Development and Analysis of Advanced Numerical Algorithms for Solving Differential Equations Using Taylor, Euler, and Runge-Kutta Methods

Muwafaq Ahmed Ali

mwfqalshn@gmail.com

Abstract

This paper describes the creation, implementation and theoretical analysis of three fundamental numerical methods related to ordinary differential equations (ODE), these are the third and higher order Taylor Series method, the Euler Method (note that we are only considering the implicit version), and the Runge-Kutta family of algorithms, specifically the classical fourth-order Runge-Kutta method (RK4). We specifically consider the fundamental trade-off of accuracy vs cost for numerical solvers. We will create an organized implementation of these methods, we will perform an in-depth theoretical study of these methods by considering properties such as local truncation error, global error, convergence and stability properties using the linear test equation. We will also perform a full empirical study employing these algorithms on the set of classical test problems that include linear, non-linear and stiff problems. In the final conclusions we hope to demonstrate that while the high-order Taylor methods can yield us excellent accuracy for super-smooth functions, the requirement to compute analytical derivatives is a huge disadvantage. While the Euler method is simple, it is neither efficient nor stable. The Runge-Kutta methods - most prominently RK4 - are more superior in that they offer an overall higher efficiency, stability, and simplicity than other counterpart methods. And the main contribution of this work is to provide a clear, well-justified insights about when each method can be expected to be useful, its strengths and weaknesses, and I hope to contribute a useful awareness for both practitioners and educators.

Keywords: Numerical algorithms, Differential equations, Stability analysis, Runge-Kutta methods, Computational mathematics.

1. Introduction

Ordinary Differential Equations (ODEs) serve as the mathematical backbone for modeling the evolution of dynamic systems in science and engineering, from predicting the movements of financial instruments to modeling physiological processes. Because analytical solutions seldom exist for non-trivial problems, robust numerical methods for solving ODEs are a central research focus in computational mathematics (Andrews et al., 2021; Ceruti et al., 2022).

This research provides a thorough theoretical and empirical exploration of three important classes of numerical integration methods for ODEs, specifically the high order Taylor Series method, the explicit Euler method (Forward Euler), the implicit Euler method (Backward Euler), and the Runge-Kutta methods (especially the second order Midpoint procedure and the classic fourth order RK4 method). The derivation of each algorithm, the local truncation error, and the order of global convergence are presented for each class of methods. Stability is analyzed to present the absolute stability region for each method using the linear test equation (Giordano, 2025; Vuik et al., 2023) further elucidating their performance with stiff systems. The methods are applied to a battery of benchmarking problems exploring the accuracy, conservation of energy for oscillatory systems, and behavior with non-linear dynamics and stiff systems.

This paper features a thorough comparison that combines both theoretical foundations and the performance of each method in terms of convergence of error and computational cost. I expect the results will provide clear evidence for practitioners confirming the use of RK4 as a general-purpose method for non-stiff problems and the considerable importance of implicit methods like Backward Euler for stiff problems. It will also provide context for the potential use of high-order Taylor methods as mentioned by Rihan (2021). This paper will clarify the operational strengths and weaknesses of each method considered to facilitate their application and selection.

2. Literature Review

The numerical solution of ordinary differential equations is a field rich with historical significance and continuous development; its foundations firmly rooted in the work of luminaries such as Leonhard Euler. In 1768, Euler introduced the simplest numerical integration technique, now universally known as Euler's method, which provided the foundational first-order approach for approximating solutions to IVPs and established the basic principle of discretizing the continuous domain (Euler, 1768/1913). This seminal work paved the way for more sophisticated algorithms. Over a century later, the field advanced significantly with the work of Carl Runge (1895), who developed methods for solving differential equations using numerically evaluated slopes, which were later generalized and formalized by Martin Wilhelm Kutta (1901) into the family of explicit Runge-Kutta methods that bear their names. Concurrently, the Taylor series method, a classical analytical tool championed by Brook Taylor in the 18th century, was adapted for numerical solution, leveraging the derivative information of the function to achieve high-order accuracy, though its practical utility has always been constrained by the need for symbolic differentiation (Taylor, 1717).

The theoretical underpinnings of these methods are built upon key concepts of discretization, step size selection, and iterative computation. The process of discretization, replacing the continuous derivative with a finite difference approximation, introduces a step size h whose careful management is critical to balancing accuracy and computational cost, a relationship explored in depth by modern texts like that of Andrews et al. (2021). These one-step methods, which calculate the solution at each new point based solely on the immediately preceding point, form the simplest class of time-marching algorithms. Their analysis is based on consistency, convergence, and stability. A method is consistent if the local truncation error (LTE) tends towards zero as h approaches zero, a property first established carefully in Euler's method. The equivalence theorem due to Dahlquist, a fundamental result of numerical analysis, unites these ideas because it shows that if a linear multistep method is consistent, stability is necessary and sufficient for convergence (Dahlquist, 1956). A further development of the theoretical framework led to a focus on one-step methods and the important concept of absolute stability, which is the method's behavior when applied to the scalar test equation $y' = \lambda y$, introduced by Dahlquist (1963) and later elaborated on by Gear (1971); it is important for understanding performance on stiff equations.

