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# Stability Analysis of a Typical Three-Species Syn-Ecology Consisting of a Commensal and Two Hosts

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**Abstract:** In this paper, the system comprises of two hosts $S_1$, $S_3$ and one commensal $S_2$ ie., $S_1$ and $S_3$ both benefit $S_2$, without getting themselves affected either positively or adversely. Further, $S_1$ and $S_3$ are neutral. Here all the three species posses limited resources. The model equations constitute a set of three first order non-linear simultaneous differential equations. Criteria for the asymptotic stability of all the eight equilibrium states are established. The system would be stable if all the characteristic roots are negative, in case they are real, and have negative real parts, in case they are complex. Trajectories of the perturbations over the equilibrium states are illustrated. Further, the numerical solutions for the growth rate equations are computed using Runge-Kutta fourth order scheme.

**Keywords:** Commensal, Equilibrium state, Host, Lyapunov's function, Stable, Trajectories, Unstable.

**2010 Mathematics Subject Classification:** 92D25; 92D40

## 1. Introduction

Mathematical models have become important tools in biological investigations with an iterative procedure of information collection. If such models are properly developed and used, they can provide insight into the relations between the physical variables and process influencing the system being studied. The resulting interplay between the experimental investigation and the theoretical model can be an essential factor in designing experiments and in the interpretation of data. There are various types of mathematical modelling. Since real-life systems are complex, mathematical formulations have been developed to reproduce the experimental results irrespective of the underlying mechanisms. Such models can be extremely useful in highlighting the performance of the biological systems, albeit the components of the model are not identifiable with the components and mechanisms of the real system. However, experimental results can be reproduced in such circumstances by arbitrarily adjusting the models to explore the relation among various systems. The insight obtained from studies of such models has proved to be of immense use in complex real-life systems. Several authors Ma [6], Moghadas [7], Murray [8] and Sze-Bi Hsu [30] were introduced the general concepts of Modeling in Biological Science. Srinivas [29] studied the competitive ecosystem of two species and three species with limited and unlimited resources. Later, Narayan [9] studied prey-predator ecological models with partial cover for the prey and alternate food for the predator. Further, Kumar [5] studied some mathematical models of ecological commensalism. The present author Prasad [10-27] investigated continuous and discrete models on two, three and four species syn-ecosystems

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Ecology relates to the study of living being in relation to their habits and habitats. This discipline of knowledge is a branch of evolutionary biology purported to explain how or to what extent the living beings are regulated in nature. Allied to the problem of population regulation is the problem of species distribution- prey-predator, competition and so on. The subject of ecology can be broadly sub-divided as auto-ecology (the study of single species populations) and syn ecology (the study of two or more communities). Syn ecological studies lead to the concept of the eco-system. This concept is a direct outcome of the intensive work of several life scientists/biologists and botanists of many generations. An eco-system may be considered as a unit that includes animals, plants and the physical environment in which these live. Significant researches in the area of theoretical ecology have been discussed by Gillman [3] and by Kot [4]. Several ecologists and mathematicians contributed to the growth of this area of knowledge. Mathematical ecology can be broadly divided into two main sub-divisions, Autecology and Synecology, which are described in the treatises of Anna Sher [1], Arumugam [2] and Sharma [28].

**Commensalism** is a symbiotic interaction between two populations where one population ($S_1$) gets benefit from ($S_2$) while the other ($S_2$) is neither harmed nor benefited due to the interaction with ($S_1$). The benefited species ($S_1$) is called the commensal and the other ($S_2$) is called the host. Some real-life examples of commensalism are presented below.

(i) The clownfish shelters among the tentacles of the sea anemone, while the sea anemone is not effected.

(ii) Sucker fish (echeneis) gets attached to the under surface of sharks by its sucker. This provides easy transport for new feeding grounds and also food pieces falling from the sharks prey, to Echeneis.

(iii) A flatworm attached to the horse crab and eating the crab's food, while the crab is not put to any disadvantage.

(iv) The interaction between Euklonia maxima and patella compressa. The patella gets its food from the plant while the Euklonia, is not harmed or damaged in the process.

