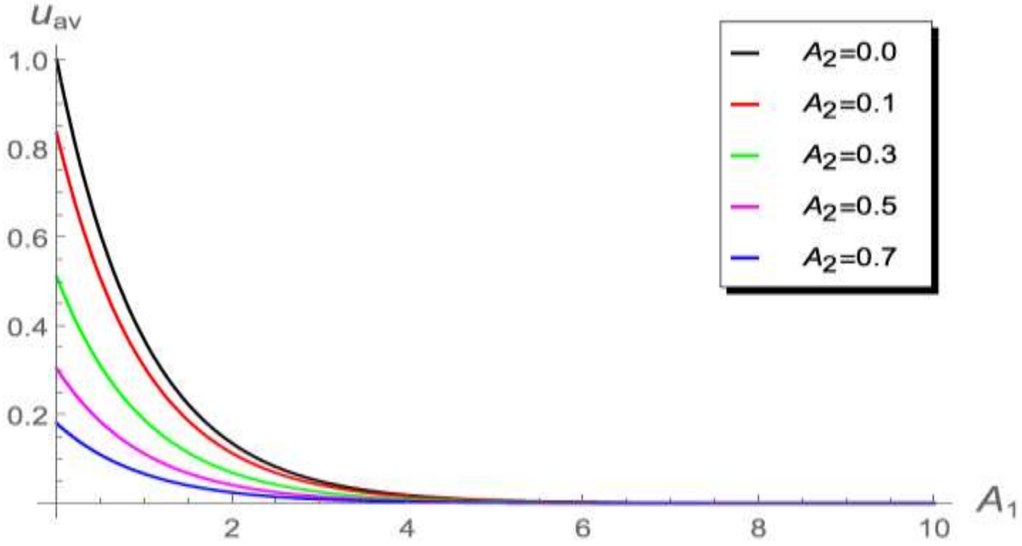

Figure 6. The f_{av} against A_1 at different values of $A_2, A_0 = 1.4$.

Conclusion:

The Laplace integral transforms of Bessel functions will be used in this essay to evaluate definite integrals. The findings demonstrate that Laplace transforms for these functions can quickly and with little computational effort determine integrals including Bessel functions in the integrand.

The two-dimensional chlorine model was discovered to have a theoretical solution. The exact response was obtained using the LT. The inverse LT of complex expressions was found using the residues approach, and the solution was expressed in terms of Bessel functions of the first and second kinds of order zero. The exact outcomes are consistent with those previously published by the separation of variables approach. However, the resulting numerical findings are superior to those given in^4 due to the limitations on the computation of the roots of Eq. (45). In previous publications, a range of fitting curves were used to quantitatively forecast the values of such roots at specific values of A_2.

This fitting method contains some numerical flaws, as shown in Tables (1-3), and the absolute errors might be substantial in several cases. An analytical expression for the dimensionless cup-mixing average concentration was provided. The results showed that the new approach offers reliable and accurate solutions to the problem.

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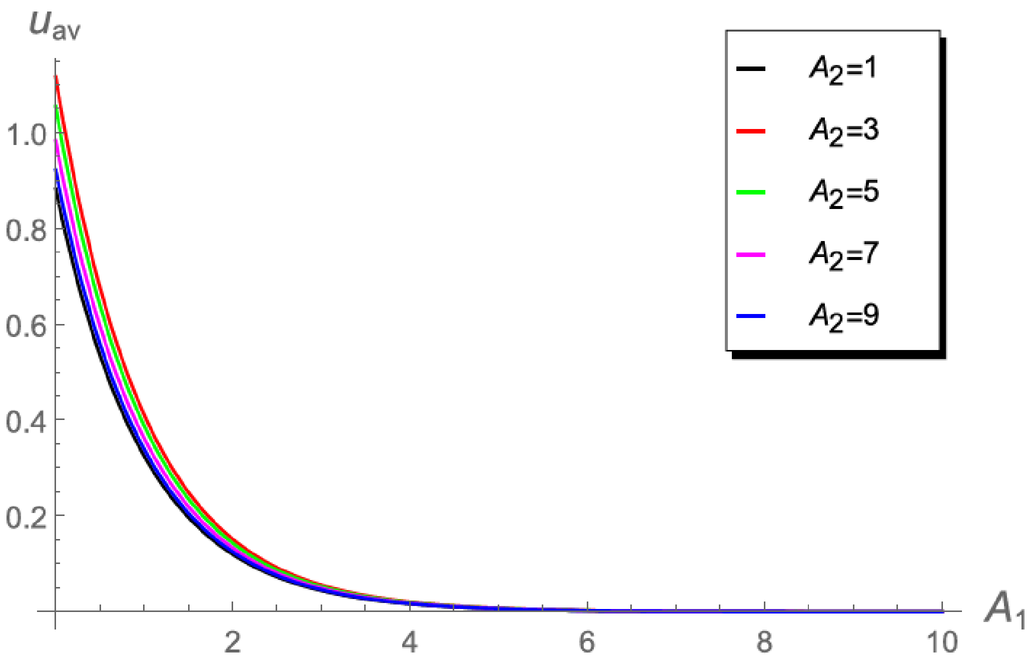

