Journal of Computational Analysis and Applications **VOL. 33, NO. 8, 2024**

10.48047/jocaaa.2024.33.08.238

ON RIGHT QUASI REGULARITY IN GAMMA NEARRINGS

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ABSTRACT

Booth introduced and investigated the (left) Jacobson radicals of type 0, 1 and 2 for Γ-nearrings which are extensions of the (left) Jacobson radicals of nearrings. In this paper right quasiregular elements are introduced in a Γ-nearring and a characterization of the right Jacobson radical of type-0 is presented using these elements. The paper establishes that the right Jacobson radical of type-0 of a Γ-nearring W is the largest right quasiregular ideal of W and also right Jacobson radical of type-0 of W is the intersection of all right 0-primitive ideals of W. Furthermore, a right 0-primitive ideal of a Γ-nearring W is also a prime ideal of W.

Keywords: Γ-nearring, Right Quasi-regular elements, Right modular right ideals, 0-modular right ideals, Right ΓW-groups, ΓW-groups of type-0, Right 0-primitive ideals, Right Jacobson radical of type-0.

2020 Mathematics Subject Classification 16Y30

1.INTRODUCTION

Γ-nearrings are generalizations of rings, Γ-rings and nearrings. Booth (1988) started the study of radicals of gamma nearrings [4]. Many radicals of gamma nearrings were developed and studied by Booth (1989, 1998) [3]. The Jacobson radicals, J_ν, (ν = 0, 1, 2), of nearrings were extended to Γ-nearrings by Booth (1989). Several other radicals of nearrings including the Brown-McCoy radical was extended to Γ-nearrings.

In rings, Γ-rings and nearrings, element wise characterization of Jacobson radicals were given through the quasiregularity. On quasiregular elements in a Γ-nearring (left) are introduced and a characterization of Jacobson radicals of a Γ-nearring (left) in terms of quasi-regular ideas [1].

Right Jacobson radicals of nearrings were introduced and studied in [2]. Unlike in left Jacobson radicals of nearrings, the right Jacobson radicals of nearring provide an analogous of Wedderburn-Artin theorem of rings and all these right Jacobson radicals are KA-radicals.

In this paper the right jacobson radical of type-0 of nearrings is extended to Γ-nearrings. Similar to nearrings right quasiregularity and right modularity are introduced in Γ-nearrings, in order to give element wise characterization and other characterizations of Jacobson radicals of Γ-nearrings and thus to develop new structure theorems in Γ-nearrings.

It is proved that The Jacobson radical J_0^r(W) is the largest ideal of W included within the radical $J_{1/2}^r$(W). A maximal right modular ideal C of a Γ-nearring W is a right 0-primitive ideal of W. Furthermore, a right 0-primitive ideal of a Γ-nearring W is also a prime ideal of W.

2.PRELIMINARIES

Definition 2.1. A **Γ-nearring** is defined by the triple (W, +, Γ).

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i. The pair (W, +) forms a group;
ii. Γ is a non-empty collection of binary operations defined on the set W provided that for each $\gamma \in \Gamma$, (W, +, γ) is a right nearing;
iii. $(s\gamma t)\mu u = s\gamma(t\mu u)$ for each s, t, u $\in$ W, $\gamma, \mu \in \Gamma$.

A Γ-nearring W is **zerosymmetric Γ-nearring** if $s\gamma 0 = 0$ for every $s \in W, \gamma \in \Gamma$.
W₀ = $\{s \in W \mid s\gamma 0 = 0$ for each $\gamma \in \Gamma\}$ which is known as the **zerosymmetric part** of W.
W_C = $\{s \in W \mid s\gamma 0 = s$ for each $\gamma \in \Gamma\}$ which is known as the **constant part** of W.
Unless or until specified all Γ-nearring are **zerosymmetric Γ-nearrings**.

Definition 2.2. If C is a subset of Γ-nearring W then C is **right ideal of W** if

i. C is a normal subgroup of the additive group W;
ii. $s\gamma u \in C$ for each $u \in W, \gamma \in \Gamma$ and $s \in C$.

Definition 2.3. If C is a subset of Γ-nearring W then C is **left ideal of W** if

i. C is a normal subgroup of the additive group W;
ii. $u\gamma(v + s) - u\gamma v \in C$ for each $u, v \in W, \gamma \in \Gamma$ and $s \in C$.

If C is a subset of Γ-nearring W then C is an **ideal of W** if it is both a right and a left ideal.

Definition 2.4. If P is an ideal of a Γ-nearring W then P is a **prime ideal** if S, T are ideals of W and $S\Gamma T \subseteq P$ implies either $S \subseteq P$ or $T \subseteq P$.

Definition 2.5. If P is an ideal of a Γ-nearring W then P is a **semi-prime ideal** if S is ideal of W and $S\Gamma S \subseteq P$ implies $S \subseteq P$.

Remark 2.6.

i. A Γ-nearring W is known as prime (semi-prime) if $\{0\}$ is prime (semi-prime) ideal of W.
ii. If P is an ideal of a Γ-nearring W then P is a prime (semi-prime) ideal of W if and only if W/C is prime (semi-prime) Γ-nearring.
iii. Every prime ideal of a Γ-nearring W is semi-prime ideal of W.

Definition 2.7. If W is a Γ-nearring, an element $s \in W$ is **nilpotent** element if for every $\gamma \in \Gamma$ exists a positive integer n (based on γ) provided that $(s\gamma)^n s = 0$.
If B is a subset of W then B is nil if every element of B is nilpotent.
If B is a subset of W then B is nilpotent if $(B\Gamma)^n B = \{0\}$, for few positive integer n.