## Notation

$S_1, S_3$ : Hosts of $S_2$

$S_2$ : Commensal for $S_1$ and $S_3$

$N_i(t)$ : The population strength of $S_i$ at time $t$, $i = 1,2,3$

$t$ : Time instant

$a_i$ : Natural growth rate of $S_i$, $i = 1,2,3$

$a_{ii}$ : Self inhibition coefficients of $S_i$, $i = 1,2,3$

$a_{21}, a_{23}$ : Interaction coefficients of $S_2$ due to $S_1$ and $S_3$

$k_i = \dfrac{a_i}{a_{ii}}$ : Carrying capacities of $S_i$, $i = 1,2,3$

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Further the variables $N_1, N_2, N_3$ are non-negative and the model parameters $a_1, a_2, a_3, a_{21}$, $a_{11}, a_{22}, a_{33}, a_{23}, k_1, k_2, k_3$ are assumed to be non-negative constants.

## 2. Basic Equations

The model equations for syn ecosystem is given by the following system of first order non-linear ordinary differential equations.

$$\frac{dN_i}{dt} = a_i N_i - a_{ii} N_i^2 \qquad (1)$$

$$\frac{dN_2}{dt} = a_2 N_2 - a_{22} N_2^{\,2} + a_{21} N_1 N_2 + a_{23} N_2 N_3 \qquad (2)$$

where i = 1, 3

## 3. Equilibrium States:

At $\frac{dN_i}{dt} = 0, i = 1,2,3$ the equations (1) & (2) have eight equilibrium states given by

(i) Fully washed-out state

$$E_1 : \bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = 0$$

(ii) States in which two of the three species are washed out and the third is not.

$$E_2 : \bar{N}_1 = 0, \bar{N}_2 = k_2, \bar{N}_3 = 0$$

$$E_3 : \bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = k_3$$

$$E_4 : \bar{N}_1 = k_1, \bar{N}_2 = 0, \bar{N}_3 = 0$$

(iii) Only one of the three species is washed out while the other two are not

$$E_5 : \bar{N}_1 = 0, \bar{N}_2 = k_2 + \frac{a_{23} k_3}{a_{22}}, \bar{N}_3 = k_3$$

$$E_6 : \bar{N}_1 = k_1, \bar{N}_2 = 0, \bar{N}_3 = k_3$$

$$E_7 : \bar{N}_1 = k_1, \bar{N}_2 = k_2 + \frac{a_{21} k_1}{a_{22}}, \bar{N}_3 = 0$$

(iv) The co-existent state (or) normal steady state

$$E_8 : \bar{N}_1 = k_1, \bar{N}_2 = k_2 + \frac{a_{21} k_1 + a_{23} k_3}{a_{22}}, \bar{N}_3 = k_3$$

## 4. Stability Analysis of the Equilibrium States:

Let $N = (N_1, N_2, N_3) = \bar{N} + U$

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where $U = (u_1, u_2, u_3)^T$ is very small perturbation upon the equilibrium point $\overline{N} = (\overline{N}_1, \overline{N}_2, \overline{N}_3)$. The basic equations (1) & (2) are quasi- linearized to obtain the equations for the perturbed state as

$$\frac{dU}{dt} = AU \tag{3}$$

where $A = \begin{bmatrix} a_1 - 2a_{11}\overline{N}_1 & 0 & 0 \\ a_{21}\overline{N}_2 & a_2 - 2a_{22}\overline{N}_2 + a_{21}\overline{N}_1 + a_{23}\overline{N}_3 & a_{23}\overline{N}_2 \\ 0 & 0 & a_3 - 2a_{33}\overline{N}_3 \end{bmatrix}$

The characteristic equation for the system is $| A - \lambda I | = 0$ (4)

The equilibrium state is stable if all the roots of the equation (4) are negative in case they are real or have negative real parts in case they are complex.

### 4.1 Fully washed out state $E_1 : \overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = 0$

The characteristic equation is $(\lambda - a_1)(\lambda - a_2)(\lambda - a_3) = 0$ (5)

The characteristic roots of (5) are $a_1, a_2, a_3$. Since all the roots are positive. Hence the fully washed out state is un**stable** and the solutions of the equations (3) are

$$u_1 = u_{10}e^{a_1 t}, u_2 = u_{20}e^{a_2 t}, u_3 = u_{30}e^{a_3 t} \tag{6}$$

where $u_{10}, u_{20}$ and $u_{30}$ are the initial values of $u_1, u_2$ and $u_3$ respectively.