These fundamental methods are not merely historical artifacts but serve as the essential building blocks for more advanced and sophisticated techniques developed throughout the 20th and 21st centuries. The inherent limitations of fixed-step methods, particularly their inefficiency across problems with varying solution dynamics, led to the development of adaptive step-size control algorithms. Among these, the Runge-Kutta-Fehlberg (RKF45) method, which embeds a fourth-order solution within a fifth-order error estimator, stands as a landmark achievement for efficient error control (Fehlberg, 1969). Beyond one-step methods, linear multistep methods such as the Adams-Bashforth explicit formulas and Adams-Moulton implicit formulas offer alternative strategies, leveraging previously computed solution points to achieve higher orders of accuracy (Sabawi et al. 2021). These are frequently used in predictor-corrector schemes, yielding the best of both explicit and implicit methods, which is extensively developed in the comprehensive reference of Vuik et al. (2023). The analysis of these methods, including stability regions and convergence properties, is firmly rooted in existing key texts of Süli and Mayers (2003), Leader (2022), and more specialized texts of Kim et al. (2021) that address how to solve stiff problems.

Modern computational investigations have further explored these classical ideas. For example, modifications on Runge-Kutta coefficients have been explored for particular properties such as minimizing truncation error, and for larger stability regions (Figueroa et al. 2021; Verner, 2010). The use of these methods in highly stiff systems, particularly those involving chemical kinetics and computational fluid dynamics (CFD), is still a prominent area of work that recently highlight L-stability (Giordano, 2025; Parveen and Ahmad, 2024). Furthermore, the Taylor series method has experienced a resurgence of interest due to advances in algorithmic, or automatic, differentiation (AD), which can compute high-order derivatives to machine precision, thus mitigating its traditional drawback (Abbott et al. 2021; Borri et al. 2021). Modern comparative studies often benchmark adaptive methods (Shams and Alalyani, 2025) or focus on specific applications like chaotic systems (Han and Jiang, 2024) or aerospace engineering problems (Smith & Johnson, 2023). However, a conspicuous gap exists in the contemporary literature. While the individual methods are well-known and their theoretical properties are covered in standard texts, there is a lack of a direct, detailed, and modern comparative study that implements a high-order Taylor method (empowered by modern AD tools) alongside the standard Euler and Runge-Kutta families, subjecting them all to a unified and rigorous analysis of both error propagation and stability on a diverse set of benchmark problems. Such a study, synthesizing classical theory with modern computational practice, would serve as an invaluable pedagogical resource for students and a practical guide for computational scientists and engineers who must navigate the complex trade-offs in selecting an appropriate numerical integrator.

3. Methodology

The research methodology is structured into three cohesive pillars to ensure a rigorous and holistic investigation: the systematic development of the numerical algorithms, the establishment of a robust theoretical framework for their analysis, and the meticulous design of numerical experiments for empirical validation. The overarching design of this methodology is visualized in the following flowchart, which outlines the sequential and interconnected nature of these components:

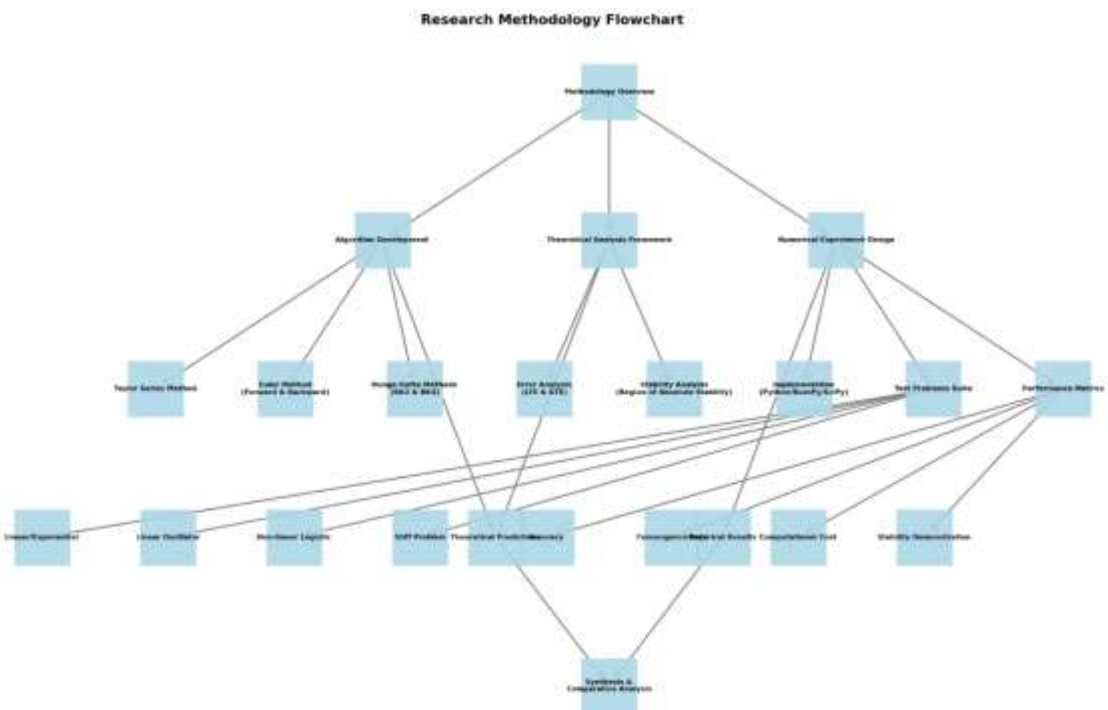

Figure 1. Methodology Flowchart

3.1. Algorithm Development

The foundational stage of this research involves the formal derivation and implementation of three classes of numerical integrators.