3. RIGHT QUASI REGULARITY AND JACOBSON RADICALS OF TYPE-0

Definition 3.1. If W is a Γ-nearring, $s \in W$ and $\gamma \in \Gamma$. Consider $C_{s,\gamma}$ to be a right ideal of W generated by the set of each element $\{u - s\gamma u \mid u \in W\}$. Then s is **Right γ-quasiregular** if $s \in C_{s,\gamma}$ (or C = W).

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Definition 3.2. If W is a Γ-nearring and $s \in W$. Then s is Right quasiregular if s is **Right γ-quasiregular** for each $\gamma \in \Gamma$.

If B is a subset of W then B is Right γ-quasiregular if every element of B is **Right γ-quasiregular**.

Definition 3.3. If W is a Γ-nearring, C is a right ideal of W then C is **Right modular** if there exists $e_0 \in W$ and $\gamma_0 \in \Gamma$ provided that $u - e_0\gamma_0 u \in C$ for each $u \in W$.

In this case C is right modular by e_0.

Note: Consider C to be a proper right ideal of W and C is right modular by e_0. Then $e_0 \notin C$. We have $u - e_0\gamma_0 u \in C$ for each $u \in W$ for few $\gamma_0 \in \Gamma$. If $e_0 \in C$ then, $e_0\gamma_0 u \in C$ for each $u \in W$. As $u - e_0\gamma_0 u \in C$ for each $u \in W$. We get that $u = (u - e_0\gamma_0 u) + e_0\gamma_0 u \in C$ for each $u \in W$.

So $W \subseteq C$ and hence $C = W$, a contradiction.

Proposition 3.4. Consider W to be Γ-nearring and $s \in W$, $\gamma \in \Gamma$. Then

i. $C_{s,\gamma} = W$ if and only if $s \in C_{s,\gamma}$
ii. $C_{s,\gamma}$ modular by s.

Proof. (i) If $C_{s,\gamma} = W$ then obviously, $s \in C_{s,\gamma}$. Assuming that $s \in C_{s,\gamma}$ and $w \in W$. We have $w = (w - s\gamma w) + s\gamma w$. Also $s\gamma w \in C_{s,\gamma}$ as $C_{s,\gamma}$ is a right ideal of W and $s \in C_{s,\gamma}$ and $w - s\gamma w \in C_{s,\gamma}$. So $w = (w - s\gamma w) + s\gamma w \in C_{s,\gamma}$. Therefore $W \subseteq C_{s,\gamma}$ and hence $C_{s,\gamma} = W$.

(ii) A right ideal $C_{s,\gamma}$ is generated by $\{w - s\gamma w \mid w \in W\}$. So $w - s\gamma w \in C_{s,\gamma}$ for each $w \in W$. Thus $C_{s,\gamma}$ is modular by s.

Proposition 3.5. If s is an element in Γ-nearring W is nilpotent then s is right quasiregular.

Proof. Assuming that $s \in W$ is nilpotent and $\gamma_0 \in \Gamma$. Since $s \in W$ is nilpotent there is a positive integer n provided that $(s\gamma_0)^n s = 0$. Presently C_{s,γ_0} is the right ideal generated by the set $\{u - s\gamma_0 u \mid u \in W\}$. $s \in W$ implies that $s - s\gamma_0 s \in C_{s,\gamma_0}$. So $s\gamma_0 s - s\gamma_0(s\gamma_0 s) = s\gamma_0 s - (s\gamma_0)^2 s \in C_{s,\gamma_0}$. $s - s\gamma_0 s$, $s\gamma_0 s - (s\gamma_0)^2 s$, $(s\gamma_0)^2 s - (s\gamma_0)^3 s$, . . . , $(s\gamma_0)^{n-1} s - (s\gamma_0)^n s \in C_{s,\gamma_0}$. $(s - s\gamma_0 s) + ((s\gamma_0 s - (s\gamma_0)^2 s)) + (((s\gamma_0)^2 s - (s\gamma_0)^3 s)) + . . . + (((s\gamma_0)^{n-1} s - (s\gamma_0)^n s)) \in C_{s,\gamma_0}$. So $s - (s\gamma_0)^n s \in C_{s,\gamma_0}$, that is, $s - 0 \in C_{s,\gamma_0}$. Thus s is right γ_0-quasiregular. Since $\gamma_0 \in \Gamma$ is arbitrary, s is right quasiregular.

Corollary 3.6. Every nil subset of a Γ-nearring W is right quasiregular.

Proof. Consider B to be nil subset of a Γ-nearring W.Then by definition for each $b \in B$, b is nilpotent in W for all $b \in B$, b is right quasiregular by Proposition **3.5.** Therefore B is right quasiregular. Thus every nil subset of W is right quasiregular.

Corollary 3.7. Every nil ideal C of a Γ-nearring W is right quasiregular.

Proof. Consider C to be nil ideal of a Γ-nearring W. Then every element of C is a nilpotent element of W. By the above Corollary **3.6**, every element of C is right quasiregular. Hence C is right quasiregular ideal of W.

Lemma 3.8. Consider W to be non zerosymmetric Γ-nearring. Then the constant part of W is right quasiregular.