**Trajectories of perturbations:**

The trajectories in the $u_1 - u_2, u_2 - u_3$ and $u_3 - u_1$ planes are given by

$$(x_1)^{\frac{1}{a_1}} = (x_2)^{\frac{1}{a_2}} = (x_3)^{\frac{1}{a_3}}$$

Where $x_1 = \left(\frac{u_1}{u_{10}}\right); x_2 = \left(\frac{u_2}{u_{20}}\right); x_3 = \left(\frac{u_3}{u_{30}}\right)$

### 4.2 Equilibrium state $E_2 : \overline{N}_1 = 0, \overline{N}_2 = k_2, \overline{N}_3 = 0$

In this state the characteristic equation is $(\lambda - a_1)(\lambda + a_2)(\lambda - a_3) = 0$ (7)

The characteristic roots of (7) are $a_1, -a_2, a_3$. Since two of the roots are positive. Hence the state is un**stable** and the solutions are

$$u_1 = u_{10}e^{a_1 t}, u_2 = \left[u_{20} - \frac{a_{21}k_2u_{10}}{a_1 + a_2} - \frac{a_{23}k_2u_{30}}{a_3 + a_2}\right]e^{-a_2 t} + \frac{a_{21}k_2u_{10}e^{a_1 t}}{a_1 + a_2} + \frac{a_{23}k_2u_{30}e^{a_3 t}}{a_3 + a_2},\ u_3 = u_{30}e^{a_3 t} \tag{8}$$

**Trajectories of perturbations:**

The Trajectories are

$$u_1-u_2:x_2=\left[1-\frac{a_{21}k_1u_{10}}{u_{20}\left(a_1+a_2\right)}-\frac{a_{23}k_2u_{30}}{u_{20}\left(a_3+a_2\right)}\right]x_1^{\frac{-a_2}{a_1}}+\frac{a_{21}k_1u_{10}}{u_{20}\left(a_1+a_2\right)}x_1+\frac{a_{23}k_2u_{30}}{u_{20}\left(a_3+a_2\right)}x_1^{\frac{a_3}{a_1}},$$

$$u_2-u_3:x_2=\left[1-\frac{a_{21}k_1u_{10}}{u_{20}\left(a_1+a_2\right)}-\frac{a_{23}k_2u_{30}}{u_{20}\left(a_3+a_2\right)}\right]x_3^{\frac{-a_2}{a_3}}+\frac{a_{21}k_1u_{10}}{u_{20}\left(a_1+a_2\right)}x_3^{\frac{a_1}{a_3}}+\frac{a_{23}k_2u_{30}}{u_{20}\left(a_3+a_2\right)}x_3,$$

$$u_3-u_1:\left(x_3\right)^{a_1}=\left(x_1\right)^{a_3}$$

### 4.3 Equilibrium state $E_3:\bar{N}_1=0,\bar{N}_2=0,\bar{N}_3=k_3$

The characteristic equation at this state is $\Rightarrow\left(\lambda-a_1\right)\left(\lambda-a_2-a_{23}k_3\right)\left(\lambda+a_3\right)=0$ (8)

The characteristic roots of (8) are $a_1,a_2+a_{23}k_3,-a_3$ .Since two of the roots are positive. Hence the state is un**stable** and the solutions of the equations are given by

$$u_1=u_{10}e^{a_1t},u_2=u_{20}e^{\left(a_2+a_3k_3\right)t},u_3=u_{30}e^{-a_3t} \tag{9}$$

**Trajectories of perturbations:**

The trajectories in the $u_1-u_2,u_2-u_3$ and $u_3-u_1$ planes are given by

$$\left(x_1\right)^{a_2+a_{23}k_3}=\left(x_2\right)^{a_1};\left(x_2\right)^{-a_3}=\left(x_3\right)^{a_2+a_{23}k_3};\left(x_3\right)^{a_1}=\left(x_1\right)^{-a_3}$$