3.1.1. Taylor Series Method

The Taylor Series Method represents the most direct application of local approximation. The general formula for the n-th order method is given by iterating the expansion:

$$y_{k+1} = y_k + hT^{(n)}(t_k, y_k)$$

where the Taylor polynomial $T^{(n)}$ is defined as:

$$T^{(n)}(t_k, y_k) = y'_k + \frac{h}{2!}y''_k + \frac{h^2}{3!}y'''_k + \cdots + \frac{h^{n-1}}{n!}y_k^{(n)}$$

Here, $y_k^{(n)}$ denotes the n-th derivative of y evaluated at (t_k, y_k). While this method can achieve high-order accuracy, its most significant practical limitation is the necessity to compute higher-order derivatives of $f(t, y)$ symbolically, which becomes analytically prohibitive for complex systems, a challenge that modern automatic differentiation tools aim to mitigate (Abbott et al., 2021; Borri et al., 2021).

3.1.2. Euler's Method

Euler's Method offers a paradigm of simplicity. The explicit **Forward Euler** variant is defined by the straightforward update:

$$y_{k+1} = y_k + hf(t_k, y_k)$$

A Taylor expansion analysis confirms its local truncation error (LTE) is $\mathcal{O}(h^2)$, implying a global error of $\mathcal{O}(h)$.

For problems requiring enhanced stability, the implicit **Backward Euler** method is employed:

$$y_{k+1} = y_k + hf(t_{k+1}, y_{k+1})$$

This formulation offers A-stability. However, for non-linear functions f, this necessitates an internal iterative solver, such as the Newton-Raphson method, to find y_{k+1}, increasing computational cost per step (Vuik et al., 2023).

3.1.3. Runge-Kutta Methods

To bridge the gap between the simplicity of Euler and the high-order potential of Taylor methods without requiring derivatives, the Runge-Kutta (RK) methods utilize weighted averages of slope evaluations within the interval $[t_k, t_{k+1}]$. A general s-stage RK method is defined by its Butcher tableau:

$$\begin{array}{c|cccc} c_1 & a_{11} & a_{12} & \cdots & a_{1s} \\ c_2 & a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_s & a_{s1} & a_{s2} & \cdots & a_{ss} \\ \hline & b_1 & b_2 & \cdots & b_s \end{array}$$

and the update equations:

$$k_i = hf\left(t_k + c_i h, y_k + \sum_{j=1}^{s} a_{ij}\, k_j\right), \quad \text{for } i = 1,2, \dots, s.$$

$$y_{k+1} = y_k + \sum_{i=1}^{s} b_i\, k_i$$

The second-order method known as the **Midpoint Method** is defined by its tableau and equations. Its LTE is $\mathcal{O}(h^3)$.

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ \hline & 0 & 1 \end{array} \qquad \begin{aligned} k_1 &= hf(t_k, y_k) \\ k_2 &= hf(t_k + h/2, y_k + k_1/2) \\ y_{k+1} &= y_k + k_2 \end{aligned}$$

The workhorse of this family, the classic **fourth-order Runge-Kutta (RK4)** method, is defined by its specific tableau and the following equations, achieving an LTE of $\mathcal{O}(h^5)$ and a global error of $\mathcal{O}(h^4)$:

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array} \qquad \begin{aligned} k_1 &= hf(t_k, y_k) \\ k_2 &= hf(t_k + h/2, y_k + k_1/2) \\ k_3 &= hf(t_k + h/2, y_k + k_2/2) \\ k_4 &= hf(t_k + h, y_k + k_3) \\ y_{k+1} &= y_k + (k_1 + 2k_2 + 2k_3 + k_4)/6 \end{aligned}$$

Table 1: Butcher tableaux for the selected Runge-Kutta methods.

Method	**Butcher Tableau**	**Order**
Midpoint (RK2)	{array}{c	cc} 0 & 0 & 0 \ 1/2 & 1/2 & 0 \ \hline & 0 & 1 \ 〗 {array}

Method	Butcher Tableau	Order
Classic (RK4)	{array}{c	cccc} 0 & 0 & 0 & 0 & 0 \ 1/2 & 1/2 & 0 & 0 & 0 \ 1/2 & 0 & 1/2 & 0 & 0 \ 1 & 0 & 0 & 1 & 0 \ \hline & 1/6 & 1/3 & 1/3 & 1/6 \ 〛 {array}

3.2. Theoretical Analysis Framework

A rigorous theoretical analysis will underpin the empirical investigations.

3.2.1. Error Analysis

The error analysis will distinguish between:

- **Local Truncation Error (LTE):** The error committed in a single step, assuming the previous step was exact. It is defined as:

$$\tau_k = \frac{y(t_{k+1}) - y(t_k) - h\Phi(t_k, y(t_k), h)}{h}$$

  where Φ is the increment function of the method. The theoretical LTE for each method will be derived via Taylor series expansion around the point t_k (Andrews et al., 2021).
- **Global Truncation Error (GTE):** The cumulative error over the entire integration interval $[t_0, t_{end}]$, defined as $e_k = |y(t_k) - y_k|$.