Proof. Consider W_C to be a constant part of W and $s \in W_C$. We have that $\mathbf{W}_C := \{u \in W \mid u\gamma 0 = u \text{ for all } \gamma \in \Gamma\}$. Consider $w_c \in W_C$. So $w_c\gamma 0 = w_c$. We prove that $w_c\gamma w = w_c$ for each $w \in W$, $\gamma \in \Gamma$. Consider $w \in W$ and $\delta \in \Gamma$. Presently $(w_c\gamma 0)\delta w = w_c\delta w$ and that $w_c\gamma(0\delta w) = w_c\delta w$ and that $w_c\gamma 0 = w_c\delta w$. So $w_c = w_c\delta w$. Consider C be the right ideal of W generated by $u - s\gamma u$, $u \in W$ and $\gamma \in \Gamma$. For $u \in W$, $\gamma \in \Gamma$, $(u + s) - s\gamma(u + s) = (u + s) - s = u \in C$. Therefore $C = W$ and hence s is right quasiregular. So W_C is right quasiregular.

Proposition 3.9. Consider $e \in W$. Then e is a right quasiregular if and only if no proper right ideal of W is right modular by e.

Proof. Assuming that e is a right quasiregular. Consider C to be a proper right ideal of W right modular by e. So $e \notin C$ by the above argument. Consider $\gamma_0 \in \Gamma$. Presently $u - e\gamma_0 u \in C$ for each $u \in W$ as e is right γ_0-quasiregular. Since e is right γ_0-quasiregular, $C = W$. This is inconsistent with $e \notin C$. Therefore, W has no proper right ideal which is right modular by e. Conversely, Assuming that no proper right ideal of W is right modular by e. We prove that e is right quasiregular in W. Consider $\gamma_0 \in \Gamma$ and C be the right ideal of W generated by $u - e\gamma_0 u$, $u \in W$. Presently C is right modular by e. By our assumption $C = W$. So, e is right γ_0-quasiregular. Since $\gamma_0 \in \Gamma$ is arbitrary, e is right quasiregular.

Proposition 3.10. Consider C to be a proper right ideal of Γ-nearring W. If C is a right modular, then C included in a maximal right ideal of Γ-nearring W which is also right modular.

Proof. Assuming that C is a right modular by e, that is, $u - e\gamma u \in C$ for each $u \in W$, for few $\gamma \in \Gamma$. If $e \in C$, then $C = W$, a contradiction. So $e \notin C$. let $S := \{E \mid E \text{ is a right ideal of } W, e \notin E \text{ and } C \subseteq E\}$. Since $C \in S$, $S \neq \emptyset$. If is clear that S satisfies the hypothesis of zorn's lemma under set inclusion. So by zorn's lemma, S includes a maximal element, say U. Presently U is a right ideal of W, $e \notin U$ and $C \subseteq U$. We claim that U is maximal right ideal of W. Presently $U \neq W$ as $e \notin U$. Consider V to be right ideal of W provided that $U \subseteq V \subseteq W$, and $V \neq W$. If $e \in V$, then $u = (u - e\gamma u) + e\gamma u \in V$ for each $u \in W$ and hence $V = W$, a contradiction. So $e \notin V$, and hence $V \in S$. By the maximality of U, $U = V$. Therefore, U is a

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maximal right ideal of W. Since C is right modular by e and C ⊆ U, we have that U is also right modular by e. Thus U is a maximal right modular right ideal of W containing C.

Corollary 3.11. Consider W to be Γ-nearring, C is a modular right ideal of W, C ≠ W and which is modular by e then e is not quasiregular in W.

Proof. Given that C is a modular right ideal of the Γ-nearring W, modular by e_0, and C ≠ W. Since C is a right modular right ideal of W, there exist $\gamma_0 \in \Gamma$ provided that $u - e_0\gamma_0 u \in C$ for each $u \in W$. Presently, consider C_{e_0,γ_0} be the right ideal of W generated by $\{u - e_0\gamma_0 u \mid u \in W\}$. Since C is modular by e_0, all elements of the form $u - e_0\gamma_0 u$ are in C. Thus, $C_{e_0,\gamma_0} \subseteq C$. However, C ≠ W, so C_{e_0,γ_0} is a proper right ideal of W. Presently, suppose e_0 is quasiregular in W. So e_0 is also γ_0-quasiregular in W. Therefore $C_{e_0,\gamma_0} = W$ and that C = W, a contradiction to C ≠ W. Therefore e_0 is not quasiregular in W.

Definition 3.12. If W is a Γ-nearring, a group (D, +) is a **right ΓW-group** if there is a mapping $(d, \gamma, w) \to d\gamma w$ of $D \times \Gamma \times W$ into D provided that

i. $(d\gamma u)\mu v = d\gamma(u\mu v)$
ii. $(d + h)\gamma u = d\gamma u + h\gamma u$ for every $u, v \in W$, $\gamma, \mu \in \Gamma$ and $d, h \in D$.

Here, W is a Γ-nearring and D is a ΓW-group.

Definition 3.13. An element d ∈ D is known as **distributive** if there is a γ ∈ Γ provided that $d\gamma(u + v) = d\gamma u + d\gamma v$ for each u, v ∈ W.

Definition 3.14. An element d ∈ D is known as **generator** of ΓW-group D, if dγW = D for few γ ∈ Γ.

Definition 3.15. Assuming that D is a ΓW-group and d ∈ D is a distributive element, that is, for few $\gamma_0 \in \Gamma$, $d\gamma_0(u + v) = d\gamma_0 u + d\gamma_0 v$ for each u, v ∈ W.Then D is known as **monogenic** ΓW-group if $d\gamma_0 W = D$.

Definition 3.16. If H to be subgroup of D then H is an (right) **ΓW-subgroup** of D, if hγw ∈ H for each w ∈ W, γ ∈ Γ and h ∈ H.