### 4.4 Equilibrium state $E_4:\bar{N}_1=k_1,\bar{N}_2=0,\bar{N}_3=0$

The characteristic equation is $\left(\lambda+a_1\right)\left[\lambda-\left(a_2+a_{21}k_1\right)\right]\left(\lambda-a_3\right)=0$ (10)

The characteristic roots are $-a_1,a_2+a_{21}k_1,a_3$ .Since two of the roots are positive. Hence the state is **unstable** and the solutions are

$$u_1=u_{10}e^{-a_1t},u_2=u_{20}e^{\left(a_2+a_{21}k_1\right)t},u_3=u_{30}e^{a_3t} \tag{11}$$

**Trajectories of perturbations:**

The trajectories in the $u_1-u_2,u_2-u_3$ and $u_3-u_1$ planes are

$$\left(x_1\right)^{a_2+a_{21}k_1}=\left(x_2\right)^{-a_1};\left(x_2\right)^{a_3}=\left(x_3\right)^{a_2+a_{21}k_1};\left(x_1\right)^{a_3}=\left(x_3\right)^{-a_1}$$

### 4.5 Equilibrium state $E_5:\bar{N}_1=0,\bar{N}_2=k_2+\frac{a_{23}k_3}{a_{22}},\bar{N}_3=k_3$

The characteristic equation is $\left(\lambda-a_1\right)\left[\lambda+\left(a_2+a_{23}k_3\right)\right]\left(\lambda-a_3\right)=0$ (12)

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$a_1, -\left(a_2 + a_{23}k_3\right), a_3$ are the characteristic roots of (12). Since two of the roots are positive. Hence the state is un**stable** and the solutions of (3) are

$$u_1 = u_{10}e^{a_1 t},\ u_2 = \left[u_{20} - \alpha_1 u_{10} - \alpha_2 u_{30}\right]e^{-(a_2+a_{23}k_3)t} + \alpha_1 u_{10}e^{a_1 t} + \alpha_2 u_{30}e^{a_3 t},\ u_3 = u_{30}e^{a_3 t} \tag{13}$$

where $\alpha_1 = \dfrac{a_{21}\left(a_2 + a_{23}k_3\right)}{a_{22}\left(a_1 + a_2 + a_{23}k_3\right)} > 0, \alpha_2 = \dfrac{a_{23}\left(a_2 + a_{23}k_3\right)}{a_{22}\left(a_2 + a_3 + a_{23}k_3\right)} > 0$

**Trajectories of perturbations:**

The trajectories are given by

$$x_2 = \left[1 - \frac{\alpha_1 u_{10}}{u_{20}} - \frac{\alpha_2 u_{30}}{u_{20}}\right] x_1^{\frac{-(a_2+a_{23}k_3)}{a_1}} + \frac{\alpha_1 u_{10}}{u_{20}} x_1 + \frac{\alpha_2 u_{30}}{u_{20}} x_1^{\frac{a_3}{a_1}},$$

$$x_2 = \left[1 - \frac{\alpha_1 u_{10}}{u_{20}} - \frac{\alpha_2 u_{30}}{u_{20}}\right] x_3^{\frac{-(a_2+a_{23}k_3)}{a_3}} + \frac{\alpha_1 u_{10}}{u_{20}} x_3^{\frac{a_1}{a_3}} + \frac{\alpha_2 u_{30}}{u_{20}} x_3,\ \left(x_3\right)^{a_1} = \left(x_1\right)^{a_3}$$

### 4.6 Equilibrium state $E_6 : \bar{N}_1 = k_1, \bar{N}_2 = 0, \bar{N}_3 = k_3$

The characteristic equation is $\left(\lambda + a_1\right)\left(\lambda - b_2\right)\left(\lambda + a_3\right) = 0$ (14)

The characteristic roots of (14) are $\lambda = -a_1\ , b_2\ , -a_3$ where $b_2 = a_2 + a_{21}k_1 + a_{23}k_3$. Since one of the roots is positive. Hence the state is un**stable** and the solutions of the equation (3) are

$$u_1 = u_{10}e^{-a_1 t}, u_2 = u_{20}e^{b_2 t}, u_3 = u_{30}e^{-a_3 t} \tag{15}$$