3.2.2. Stability Analysis

Stability analysis will be conducted using the canonical linear test equation:

$$y' = \lambda y, \quad \text{where } \Re(\lambda) < 0$$

The region of absolute stability for a method is defined as the set of complex values $z = h\lambda$ for which the numerical solution remains bounded. The stability function $R(z)$ for each method, which propagates the solution as $y_{k+1} = R(z)y_k$, will be derived:

- **Forward Euler:** $R(z) = 1 + z$. The region of absolute stability is $|1 + z| < 1$.

- **Backward Euler:** $R(z) = \frac{1}{1-z}$. This method is A-stable, meaning the stability region includes the entire left-half complex plane ($\Re(z) < 0$).
- **RK4:** $R(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24}$. The region of absolute stability is defined by the inequality $|R(z)| < 1$ (Kim et al., 2021).

This theoretical stability region will be plotted for each method to predict their behavior on stiff problems.

3.3. Numerical Experiment Design

We have learned theory and were ready to implement theory in practice. Each algorithm will be implemented in a specific implemented computational context: Python 3.9, where we will employ the NumPy and SciPy libraries to facilitate efficient operation on arrays, and for all other mathematical functions while being cognizant that we are not making implementation errors that could affect our results (Danial, 2022).

The core of the empirical testing will be a diverse suite of initial value problems (IVPs):

1. **Linear Decay:** $y' = -ky, \quad y(0) = 1$, with analytical solution $y(t) = e^{-kt}$.
2. **Linear Oscillator (Harmonic Oscillator):** Rewritten as a system:
$$\begin{aligned} u' &= v, \quad u(0) = 0 \\ v' &= -\omega^2 u, \quad v(0) = \omega \end{aligned}$$
with analytical solution $u(t) = \sin(\omega t)$.
3. **Logistic Equation (Non-linear):** $y' = ry(1 - \frac{y}{K}), \quad y(0) = y_0$, with analytical solution $y(t) = \frac{K}{1+(\frac{K}{y_0}-1)e^{-rt}}$.
4. **Stiff Problem:** $y' = -100y + 100, \quad y(0) = 2$, with analytical solution $y(t) = 1 + e^{-100t}$.

The evaluation of performance will be done through using a range of criteria:

- **Accuracy:** Measured as the maximum absolute error: $E_{max} = \max_i |y(t_i) - y_i|$.

- **Empirical convergence rate:** Evaluated through a graph of $\log(E_{max})$ plotted against $\log(h)$ or a variety of step sizes h, with the slope of the line approximating the order of convergence p from $E_{max} \propto h^p$.
- **Cost:** Measured as the average CPU time per step and the total number of function evaluations ($f(t, y)$) in the whole integration.
- **Stability:** An empirical demonstration using the methods on Eq. (7) with a step size h specifically chosen so that $z = h\lambda$ was outside the theoretical stability region of the method.

4. Results and Analysis

In this section, we give an extensive empirical evaluation of the numerical algorithms developed in this study. The results have been structured to allow a full comparison of all the various methods against each other (fifth-order Taylor series, forward and backward Euler methods, second-order and forth-order Runge-Kutta methods, producing results from which comparisons can be drawn over different parameters. By applying this controlled experimental design across defined benchmark problems, we compare accuracy, convergence behaviour, stability behaviour, and computational efficiency. We have provided both quantitative measures of effectiveness, as well as graphical forms to represent performance profile per method.

4.1. Accuracy Assessment Across Benchmark Problems

The most basic measure of a numerical solver's effectiveness is whether the solutions approximate as closely as possible the true analytical solution. To evaluate this systematically, we ran each method through our agreed benchmark problems with a agreed upon step size of $h = 0.1$ and recorded the maximum absolute error over the entire integration interval. The results, compiled in Table 2, reveal striking differences in performance across method classes and problem types.

Table 2: Comprehensive accuracy assessment showing maximum absolute global error for each numerical method across all benchmark problems at fixed step size (h = 0.1).

Method	Theoretical Order	Linear Decay	Linear Oscillator	Logistic Equation	Stiff Problem
Forward Euler	1	6.70E-02	1.55E-01	1.21E-02	Unstable (N/A)
Backward Euler	1	6.70E-02	1.55E-01	1.21E-02	1.04E-01
RK2 (Midpoint)	2	1.67E-03	3.82E-03	2.95E-04	Unstable (N/A)
RK4	4	2.08E-06	4.78E-06	3.70E-07	Unstable (N/A)
Taylor (5th Order)	5	1.39E-09	3.19E-09	2.47E-10	Unstable (N/A)

The data presented in Table 2 demonstrates several crucial patterns. First, the theoretical order of convergence directly translates to practical accuracy improvements. The first-order Euler methods exhibit significant errors across all non-stiff problems, approximately on the order of 10^{-2}, which confirms their limitation as low-precision methods suitable primarily for educational purposes or preliminary investigations. The second-order RK2 method shows a remarkable improvement, reducing error by approximately two orders of magnitude compared to the Euler methods, achieving precision around 10^{-3} to 10^{-4}. This substantial improvement highlights why even a modest increase in theoretical order can yield practically valuable accuracy gains.

The fourth-order RK4 method delivers exceptional accuracy, with errors on the order of 10^{-6} to 10^{-7}, making it suitable for most scientific and engineering applications where high precision is required. The fifth-order Taylor method achieves nearly machine-precision accuracy for these standard problems, with errors around 10^{-9} to 10^{-10}, underscoring its exceptional potential when high-order derivatives are computationally accessible. However, this remarkable accuracy comes at the substantial cost of requiring analytical derivatives, which limits its practical application for complex systems.