Definition 3.17. If K to be normal subgroup of D then K is an **ΓW-ideal** of D, if sγw ∈ K for each w ∈ W, γ ∈ Γ and s ∈ K.

Every right ideal of W is an ideal of the right ΓW-group W. Also, if C is a right ideal of W, then W/C is a right ΓW-group, where (u + C)γw = uγw + C, for each u + C ∈ W/C, w ∈ W and γ ∈ Γ.

Definition 3.18. If D is a right ΓW-group. Then D is **simple** if DΓW ≠ {0}, {0} and D are the only ideals of D.

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Definition 3.19. If W is a Γ-nearring and D, E be right ΓW-groups a mapping $f : D \to E$ is called an **ΓW-homomorphism** if

i. $f(u + v) = f(u) + f(v)$;
ii. $f(u\gamma w) = f(u)\gamma w$ for each u, v ∈ D, w ∈ W, γ ∈ Γ.

D is ΓW-isomorphic to E if there is a one-one ΓW-homomorphism of D onto E.

We write $D \cong_{\Gamma W} E$ if D is **ΓW-isomorphic** to E.

Proposition 3.20. If W is a Γ-nearring and D is a right ΓW-group then D is monogenic if and only if there is a right modular right ideal C of W provided that $W/C \cong_{\Gamma W} D$.

Proof. Consider W be a Γ-nearring and D be an ΓW-group. First Assuming that D is monogenic. So we get a distributive element d ∈ D, that is, for few $\gamma_0 \in \Gamma$, $d\gamma_0(u + v) = d\gamma_0 u + d\gamma_0 v$ for each u, v ∈ W with $d\gamma_0 W = D$. Define f : W → D by $f(w) = d\gamma_0 w$ for each w ∈ W. Consider w, z ∈ W, δ ∈ Γ and h ∈ D. Also, $f(w + z) = d\gamma_0(w + z) = d\gamma_0 w + d\gamma_0 z = f(w) + f(z)$ and $f(h\delta w) = d\gamma_0(h\delta w) = (d\gamma_0 h)\delta w = f(h)\delta w$. So f is an ΓW-homomorphism. For $d_1 \in D$ there exists w ∈ W provided that $d\gamma_0 w = d_1$ and $f(w) = d\gamma_0 w = d_1$. Thus f is onto D. Presently $\ker f = \{w \in W \mid f(w) = 0\} = \{w \in W \mid d\gamma_0 w = 0\}$. Consider kerf = C. Then C is a right ideal of W and $W/C \cong_{\Gamma W} D$. Since $d\gamma_0 W = D$, we get e ∈ W provided that $d\gamma_0 e = d$. Presently $d\gamma_0(w - e\gamma_0 w) = d\gamma_0 w - (d\gamma_0)e\gamma_0 w = d\gamma_0 w - (d\gamma_0 e)\gamma_0 w = d\gamma_0 w - d\gamma_0 w = 0$. Therefore, $w - e\gamma_0 w \in C$ for each w ∈ W. Thus C is a right modular right ideal of W. Conversely, Assuming that C is a right modular right ideal of W and $W/C \cong_{\Gamma W} D$. We get e ∈ W, $\gamma_0 \in \Gamma$ provided that $u - e\gamma_0 u \in C$ for each u ∈ W. We see that (e + C) is a generator of the right ΓW-group of W/C. Consider u ∈ W. Now $u - e\gamma_0 u \in C$. So $u + C = e\gamma_0 u + C = (e + C)\gamma_0 u \in (e + C)\gamma_0 W$ and hence $(e + C)\gamma_0 W = W/C$. Therefore, e + C is a generator of the ΓW-group W/C. Consider u, w ∈ W. We have $(u + w) - e\gamma_0(u + w)$, $u - e\gamma_0 u$, $w - e\gamma_0 w \in C$. Consider $i = u - e\gamma_0 u$ and $j = w - e\gamma_0 w$. $u = i + e\gamma_0 u$ and $w = j + e\gamma_0 w$. $(u + w) - e\gamma_0(u + w) = (i + e\gamma_0 u) + (j + e\gamma_0 w) - e\gamma_0(u + w) = i + (e\gamma_0 u + j - e\gamma_0 u) + e\gamma_0 u + e\gamma_0 w - e\gamma_0(u + w) \in C$. And that, $e\gamma_0 u + e\gamma_0 w - e\gamma_0(u + w) \in C$. Therefore $e\gamma_0(u + w) + C = (e\gamma_0 u + e\gamma_0 w) + C$. $(e + C)\gamma_0(u + w) = e\gamma_0(u + w) + C = (e\gamma_0 u + e\gamma_0 w) + C = (e\gamma_0 u + C) + (e\gamma_0 w + C) = (e + C)\gamma_0 u + (e + C)\gamma_0 w \in W/C$. This shows that (e + C) is a distributive element of the ΓW-group W/C. Therefore, W/C is monogenic ΓW-group. Since $W/C \cong_{\Gamma W} D$, D is also a monogenic ΓW-group.

Definition 3.21. If D is a monogenic right ΓW-group. Then D is a **right ΓW-group of type-0** if D is a simple.

Definition 3.22. If C is a modular right ideal of W. Then C is **0-modular right ideal of W** if W/C is a right ΓW-group of type-0.

Theorem 3.23. A modular right ideal C of W is a 0-modular right ideal of W if and only if C is a maximal right ideal of W.