**Trajectories of perturbations:**

The trajectories in the $u_1 - u_2$, $u_2 - u_3$ and $u_3 - u_1$ planes are

$$\left(x_1\right)^{b_2} = \left(x_2\right)^{-a_1}; \left(x_2\right)^{-a_3} = \left(x_3\right)^{b_2}; \left(x_3\right)^{a_1} = \left(x_1\right)^{a_3}$$

### 4.7 Equilibrium state $E_7 : \bar{N}_1 = k_1, \bar{N}_2 = k_2 + \dfrac{a_{21}k_1}{a_{22}}, \bar{N}_3 = 0$

The characteristic equation is $\left(\lambda + a_1\right)\left[\lambda + \left(a_2 + a_{21}k_1\right)\right]\left(\lambda - a_3\right) = 0$ (16)

The characteristic roots of (16) are $\lambda = -a_1\ , -\left(a_2 + a_{21}k_1\right), a_3$ .Since one of the roots is positive. Hence the state is un**stable** and the solutions are

$$u_1 = u_{10}e^{-a_1 t}, u_2 = \left[u_{20} - \beta_1 u_{10} - \beta_2 u_{30}\right]e^{-(a_2+a_{21}k_1)t} + \beta_1 u_{10}e^{-a_1 t} + \beta_2 u_{30}e^{a_3 t}, u_3 = u_{30}e^{a_3 t} \tag{17}$$

where $\beta_1 = \dfrac{a_{21}\left(a_2 + a_{21}k_1\right)}{a_{22}\left(a_2 - a_1 + a_{21}k_1\right)}, \beta_2 = \dfrac{a_{23}\left(a_2 + a_{21}k_1\right)}{a_{22}\left(a_2 + a_3 + a_{21}k_1\right)} > 0$; with $a_2 \neq a_1 + a_{21}k_1$

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**Trajectories of perturbations:**

The Trajectories are

$$x_2=\left[1-\frac{\beta_1 u_{10}}{u_{20}}-\frac{\beta_2 u_{30}}{u_{20}}\right]x_1^{\frac{(a_2+a_{21}k_1)}{a_1}}+\frac{\beta_1 u_{10}}{u_{20}}x_1+\frac{\beta_2 u_{30}}{u_{20}}x_1^{\frac{-a_3}{a_1}},$$

$$x_2=\left[1-\frac{\beta_1 u_{10}}{u_{20}}-\frac{\beta_2 u_{30}}{u_{20}}\right]x_3^{\frac{-(a_2+a_{21}k_1)}{a_3}}+\frac{\beta_1 u_{10}}{u_{20}}x_3^{\frac{-a_1}{a_3}}+\frac{\beta_2 u_{30}}{u_{20}}x_3,(x_3)^{-a_1}=(x_1)^{a_3}$$

## 4.8 Equilibrium state $E_8:\overline{N_1}=k_1,\overline{N_2}=k_2+\dfrac{a_{21}k_1+a_{23}k_3}{a_{22}},\overline{N_3}=k_3$

The characteristic equation is $(\lambda+a_1)(\lambda+c_2)(\lambda+a_3)=0$ (18)

The characteristic roots of (18) are $-a_1,-c_2,-a_3$ where $c_2=a_2+a_{21}k_1+a_{23}k_3>0$. Since all the roots are negative. Hence the state is **stable** and the solutions of the equation (3) are

$$u_1=u_{10}e^{-a_1t},u_2=\left[u_{20}-\gamma_1u_{10}-\gamma_2u_{30}\right]e^{-c_2t}+\gamma_1u_{10}e^{-a_1t}+\gamma_2u_{30}e^{-a_3t},u_3=u_{30}e^{-a_3t} \tag{19}$$

Where $\gamma_1=\dfrac{a_{21}c_2}{a_{22}(c_2-a_1)};\gamma_2=\dfrac{a_{23}c_2}{a_{22}(c_2-a_3)};$with $c_2\neq a_1,a_3$

**Trajectories of perturbations:**

$$x_2=\left[1-\frac{\gamma_1 u_{10}}{u_{20}}-\frac{\gamma_2 u_{30}}{u_{20}}\right]x_1^{\frac{c_2}{a_1}}+\frac{\gamma_1 u_{10}}{u_{20}}x_1+\frac{\gamma_2 u_{30}}{u_{20}}x_1^{\frac{a_3}{a_1}},$$

$$x_2=\left[1-\frac{\gamma_1 u_{10}}{u_{20}}-\frac{\gamma_2 u_{30}}{u_{20}}\right]x_3^{\frac{c_2}{a_3}}+\frac{\gamma_1 u_{10}}{u_{20}}x_3^{\frac{a_1}{a_3}}+\frac{\gamma_2 u_{30}}{u_{20}}x_3,(x_3)^{a_1}=(x_1)^{a_3}$$ are the Trajectories.