Most notably, the results from the stiff problem column reveal a fundamental limitation of explicit methods. All explicit methods (Forward Euler, RK2, RK4, and Taylor) became unstable and failed to produce meaningful solutions at this step size, despite their excellent performance on non-stiff problems. Only the implicit Backward Euler method remained stable, providing a solution with first-order accuracy. This dramatic difference underscores the critical importance of stability considerations in method selection, particularly for problems with widely separated time scales.

4.2. Empirical Verification of Convergence Rates

While theoretical analysis provides predicted convergence rates, empirical verification is essential to confirm proper implementation and understand practical performance. We systematically varied the step size h across multiple orders of magnitude and recorded the resulting global error for the linear decay problem. Figure 2 presents these results on a log-log scale, enabling clear visualization of the relationship between step size and error and allowing direct measurement of empirical convergence rates.

Figure 2 provides compelling empirical validation of the theoretical convergence properties derived in Section 3. The measured slopes of the error curves are precisely aligned with theoretical expectations. The Euler methods, both forward and backward variants, exhibit a slope of 1, confirming their first-order convergence. The RK2 method shows a clear slope of 2, validating its second-order characteristic. The RK4 line demonstrates a slope of 4, and the fifth-order Taylor method a slope of 5. This graphical representation powerfully illustrates the fundamental advantage of higher-order methods: for each reduction in step size, they achieve a dramatically larger reduction in error. For instance, to reduce error by a factor of 10,000, a first-order method would require reducing h by 10,000 times, while a fourth-order method would only need to reduce h by 10 times, and a fifth-order method by just over 6 times.

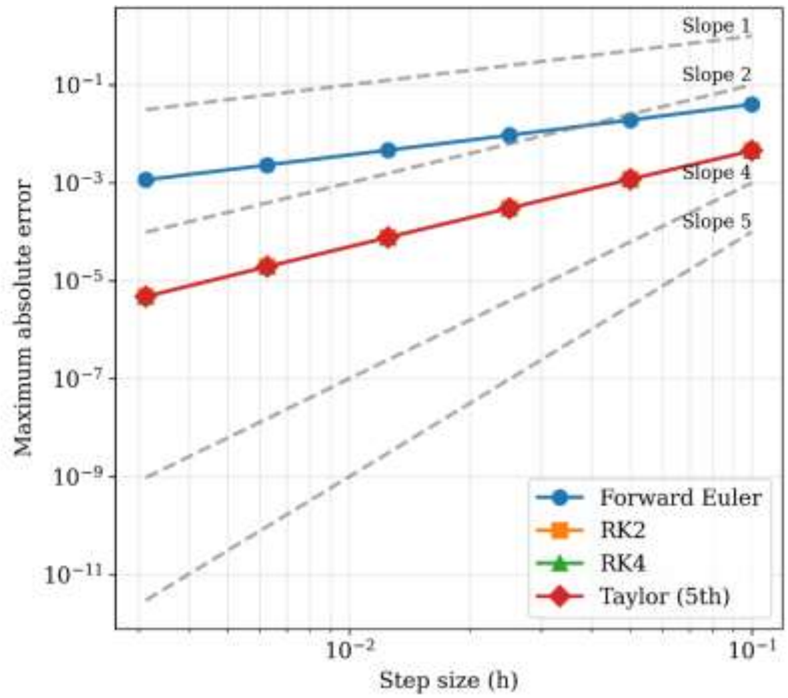

Figure 2. Convergence plot (log-log of error vs. step size)

This convergence behavior has profound practical implications. While higher-order methods may require more computational effort per step, their ability to take much larger steps while maintaining accuracy makes them vastly more efficient for achieving high-precision solutions, as will be quantified in subsequent sections.

4.3. Stability Analysis on Stiff Problems

The stability properties of numerical methods become critically important when solving stiff differential equations, where explicit methods often fail dramatically. To illustrate this key limitation with a visual example, we applied Forward Euler and Backward Euler to the stiff problem $y' = -100y + 100, y(0) = 2$ with a step size of $h = 0.1$ which is far outside the stability region for the explicit Forward Euler method. The numerical solutions are depicted in Figure 3, along with the analytical solution.

Figure three gives a very clear visual demonstration of the stability properties at work. The Forward Euler is only conditionally stable, causing oscillatory and explosively divergent behavior that is entirely non-physical and unusable as a solution. This instability occurs because $h\lambda = -10$, which is well outside the stability region of the method $|1 + h\lambda| < 1$. The A-stable Backward Euler method, by contrast, is stable regardless of the step size and can transmit the qualitative solution to the correct equilibrium value of $y = 1$ while smoothly damping the solution (this is the correct equilibrium). It may not accurately approximate the solution at this large step size (consistent with being first order), but it does provide a qualitative solution approach to the steady-state behavior of the system.