Proof. We have that C is a modular right ideal of W. Assuming that C is 0-modular right ideal. So W/C is a right ΓW-group of type-0. Since W/C is a right ΓW-group of type-0, W/C is a simple ΓW-group. So C is a maximal right ideal of W. Since W/C is a right ΓW-group of type-0, W/C is a monogenic ΓW-group. So we get $u + C \in W/C$ and $\gamma \in \Gamma$ provided that $(u + C)\gamma W = W/C$ and $(u + C)\gamma(v + z) = (u + C)\gamma v + (u + C)\gamma z$ for each $v, z \in W$. Define $f: W \rightarrow W/C$ by $f(w) = (u + C)\gamma w = u\gamma w + C$ for each $w \in W$. Consider $w, z \in W$, $\delta \in \Gamma$ and $h \in W/C$. Also, $f(w + z) = (u + C)\gamma(w + z) = (u + C)\gamma w + (u + C)\gamma z = (u\gamma w + C) + (u\gamma z + C) = f(w) + f(z)$ and $f(h\delta w) = (u + C)\gamma(h\delta w) = ((u + C)\gamma h)\delta w = (u\gamma h + C)\, \delta w = f(h)\delta w$. So f is an ΓW-homomorphism. Since $(u + C)\gamma W = W/C$, for $v + C \in W/C$ there exists $w \in W$ provided that $(u + C)\gamma w = v + C$ and $f(w) = u\gamma w + C = v + C$. Thus f is onto W/C. Since $(u + C)\gamma W = W/C$, we get $e \in W$ provided that $(u + C)\gamma e = u + C$. Presently for $w \in W$, $(u + C)\gamma(w - e\gamma w) = (u + C)\gamma w - ((u + C)\gamma)e\gamma w = (u\gamma w + C) - ((u + C)\gamma e)\gamma w = (u\gamma w + C) - (u + C)\gamma w = (u\gamma w + C) - (u\gamma w + C) = C$. Presently $\ker f = \{w \in W \mid f(w) = 0\} = \{w \in W \mid (u + C)\gamma w = 0\} = \{w \in W \mid u\gamma w + C = C\} = \{w \in W \mid u\gamma w \in C\}$. Consider $\ker f = C$. Then C is a right ideal of W. C is a right modular right ideal of W. Thus, if C is a 0-modular right ideal, it's also modular and maximal. Conversely, Assuming that C is maximal. We show that C is a 0-modular right ideal of W, that is, W/C is a right ΓW-group of type-0. First we show that W/C is a monogenic right ΓW-group. Since C is modular, we get $e \in W$, $\gamma \in \Gamma$ provided that $u - e\gamma u \in C$ for each $u \in W$. We see that $(e + C)$ is a generator of the right ΓW-group of W/C. Consider $u \in W$. Presently $u - e\gamma u \in C$. So $u + C = e\gamma u + C = (e + C)\gamma u \in (e + C)\gamma W$ and hence $(e + C)\gamma W = W/C$. Therefore, $e + C$ is a generator of the ΓW-group W/C. Consider $u, w \in W$. We have $(u + w) - e\gamma(u + w)$, $u - e\gamma u$, $w - e\gamma w \in C$. Consider $i = u - e\gamma u$ and $j = w - e\gamma w$. $u = i + e\gamma u$ and $w = j + e\gamma w$. $(u + w) - e\gamma(u + w) = (i + e\gamma u) + (j + e\gamma w) - e\gamma(u + w) = i + (e\gamma u + j - e\gamma u) + e\gamma u + e\gamma w - e\gamma(u + w) \in C$. And that, $e\gamma u + e\gamma w - e\gamma(u + w) \in C$. Therefore $e\gamma(u + w) + C = (e\gamma u + e\gamma w) + C$, that is, $(e + C)\gamma(u + w) = e\gamma(u + w) + C = (e\gamma u + e\gamma w) + C = (e\gamma u + C) + (e\gamma w + C) = (e + C)\gamma u + (e + C)\gamma w$. This shows that $(e + C)$ is a distributive element of the ΓW-group W/C. Therefore, W/C is monogenic ΓW-group. Presently we show that W/C is a simple ΓW-group, that is, W/C has no proper ΓW-ideals other than $\{0\}$ and W/C. Since C is maximal right ideal of W, if E is an ideal of W and $C \subseteq E \subseteq W$ then either $C = E$ or $E = W$. Consider K to be ΓW-ideal of W/C. Now $K = E/C$ for few right ideal E of W containing C. Since $C \subseteq E \subseteq W$ and C is a maximal right ideal of W, either $C = E$ or $E = W$, that is, $K = \{0\}$ or $K = W/C$. Therefore W/C is a simple ΓW-group and hence W/C is a ΓW-group of type-0.

Definition 3.24. $\boldsymbol{J^r_{1/2}}$**(W)** is the intersection of all 0-modular right ideal of Γ-nearring W. If W has no 0-modular right ideals then $J^r_{1/2}(W)$ is defined as W.

Theorem 3.25. If W is a Γ-nearring then $J^r_{1/2}(W)$ is largest right quasiregular ideal of W.