Figure 7. The u_{av} against A_1 at different values of $A_2, A_0 = 1.4 \times 10^{-3}$.

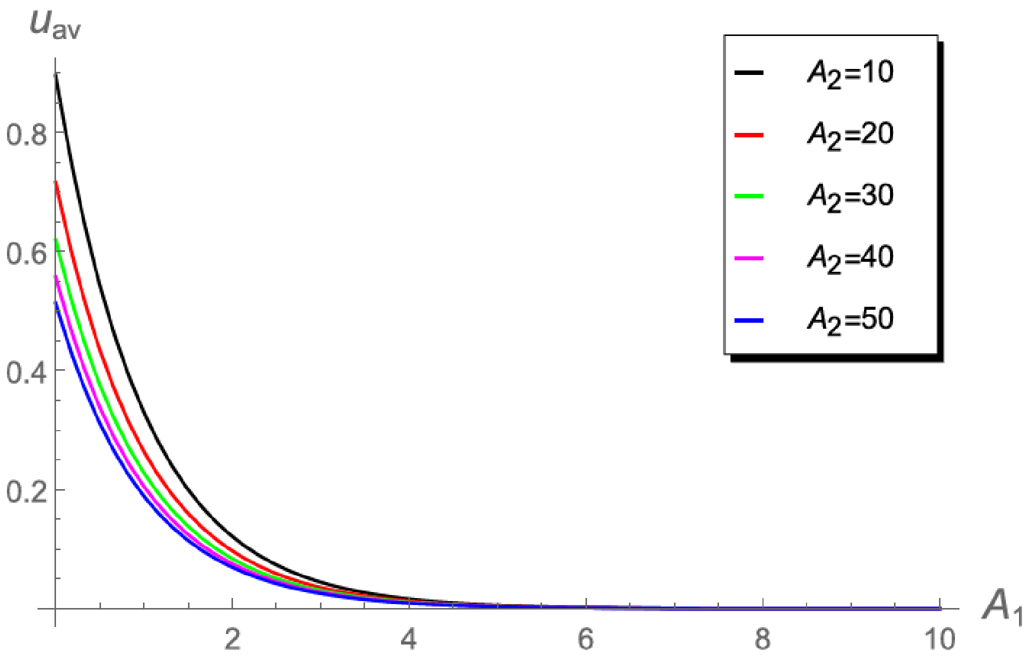

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Figure 8. The u_{av} against A_1 at different values of $A_2, A_0 = 1.4 \times 10^{-2}$.