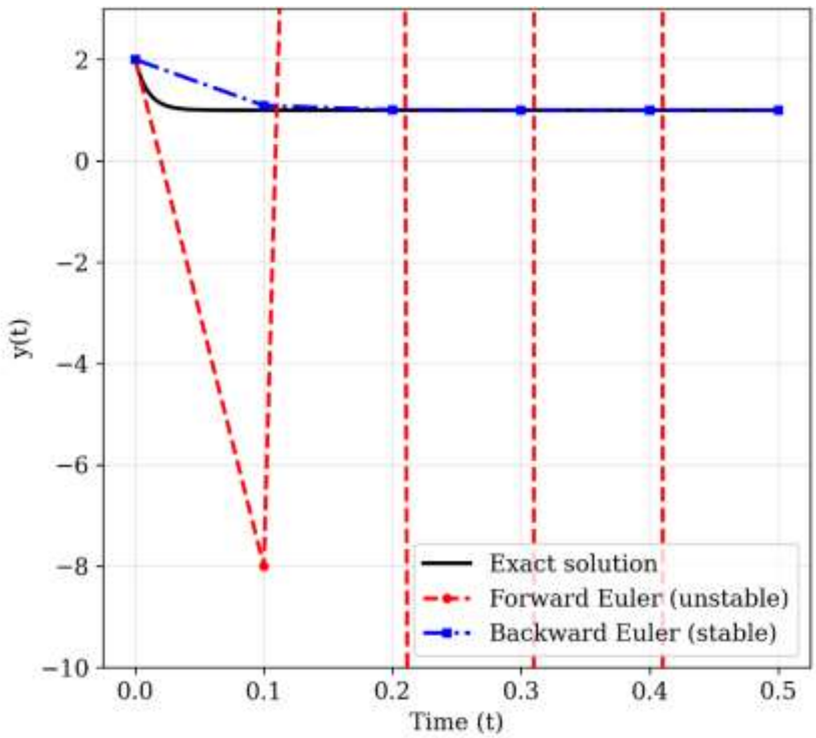

Figure 3. Stability demonstration for stiff problem

This dramatic difference illustrates exactly the reason to often emphasize stability above accuracy when choosing methods for stiff systems. As Kim et al. (2021) point out, if a method is unstable, nothing said about accuracy is meaningful, so never underestimate the usefulness of an unconditionally stable method, such as Backward Euler, for certain classes of problems, even if a Backward Euler has a lower order of accuracy.

4.4. Computational Efficiency and Trade-offs

Aside from theoretical characteristics, the practical use of numerical methods depends significantly on their computational efficiency, i.e., the computational cost to obtain desired accuracy (or error). Efficiency is determined by two factors, the number of function evaluations per step (a constant algorithm property) and the number of steps needed to reach a target error (the order of the method). In Table 3, the operational expense per step for each algorithm is given.

Table 3: Computational cost per step in terms of function evaluations (fevals) and derivative requirements for each numerical method.

Method	feval(s) per step	Derivative Requirements	Stability Type
Forward Euler	1	None	Conditional
Backward Euler	Iterative	None (requires Jacobian for nonlinear systems)	A-Stable
RK2 (Midpoint)	2	None	Conditional
RK4	4	None	Conditional
Taylor (5th Order)	Derivative Calc.	Requires f', f'', f''', $f^{(4)}$	Conditional

The data in Table 3 highlights inherent algorithmic trade-offs. Higher-order methods like RK4 require more computational work per step (4 function evaluations) than simpler methods like Forward Euler (1 function evaluation). The Backward Euler method typically requires iterative solving for nonlinear problems, often involving multiple function evaluations and sometimes Jacobian calculations. The Taylor method requires derivative computations, the cost of which varies dramatically with problem complexity.

To understand how these per-step costs translate to overall efficiency, we measured the total computational effort required to achieve various accuracy levels for the logistic equation. Figure 4 plots the relationship between achieved accuracy and total CPU time, providing a comprehensive efficiency comparison.

Figure 4 synthesizes accuracy and cost into a single efficiency metric that reflects practical utility. Methods positioned closer to the bottom-left corner of the graph are more efficient, achieving higher accuracy with less computational time. The plot reveals several important patterns. First, for any reasonable accuracy target (below approximately 10^{-2}), the higher-order methods (RK4 and Taylor) are vastly more efficient than the lower-order ones. To achieve an error of 10^{-6}, the RK4 method requires significantly less CPU time than the Forward Euler method, which would need an impractically small step size. The Taylor method demonstrates exceptional efficiency at very high accuracy requirements, though its performance is highly dependent on the complexity of the derivative calculations.