## 5. Liapunov's function for global stability:

In section 4, we discussed the local stability of all eight equilibrium states. From which only the normal steady state is **stable** and rest of them are **unstable**. We now examine the global stability of dynamical system (1) and (2) at this state by suitable Liapunov's function.

***Theorem***: The normal steady state is globally asymptotically stable.

***Proof:*** Let us consider the following Liapunov's function

$$V(N_1,N_2,N_3)=N_1-\bar{N}_1-\bar{N}_1\ \ln\left(\frac{N_1}{\bar{N}_1}\right)+l_1\left[N_2-\bar{N}_2-\bar{N}_2\ln\left(\frac{N_2}{\bar{N}_2}\right)\right]+l_2\left[N_3-\bar{N}_3-\bar{N}_3\ln\left(\frac{N_3}{\bar{N}_3}\right)\right]$$

where $l_1$& $l_2$ are suitable positive constants to be determined as in the subsequent steps.

The line derivative of L, we get

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$$\frac{dL}{dt}=\left(\frac{\left(N_1-\overline{N_1}\right)}{N_1}\right)\frac{dN_1}{dt}+l_1\left(\frac{N_2-\overline{N_2}}{N_2}\right)\left(\frac{dN_2}{dt}\right)+l_2\left(\frac{N_3-\overline{N_3}}{N_3}\right)\left(\frac{dN_3}{dt}\right)$$

$$\frac{dL}{dt}=\left(N_1-\overline{N_1}\right)\left(a_1-a_{11}N_1\right)+l_1\left(N_2-\overline{N_2}\right)\left(a_2-a_{22}N_2+a_{21}N_1+a_{23}N_3\right)+l_2\left(N_3-\overline{N_3}\right)\left(a_3-a_{33}N_3\right)$$

$$=-a_{11}\left(N_1-\overline{N_1}\right)^2-l_1a_{22}\left(N_2-N_2\right)^2+l_1a_{21}\left(N_1-\overline{N_1}\right)\left(N_2-\overline{N_2}\right)+l_1a_{23}\left(N_2-\overline{N_2}\right)\left(N_3-\overline{N_3}\right)-l_2a_{33}\left(N_3-\overline{N_3}\right)^2$$

Choosing $l_1=\frac{4a_{11}a_{22}}{a_{21}^2}>0, l_2=\frac{a_{11}a_{23}^2}{a_{21}^2a_{33}}>0$ & with some algebraic manipulations, we get

$$\frac{dL}{dt}=-\sqrt{a_{11}}\left[\left(N_1-\overline{N_1}\right)+\frac{2a_{22}}{a_{21}}\left(N_2-\overline{N_2}\right)+\frac{a_{23}}{a_{21}}\left(N_3-\overline{N_3}\right)\right]^2<0$$

Hence, the steady state is globally asymptotically stable.

## 6. Numerical approach

The numerical solutions of the growth rate equations (1) and (2) computed employing the fourth order Runge-Kutta method for specific values of the various parameters that characterize the model and the initial conditions. The results are illustrated in Figures 1 to 4.