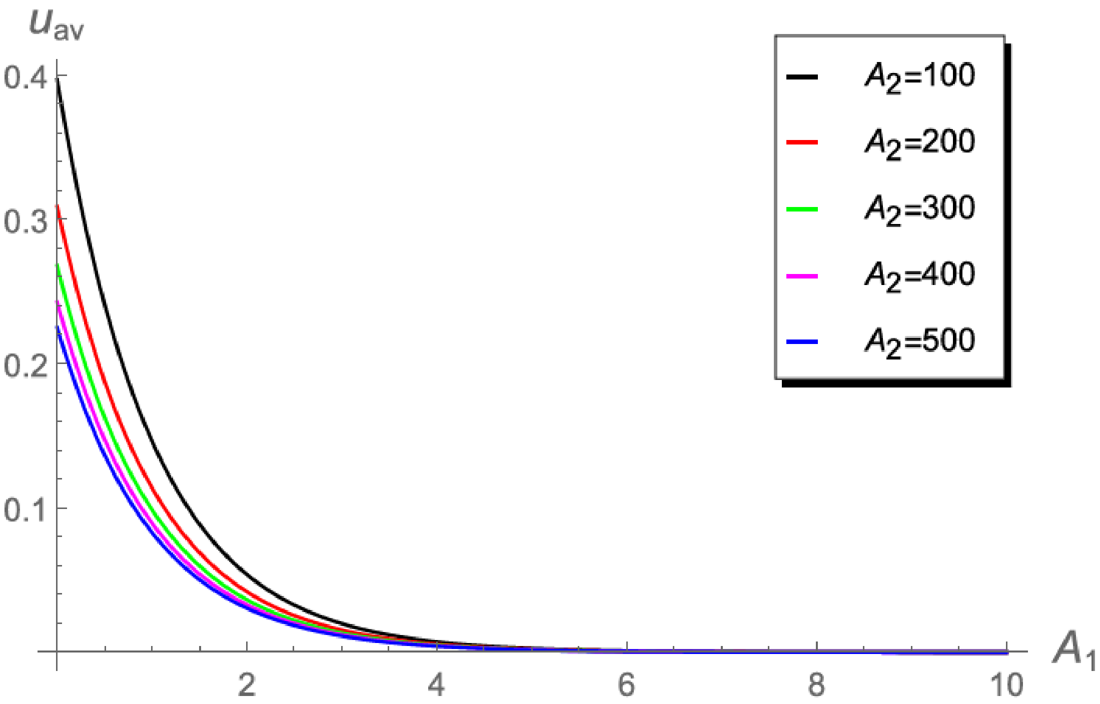

Figure 9. The u_{av} against A_1 at higher values of $A_2, A_0 = 1.4 \times 10^{-2}$.

References:

1. Lokenath Debnath: Integral Transforms and their applications, second edition, Chapman and Hall/CRC Taylor and Francis Group, 2007

2. Sneddon, I.N., The Use of Integral transforms, McGraw-Hill Book Company, New York, 1972.

3. Rainville E.D. Special Functions, Macmillan, New York, 1960.

4. Srivastava H.M. and manocha H.L.: A treatise on generating functions, John Wiley and Sons, New York, Ellis Horwood, Chichester, 1984.

10.48047/jocaaa.2023.31.03.27

5. Mclachlan, N.W. Bessel Function for engineers, Longman, Oxford (1955). Clark, R. M., Reed, E. J. & Hoff, J. C. Analysis of inactivation of Giardia lamblia by chlorine. ASCE J. Environ. Eng. 115, 80-90 (1989).

6. Le Chevallier, M. W., Cawthon, C. & Lee, R. G. Inactivation of biofilm bacteria. Appl. Environ. Microbiol. 54, 2492-2499 (1988).

7. Arnold, B. F. & Colford, J. M. Treating water with chlorine at point-of-use to improve water quality and reduce child diarrhea in developing countries: A systematic review and meta-analysis. Am. J. Trop. Med. Hyg. 76, 354-364 (2007).

8. Biswas, P., Chungsying, L. & Clark, R. M. A model for chlorine concentration decay in pipes. Water Res. 27(12), 1715-1724 (1993).

9. Jakubowski, J. & Wisniewolski, M. On matching diffusions, Laplace transforms and partial differential equations. Stoch. Process. Appl. 125(10), 3663-3690 (2015).

10. Ebaid, A. & Al Sharif, M. Application of Laplace transform for the exact effect of a magnetic field on heat transfer of carbonnanotubes suspended nanofluids. Z. Nat. 70(6), 471-475 (2015).

11. Ebaid, A., Wazwaz, A. M., Alali, E. & Masaedeh, B. Hypergeometric series solution to a class of second-order boundary value problems via Laplace transform with applications to nanouids. Commun. Theor. Phys. 67, 231 (2017).

12. Khaled, S. M. The exact effects of radiation and joule heating on magnetohydrodynamic Marangoni convection over a flat surface. Therm. Sci. 22, 63-72 (2018).

13. Ebaid, A., Alali, E. & Saleh, H. The exact solution of a class of boundary value problems with polynomial coefficients and its applications on nanofluids. J. Assoc. Arab Univ. Basic Appl. Sci. 24, 156-159 (2017).

14. Bakodah, H. O. & Ebaid, A. Exact solution of Ambartsumian delay differential equation and comparison with Daftardar-Gejji and Jafari approximate method. Mathematics 6, 331 (2018).

15. Sujit Handibag, B. & Karande, D. Laplace substitution method for solving partial differential equations involving mixed partial derivatives. Int. J. Comput. Eng. Res. 2(4), 1049-1052 (2012).