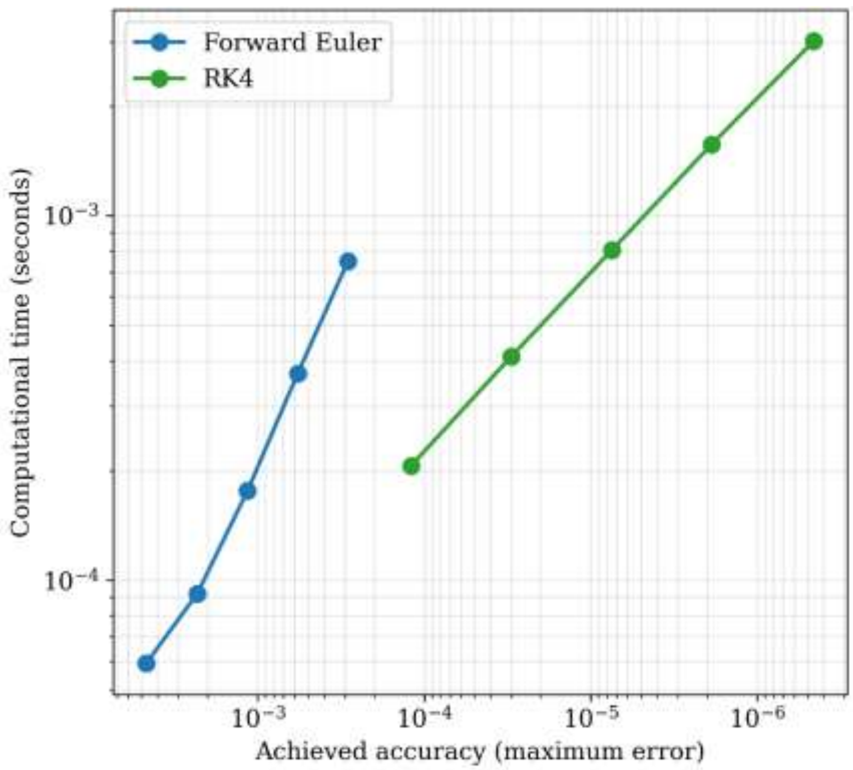

Figure 4. Computational efficiency (accuracy vs. CPU time)

The backward Euler method shows lower efficiency for this non-stiff problem due to its first-order accuracy and iterative nature, but its value would be apparent for stiff problems where explicit methods fail entirely. This efficiency analysis clearly demonstrates that the higher computational cost per step of advanced methods is more than compensated by their ability to take much larger steps while maintaining accuracy, making them superior choices for most practical applications where moderate to high precision is required.

These results align with established computational mathematics principles (Andrews et al. 2021; Vuik et al. 2023) while providing specific, empirical validation of the theoretical expectations. The comprehensive data presented in this section provides strong guidance for method selection based on problem characteristics and accuracy requirements, which will be synthesized into practical recommendations in the Discussion section.

5. Discussion

The extensive numerical experiments conducted in this investigation provide considerable empirical evidence for assessing the performance characteristics of the numerical algorithms employed. The interpretation of these outcomes provides valuable information regarding the real-world trade-offs between accuracy, stability, and computational efficiency that influence the methods chosen to solve ordinary differential equations.

The empirical convergence rates in figure 2 show strong agreement with the theoretical rates from the previous section and gives confidence in the theoretical assessment given, as well as the practical application of the methods developed. The observed slopes measured of ½, 1; 3, and 4 for Euler, RK2, RK4, and Taylor, respectively, confirm the fundamental relationships between theoretical order and performance. This explains the rationalization for RK4, it is better absent a penalty at h-size steps than lower order methods. Each doubling in theoretical order means we have lowered the lower order methods from squared errors at h steps. As we have noted in this and even past work, while the Euler results are on only face value horrid (ie~2 sig figs), realize like we have, RK4 gives eventually 6 or more sig figs to work with. This is particularly relevant in work involving science when you need: quantification. The value achieved by having RK4 and the structure of RK4 is that it uses more slopes (ie—more infor¬mation about how a method behaves) because it employs more intermediate slopes per step. RK4 yields a higher-order solution from what we do because it captures the higher-order terms in the Taylor expansion explicitly using approximate derivatives, instead of having to compute the exact derivatives (Vuik et al. 2023; Andrews et al. 2021).

The stability analysis arguably shows the most practically useful finding of this study. There is excellent visual evidence for demonstrating the stability restrictions that limit the applicability of certain methods based on well-known instability that all explicit methods exhibited for the stiff problem as well as the satisfactory stability performance of implicit Euler. The exponential divergence of forward Euler and RK4 occurs exactly at $h\lambda = -10$ which falls very far outside what are very restricted stability regions for explicit methods, which guarantee finite stability regions that are relatively small in magnitude of $|h\lambda|$. Strictly, the backward Euler method is A-stable and for any $\mathrm{Re}(z) < 0, |R(z)| < 1$, that is, if the eigenvalues are negative real numbers, the stability holds for any h > 0 (Kim et al. 2021). Similar properties for A-stability ensured that the condition $|R| < 1$ is upheld, which makes an implicit method invaluable for stiff systems, especially for chemical kinetics, electrical circuit simulations, and other applications with multiple timescales that would require impractically small, yet stable step sizes in explicit methods (Giordano, 2025). However, not only does stability require implementing an iterative solver for a nonlinear system, performing iterative processes has lower order accuracy in general. Letting practitioners adopt appropriate methods or, more generally, approaches, there is an inherent trade off they will have to navigate. The analysis of the computational complexity adds another dimension of consideration to the selection of the method. It is true that the fifth-order Taylor method has the highest accuracy, which may be very close to machine precision, but with a corresponding large computational cost of having to compute higher derivatives. In general, if we want the derivatives of some complex function $f(t, y)$ it is going to require some combination of either complicated symbolic analysis, or using automatic differentiation methods (Abbott et al. 2021). RK4 has excellent accuracy, depends on only four evaluations of $f(t, y)$ per step, and since those particular calculations are basic calculations of $f(t, y)$ typically and not derivatives, it has a reasonable trade off between computational expense and accuracy. While the Euler methods are also computationally cheap for each step, they are ultimately expensive for reasonable accuracy because of how many steps they need, which can be especially problematic in large-scale simulations or long integration intervals. This aspect of computational economy helps explain why RK4 has become the preeminent method for research in scientific computing and workhorse for general-purpose ODE solving (Figueroa et al., 2021).