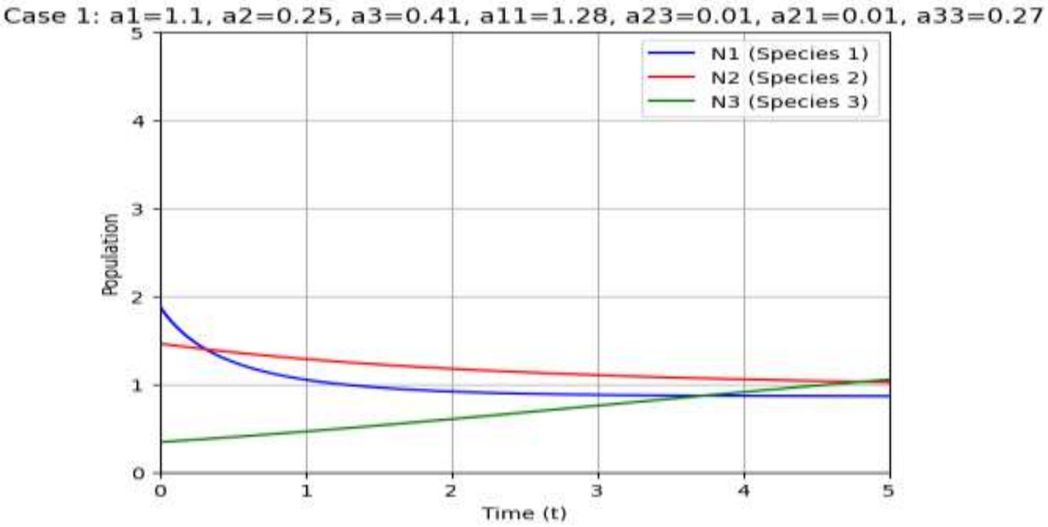

Figure 1

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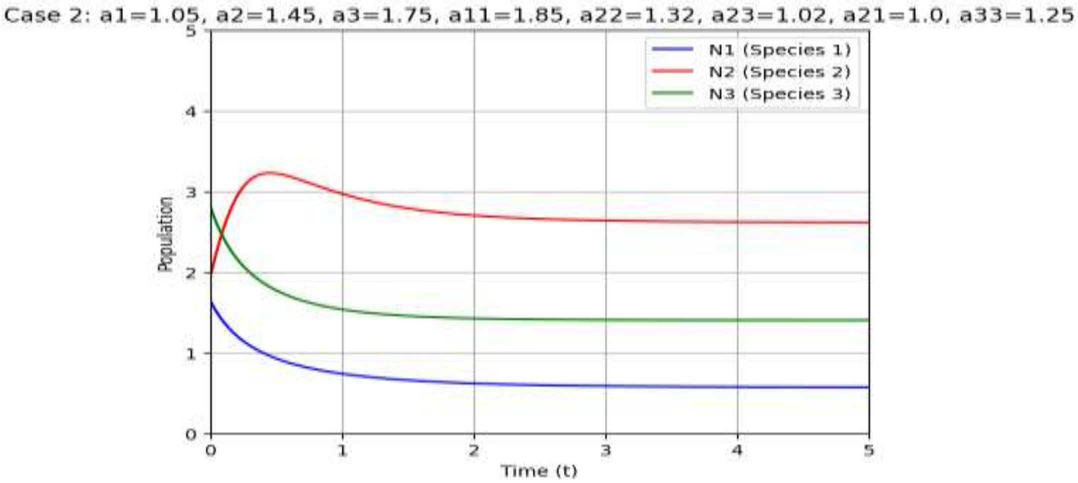
Case 2: a1=1.05, a2=1.45, a3=1.75, a11=1.85, a22=1.32, a23=1.02, a21=1.0, a33=1.25

Figure 2

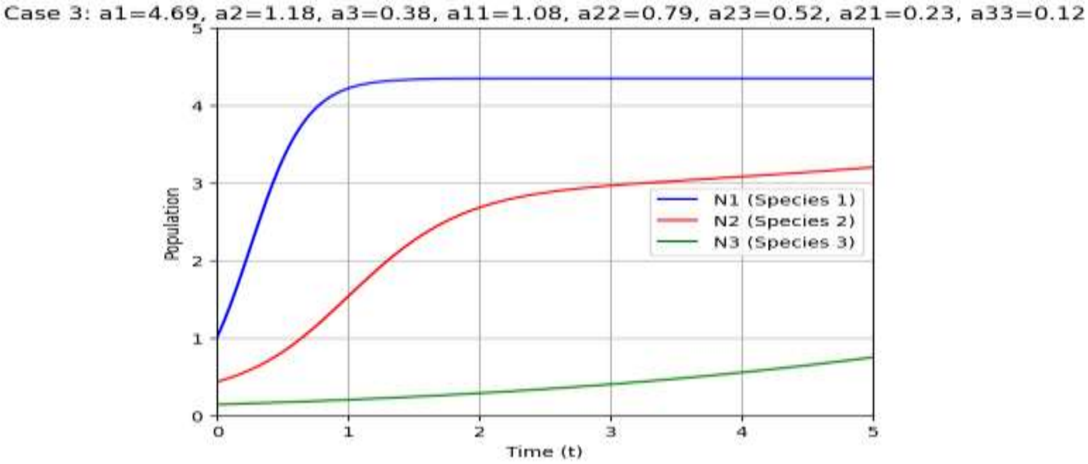
Case 3: a1=4.69, a2=1.18, a3=0.38, a11=1.08, a22=0.79, a23=0.52, a21=0.23, a33=0.12