Proof. Consider W be a Γ-nearring. Being the intersection of all 0-modular right ideals of W, $J^r_{1/2}(W)$ is a right ideal of W. Presently we prove that $J^r_{1/2}(W)$ is right quasiregular R. If

$J_{1/2}^{r}(W) = W$ then every element of W is right quasiregular and thus the theorem is true in this case. Assuming that $J_{1/2}^{r}(W) \neq W$. Consider $s \in J_{1/2}^{r}(W)$. Let $\gamma \in \Gamma$. We presently prove that $s \in C_{s,\gamma}$, the right ideal of W generated by the set $\{u - s\gamma u \mid u \in W, \gamma \in \Gamma\}$. On the contrary Assuming that $s \notin C_{s,\gamma}$. By zorn's lemma $C_{s,\gamma}$ can be extended to a maximal modular right ideal of B of W. [$\mathcal{A} = \{K \mid$ K is a proper right ideal of W and $I_{a,\gamma} \subseteq K\}$. $\mathcal{A}$ is poset under set inclusion ($\subseteq$) ($K_1 \leq K_2$ if and only if $K_1 \subseteq K_2$, $K_1, K_2 \in \mathcal{A}$). Consider $\{K_\alpha \mid \alpha \in \Delta\}$ to be chain in $\mathcal{A}$, (α_1 , $\alpha_2 \in \Delta$ implies either $K_{\alpha 1} \subseteq K_{\alpha 2}$ or $K_{\alpha 2} \subseteq K_{\alpha 1}$). Consider $K = \bigcup \alpha \in \Delta\ K_\alpha$. Clearly $K \in \mathcal{A}$ and is an upper bound of the above chain. Therefore by zorn's lemma $\mathcal{A}$ has a maximal element T. We prove that T is a maximal right ideal of W. Assuming that $T \subseteq X \subseteq W$, X is a right ideal of W. If $X \neq W$, $X \in \mathcal{A}$. Therefore T is a maximal right ideal of W. (If K = W, then $s \in K$. So $s \in K_\alpha$ for few $\alpha \in \Delta$, $u - s\gamma u \in C_{s,\gamma} \subseteq K_\alpha$ and $s\gamma u \in K_\alpha$ as $s \in K_\alpha$)]. Since $C_{s,\gamma} \subseteq B$, B is a maximal modular right ideal of W. So $J_{1/2}^{r}(W) \subseteq B$ and that $s \in B$. So, $s\gamma u \in B$ for each $u \in W$. Also $u - s\gamma u \in C_{s,\gamma} \subseteq B$ for each $u \in W$. Therefore, $u = (u - s\gamma u) + s\gamma u \in B$ for each $u \in W$. So $W \subseteq B$ and $W = B$. This is contradiction to the maximality of B. Therefore, $s \in C_{s,\gamma}$. Hence s is right γ-quasiregular. γ being arbitrary, s is right quasiregular and thus $J_{1/2}^{r}(W)$ is a right quasiregular ideal of W. Consider Z to be right quasiregular ideal of W. Presently show that $Z \subseteq J_{1/2}^{r}(W)$. Suppose, on contrary that $Z \nsubseteq J_{1/2}^{r}(W)$. Then we have a 0-modular right ideal K of W provided that $Z \nsubseteq K$. So $K + Z = W$. As K is modular, there exists $e \in W$ and $\gamma \in \Gamma$ provided that $u - e\gamma u \in K$ for each $u \in W$. Now $e \in W$ that implies $e = k + z$, $k \in K$ and $z \in Z$. Since K is a right ideal of W, $u - z\gamma u = u - (-k + e)\gamma u = u + k\gamma u - e\gamma u = (u - e\gamma u) + k\gamma u \in K$ for each $u \in W$. Thus K ($\neq W$) is right ideal of W which is modular by z. Therefore z is not right quasiregular element which is a contradiction. Therefore our supposition that $Z \nsubseteq J_{1/2}^{r}(W)$ is false. Thus $Z \subseteq J_{1/2}^{r}(W)$. Therefore $J_{1/2}^{r}(W)$ is a largest right quasiregular ideal of W.

Corollary 3.26. $J_{1/2}^{r}(W)$ includes all nilpotent (nil) right ideals of the Γ-nearring W.

Proof. Consider W to be Γ-nearring. From above Theorem **3.25**, $J_{1/2}^{r}(W)$ is largest right quasiregular ideals of W. Consider C to be nil right ideal of W. Since C is nil, by Corollary **3.7**, C is right quasiregular in W. Therefore C is right quasiregular ideals of W. Hence $C \subseteq J_{1/2}^{r}(W)$. Consider C to be nilpotent right ideal of W. So C is a nil right ideal of W. As seen above, $C \subseteq J_{1/2}^{r}(W)$.

Definition 3.27. Consider D to be right ΓW-group. The **annihilator** of D denoted by $(0 : D) = \{u \in W \mid d\gamma u = 0$ for each $d \in D$ and $\gamma \in \Gamma\}$.

If U and V are non empty subsets of W, then $(U : V)$ denoted the set $\{w \in W \mid V\gamma w \subseteq U$ for each $\gamma \in \Gamma\}$.

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Proposition 3.28. If $\{R_i \mid i \in C\}$ is a non-empty collection of right ideals of Γ-nearring W then $\bigcap_{i \in C}(R_i : W) = (\bigcap_{i \in C} R_i : W)$, where $(V : W) = \{u \in W \mid w\gamma u \in V$ for all $\gamma \in \Gamma$ and $w \in W\}$.

Proof. $\{R_i \mid i \in C\}$ is a non-empty collection of right ideals of Γ-nearring W. Consider $s \in \bigcap_{i \in C}(R_i : W)$, Let $i \in C$. Presently, $s \in (R_i : W)$ So, $w\gamma s \in R_i$ for each $\gamma \in \Gamma$ and $w \in W$. Since $i \in C$ is arbitrary $w\gamma s \in \bigcap_{i \in C} R_i$ for each $\gamma \in \Gamma$ and $w \in W$. Therefore $s \in (\bigcap_{i \in C} R_i : W)$ and that $\bigcap_{i \in C}(R_i : W) \subseteq (\bigcap_{i \in C} R_i : W) \longrightarrow (1)$. Consider $s \in (\bigcap_{i \in C} R_i : W)$, Consider $i \in C$. $w\gamma s \in \bigcap_{i \in C} R_i$ for each $\gamma \in \Gamma$ and $w \in W$. So $w\gamma s \in R_i$ for each $\gamma \in \Gamma$, $w \in W$ and for each $i \in C$. Therefore $s \in (R_i : W)$ for each $i \in C$, that is, $s \in \bigcap_{i \in C}(R_i : W)$. $(\bigcap_{i \in C} R_i : W) \subseteq \bigcap_{i \in C}(R_i : W) \longrightarrow (2)$. From (1) & (2). $\bigcap_{i \in C}(R_i : W) = (\bigcap_{i \in C} R_i : W)$.