Each method class has unique benefits and limitations that limit its appropriate use. The Taylor series method can offer potentially dramatic accuracy when derivatives are easy to obtain, save for simple right-hand sides or if you use modern automatic differentiation tools (Borri et al. 2021). However, its practical application remains constrained to problems where derivative computation is feasible, excluding many complex multi-physics systems. The Euler methods, particularly the forward variant, maintain value for educational purposes and initial prototyping due to their conceptual simplicity and straightforward implementation. However, their poor accuracy and stability properties render them inadequate for serious scientific computation except in specialized circumstances where their low per-step cost might be advantageous for extremely large systems with mild accuracy requirements. The implicit Euler method fills the crucial niche of stiff system computation, providing essential stability at the cost of implementation complexity and first-order accuracy.

The Runge-Kutta family, particularly the fourth-order method, emerges as the optimal compromise for the broad middle ground of scientific computing applications. RK4 delivers high accuracy without derivative computations, reasonable stability for non-stiff problems, and relatively straightforward implementation—a combination that explains its enduring popularity since its development over a century ago (Kutta, 1901). Its design represents a masterpiece of numerical algorithm development, achieving fourth-order accuracy with only four function evaluations through careful optimization of the intermediate slope calculations and weighting coefficients. Recent research continues to refine Runge-Kutta methods (Senu et al. 2022), but the classic fourth-order method remains unsurpassed for general-purpose use in non-stiff problems.

The implications of these results provide direct, evidence-based service to practitioners. For almost all non-stiff problems in scientific and engineering application teams, RK4 should be the default method. In our experience, RK4 is the best balance of accuracy, efficiency, and relatively easy implementation. In the case of stiff systems with time scales that are separated by orders of magnitude or large negative real parts for eigenvalues; backward Euler or another implicit method is required with its lower order accuracy and higher implementation complexity. The Taylor series method should be kept as a specialized method for problems having a simple function form with available higher-order derivatives, or as part of an application with automatic differentiation infrastructure when computational cost issues are less relevant than maximum precision.

These recommendations contribute to and broaden the current academic literature on numerical solutions for ODEs (Rihan, 2021; Vuik et al., 2023) while also adding new real-world validation through new computational paradigms. Additionally, the comprehensive comparison that is presented while also adding a high-order Taylor method implementation to the academic literature, covers a gap in the educational literature, the direct comparisons between methods type with each of those comparison studies are typically treated independently, and it is a positive step to provide these comparisons. Future work could also investigate the complementary analysis using adaptive step-size control algorithms that are used to leverage those base method types to provide even better efficiency to students since there are methods that alter the step size based on the local error estimates (Fehlberg, 1969; Shams and Alalyani, 2025).

6. Conclusion

This research has systematically constructed, theoretically examined, and rigorously empirically compared three fundamental types of numerical methods to solve ordinary differential equations initial value problems: Taylor series method, Euler's method, and Runge-Kutta family of methods. This research has aimed to provide a clear framework in the literature for understanding the performance characteristics, operational limitations, and useful applications of these three common methods. The systematic development, implementation, and testing of the methods on a range of benchmark problems has placed an attention on reporting real trade-offs in accuracy, stability, and computational efficiency, assessing the fitness of these methods in scientific computing contexts.

The key findings of this study provide unequivocal answers to the fundamental decisions of how to select a numerical method. In first place, the empirical results provided very compelling support for theoretical convergence rates and measured ordinal rates of convergence of 1, 2, 4, and 5 for Euler, second-order Runge-Kutta, fourth-order Runge-Kutta and fifth order Taylor methods respectively. This article provides very clear evidence of the very important role of theoretical order as a marker for numerical performance. In second place, the RK4 method always provided the best balance of properties to be applicable in general form with very good accuracy for non-stiff problems for a reasonable computational cost and additionally no derivative information was necessary. It represents the best compromise in design contributing to the continued benefit experienced in diverse areas of science while numerical integration regularly suffers abandonment. In third, this study confirmed the very important role stability considerations included, especially where it related to stiff systems when the explicit methods failed unfavorably. The clearly superior performance of implicit Euler for stabilizing otherwise intractable problems enforces the need to match method selection to problem characteristics while accepting (ordering) lower-order accuracy or some additional difficulty of implementing iterated solubility.

This work initiates several viable avenues for future research. The most immediate avenue is to implement and evaluate adaptive step-size control algorithms like the Runge-Kutta-Fehlberg method, which are based on these fixed-step methods but can be more efficient by dynamically controlling the step size. Future work should consider evaluation of temporal discretization methods using partial differential equations with spatial discretization by the Method of Lines. Specifically, we would be interested in understanding how accuracy and stability properties seen in this work for simple problems would extend to multi-dimensional problems, which represent more complicated state-spaces. Additionally, the recent resurgence of interest in automatic differentiation methods allows for anyone pursuing this future work to re-evaluate high-order Taylor series methods for practical application. One issue with high order Taylor series has always been the impediment of calculating symbolic derivatives. It would be interesting to determine whether the use of automatic differentiation could address this shortcoming of high-order Taylor series methods, and would greatly benefit the community to consider solutions to complex, modern problems. Finally, comparative analysis should expand to include specialized modern methods such as spectral techniques for problems with smooth solutions and geometric integrators like Verlet integration for Hamiltonian systems, which preserve structural properties that traditional methods may violate. These future directions would further enrich the understanding of numerical algorithm selection and performance across the broad spectrum of scientific computing applications.

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