Figure 3

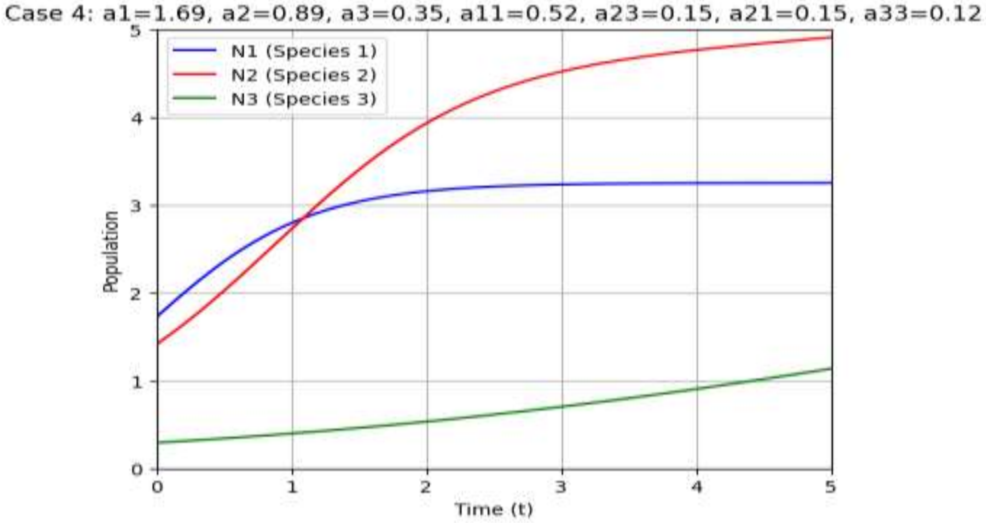
Case 4: a1=1.69, a2=0.89, a3=0.35, a11=0.52, a23=0.15, a21=0.15, a33=0.12

Figure 4

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## 7. Observations of the above graphs:

**Case 1:** In this case the first species increase initially. The third species has the least natural birth rate. $N_1$ and $N_2$: Show declining but stabilizing populations, likely due to self-limiting growth and minimal external influence. This is illustrated in Figure 1.

**Case 2:** In this case the third species dominates over the second species up to the time instant $t^* = 0.17$ after which the dominance is reversed. The first species has the least natural birth rate. $N_1$ suffers the most, declining to the lowest stable level. (Figure 2).

**Case 3:** In this case the first species dominates over the second and third species. The third species has the least natural birth rate. $N_1$ grows fastest and stabilizes highest, followed by $N_2$, and then $N_3$. This is shown in Figure 3.

**Case 4:** In this case the first species dominates over the second species up to the time instant $t^* = 1.18$ after which the dominance is reversed. The first species increase initially and then the second species increases over the first and third species. Further, the third species has the least natural birth rate. (Figure 4).

## 8. Conclusion

In this paper, we discussed the stability analysis of three species ecological commensalism. The model equations constitute a set of three first order non-linear coupled differential equations. All possible equilibrium states of the model are identified and the local stability is discussed. It is observed that, in all eight equilibrium states, only the normal state is locally stable. Further, the global stability of the system is established with the aid of suitably constructed Liapunov's function and the growth rates of the species are numerically estimated using Runge-Kutta fourth order method.

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