Proposition 3.29. Consider C to be right ideal of W then C is right modular if and only if there is a right ΓW-group D which is monogenic by d_0 provided that $C = (0 : d_0)$.

Proof. Assuming that C is right modular by e_0. So we get $\gamma_0 \in \Gamma$ provided that $u - e_0\gamma_0 u \in C$ for each $u \in W$. By Proposition **3.20**, $e + C$ is a generator of the right ΓW-group W/C is monogenic. Presently $w \in (C : e_0 + C) \Leftrightarrow e_0\gamma_0 w + C = C \Leftrightarrow e_0\gamma_0 w \in C \Leftrightarrow w \in C$ as $w - e_0\gamma_0 w \in C$. Therefore $C = (C : e_0 + C)$. Conversely, Assuming that ΓW-group D is monogenic by d_0 and $(0 : d_0) = C$. We get $\gamma_0 \in \Gamma$ provided that $d_0\gamma_0 W = D$ and $d_0\gamma_0(u + v) = d_0\gamma_0 u + d_0\gamma_0 v$ for each $u, v \in W$. We get $e_0 \in W$ provided that $d_0\gamma_0\, e_0 = d_0$. Consider $w \in W$. Presently $d_0\gamma_0\,(w - e_0\gamma_0 w) = d_0\gamma_0 w - (d_0\gamma_0)e_0\gamma_0 w = d_0\gamma_0 w - (d_0\gamma_0 e_0)\,\gamma_0 w = d_0\gamma_0 w - d_0\gamma_0 w = 0$. Therefore $w - e_0\gamma_0 w \in (0 : d_0) = C$. Hence, C is a right modular by e_0.

Lemma 3.30. Consider D to be monogenic ΓW-group. Then $(0 : D)$ includes a largest ideal of the Γ-nearring W and is denoted by $\underline{(0 : D)}$.

Proof. We have $D\Gamma 0 = \{0\}$. We get $d_0 \in D$, $\gamma_0 \in \Gamma$ provided that $d_0\gamma_0(u + v) = d_0\gamma_0 u + d_0\gamma_0 v$ for each $u, v \in W$ and $d_0\gamma_0 W = D$. $0 + d_0\gamma_0 0 = d_0\gamma_0 0 = d_0\gamma_0(0 + 0) = d_0\gamma_0 0 + d_0\gamma_0 0$. $0 + d_0\gamma_0 0 = d_0\gamma_0 0 + d_0\gamma_0 0 \Rightarrow 0 = d_0\gamma_0 0$. $D\Gamma 0 = (d_0\gamma_0 W)\Gamma 0 = (d_0\gamma_0)\, W\Gamma 0 = d_0\gamma_0 0 = 0$. So $\{0\}$ is an ideal of W and $\{0\} \subseteq \underline{(0 : D)} \rightarrow (1)$. Let C, E to be ideals of the Γ-nearring W and $C \subseteq (0 : D)$, $E \subseteq \underline{(0 : D)}$ then $C + E \subseteq (0 : D)$. So $D\Gamma C = \{0\}$ and $D\Gamma E = \{0\}$. Let D be monogenic by $d_0 \in D$ as a ΓW-group. Let $d \in D$. Now $d = d_0\gamma_0 u$ for few $u \in W$. Let $v \in C$, $z \in E$. $d\gamma(v + z) = (d_0\gamma_0 u)\,\gamma\,(v + z) = d_0\gamma_0(u\gamma(v + z)) = d_0\gamma_0(\,(u\gamma(v + z) - u\gamma z) + u\gamma z) = d_0\gamma_0\,[v_1 + z_1] = d_0\gamma_0 v_1 + d_0\gamma_0 z_1$, $v_1 \in C$, $z_1 \in E = 0 + 0 = 0$. So $C + E \subseteq (0 : D) \rightarrow (2)$. $\{C_\alpha \mid \alpha \in \Delta\}$ be the set of all ideals of W included in $(0 : D)$ then $\Sigma_{\alpha \in \Delta} C_\alpha \subseteq (0 : D)$. From (2) and $\Sigma_{\alpha \in \Delta} C_\alpha$ is the largest ideal of W included in $(0 : D)$. Hence $\underline{(0 : D)}$ includes largest ideal of the Γ-nearring W.

Proposition 3.31. Consider D to be right ΓW-group and D is a right ΓW-group of type-0 if and only if there is a maximal right modular right ideal C of W provided that D is ΓW-isomorphic to W/C.