16. Dogan, N. Solution of the system of ordinary differential equations by combined Laplace transform-Adomian decomposition method. Math. Comput. Appl. 17(3), 203-211 (2012).

17. Ra, P. Application of Laplace transforms to solve ODE using MATLAB. J. Inform. Math. Sci. 7(2), 93-97 (2015).

18. Handibag, S. S. & Karande, B. D. Laplace substitution method for n^{th}-order linear and non-linear PDEs involving mixed partial derivatives. Int. Res. J. Eng. Technol. 2(9), 378-388 (2015).

10.48047/jocaaa.2023.31.03.27

19. Alshikh, A. A. & Mahgob, M. M. A comparative study between laplace transform and two new integrals "Elzaki" transform and "Aboodh" transform. Pure Appl. Math. J. 5(5), 145-150 (2016).

20. Atangana, A. & Alkaltani, B. S. A novel double integral transform and its applications. J. Nonlinear Sci. Appl. 9, 424-434 (2016).

21. Zhou, H., Yang, L. & Agarwal, P. Solvability for fractional p-Laplacian differential equations with multipoint boundary conditions at resonance on infinite interval. J. Appl. Math. Comput. 53, 51-76. https://doi.org/10.1007/s12190-015-0957-8 (2017).

22. Liang, X., Gao, F., Gao, Y. N. & Yang, X. J. Applications of a novel integral transform to partial differential equations. J. Nonlinear Sci. Appl. 10, 528-534 (2017).

23. Venkata Pavani, P., Lakshmi Priya, U. & Amarnath Reddy, B. Solving differential equations by using Laplace transforms. Int. J. Res. Anal. Rev. 5(3), 1796-1799 (2018).

24. Agarwal, P., Ntouyas, S. K., Jain, S., Chand, M. & Singh, G. Fractional kinetic equations involving generalized k-Bessel function via Sumudu transform. Alexandria Eng. J. 57(3), 1937-1942 (2018).

25. Restrepo, J. E., Piedrahita, A. & Agarwal, P. Multidimensional Fourier transform and fractional derivative. Proc. Jangjeon Math. Soc. 22(2), 269-277. https://doi.org/10.17777/PJMS2019.22.2.269 (2019).

26. Faraj, B. M. & Ahmed, F. W. On the MATLAB technique by using Laplace transform for solving second order ODE with initial conditions exactly. Matrix Sci. Math. 3(2), 8-10 (2019).

27. Mousa, A. & Elzaki, M. Solution of volterra integro-differential equations by triple Laplace transform. Irish Interdiscip. J. Sci. Res. 3(4), 67-72 (2019).

28. Dhunde, R. R. & Waghmare, G. L. Double Laplace iterative method for solving nonlinear partial differential equations. NTMSCI 7(2), 138-149 (2019).

29. Ziane, D., Cherif, M. H., Cattani, C. & Belghaba, K. Yang-Laplace decomposition method for nonlinear system of local fractional partial differential equations. Appl. Math. Nonlinear Sci. 4(2), 489-502 (2019).

30. Mastoi, S., Othman, W. A. & Nallasamy, K. Randomly generated grids and Laplace Transform for partial differential equations. Int. J. Disaster Recovery Bus. Contin. 11(1), 1694-1702 (2020).

31. Shah, N. A., Agarwal, P., Chung, J. D., El-Zahar, E. R. & Hamed, Y. S. Analysis of optical solitons for nonlinear Schrödinger equation with detuning term by iterative transform method. Symmetry 12(11), 1850. https://doi.org/10.3390/sym12111850 (2020).

32. Zhang, H., Nadeem, M., Rauf, A. & Hui, Z. G. A novel approach for the analytical solution of nonlinear time-fractional differential equations. Int. J. Numer. Methods Heat Fluid Flowhttps://doi.org/10.1108/HFF-02-2020-0077 (2020).

33. Ebaid, A. et al. A novel exact solution for the fractional Ambartsumian equation. Adv. Differ. Equ. 2021, 88. https://doi.org/10. 1186/s13662-021-03235-w (2021).

10.48047/jocaaa.2023.31.03.27

34. Ebaid, A., Alharbi, W., Aljoufi, M. D. & El-Zahar, E. R. The exact solution of the falling body problem in three-dimensions: Comparative study. Mathematics 8(10), 1726. https://doi.org/10.3390/math8101726 (2020).

35. Spiegel, M. R. Laplace Transforms (McGraw-Hill. Inc., 1965).