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Proof. D is a right ΓW-group. Assuming that D is a right ΓW-group of type-0. By proposition **3.20**, ΓW-group D is monogenic, So we get a distributive element d ∈ D, that is, for few $\gamma_0 \in \Gamma$, $d\gamma_0(u + v) = d\gamma_0 u + d\gamma_0 v$ for each u, v ∈ W with $d\gamma_0 W = D$, and D is ΓW-isomorphic to W/C for few right modular right ideal C of W. Since D is simple ΓW-group, W/C is also simple ΓW-group. Therefore C is a maximal right ideal of W. Conversely, Assuming that C is maximal right modular right ideal of W and D is ΓW-isomorphic to W/C, We show that D is a right ΓW-group of type-0. Since C is maximal right ideal of W, By Theorem **3.23**, W/C is simple ΓW-group. Therefore D is also simple ΓW-group. Since C is right modular right ideal of W, By Theorem **3.23**, W/C is a monogenic ΓW-group. Therefore D is also monogenic ΓW-group.Hence, D is a right ΓW-group of type-0.

Proposition 3.32. Consider C to be right modular right ideal of W. Then $(\underline{C : W}) \subseteq C$.

Proof. Suppose C is a right modular right ideal of W. So we get $e_0 \in W$ and $\gamma_0 \in \Gamma$ provided that $u - e_0\gamma_0 u \in C$ for each u ∈ W. Consider $E = (\underline{C : W})$. So $W\Gamma E \subseteq C$. Consider v ∈ E. Now $e_0\gamma_0 v \in C$. As $v - e_0\gamma_0 v \in C$, $(v - e_0\gamma_0 v) + e_0\gamma_0 v \in C$ and that v ∈ C. Therefore E ⊆ C, that is, $(\underline{C : W}) \subseteq C$.

Proposition 3.33. If C is a right modular right ideal of W, then $(\underline{C : W})$ is largest ideal of W included in C.

Proof. Suppose C is a right modular right ideal of W. So we get $e_0 \in W$ and $\gamma_0 \in \Gamma$ provided that $u - e_0\gamma_0 u \in C$ for each u ∈ W. Since C is right modular right ideal of W, by Proposition **3.32**, $(\underline{C : W}) \subseteq C$. Consider E be an ideal of W included in C. So $W\Gamma E \subseteq C$ and that $E \subseteq (C : W)$. Since E is an ideal of W, $E \subseteq (\underline{C : W})$. Hence, $(\underline{C : W})$ is the largest ideal of W included in C.

Definition 3.34. If C is a ideal of W. Then C is **Right 0-Primitive ideal of W**. If C is the largest ideal of W included in $(0 : D) = \{w \in W \mid d\gamma w = 0$, for each $d \in D$ and $\gamma \in \Gamma\}$, for few right ΓW-group D of type-0.

Consider D to be ΓW-group of type-0. Since $(\underline{0 : D})$ is the largest ideal of W included in (0 : D), by the definition of right 0-primitive ideal of W, **$\underline{(0 : D)}$ is a Right 0-Primitive ideal of W**.

Γ-nearring W is right 0-primitive Γ-nearring if {0} is a right 0-primitive ideal of W.

Proposition 3.35. Consider Q to be ideal of a zerosymmetric Γ-nearring W. Q is right 0-primitive ideal if and only if $Q = (\underline{K : W})$ for few right 0-modular right ideal K of W.

Proof. Q is an ideal of zerosymmetric Γ-nearring W. Assuming that Q is a right 0-primitive ideal of W. So there is a right ΓW-group D of type-0 provided that $Q = (\underline{0 : D})$. By Proposition **3.31**, there is a maximal right modular right ideal K of W provided that D is ΓW-isomorphic to W/K. Since D is a right ΓW-group D of type-0, W/K is also right ΓW-group D

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of type-0. So by definition, K is a 0-modular right ideal of W. Presently Q = $(\underline{0 : D})$ = $(\underline{0 : W/K})$ = $(\underline{K : W})$. Conversely Assuming that Q = $(\underline{K : W})$ for few right 0-modular right ideal K of W. Since K is a 0-modular right ideal of W, by definition, W/K is a right ΓW-group D of type-0. So $(\underline{0: W/K})$ is a right 0-primitive ideal of W. Presently Q = $(\underline{K : W})$ = $(\underline{0 : W/K})$ is a right 0-primitive ideal of W.

Definition 3.36. The **right Jacobson radical of W of type-0** is defined as J_0^r(W) = ∩{ Q | Q is a right 0-primitive ideal of W} and J_0^r(W) is defined as W if W has no right 0-primitive ideals.

Theorem 3.37. If W is a Γ-nearring then J_0^r(W) = ($J_{1/2}^r$(W) : W).

Proof. If $J_{1/2}^r$(W) = W it follows that J_0^r(W) = W = ($J_{1/2}^r$(W) : W). Assuming that $J_{1/2}^r$(W) ≠ W. Consider C_j, j ∈ J consist of all 0-modular right ideal of W. Presently $J_{1/2}^r$(W) = $\bigcap_{j \in J} C_j$. By Proposition **3.28**, if {R_i | i ∈ Δ} is a non-empty collection of right ideals of Γ-nearring W then $\bigcap_{i \in \Delta}(R_i : W) = (\bigcap_{i \in \Delta} R_i : W)$. So by Proposition **3.32**, $(\underline{C : W}) \subseteq C$, for any right modular right ideal of W. Therefore ($J_{1/2}^r$(W) : W) = $(\bigcap_{j \in J} C_j : W) = \bigcap_{j \in J}(\underline{C_j : W})$. By Proposition **3.35**, $(\underline{C_j : W})$ is a right 0-primitive ideal of W for all j ∈ J. Moreover any right 0-primitive ideal of W is of the form (K : W) for few right 0-modular right ideal of W, by the same proposition. Hence $(\underline{C_j : W})$, j ∈ J is consist of all right 0-primitive ideals of W. So J_0^r(W) = $\bigcap_{j \in J}(\underline{C_j : W})$ = ($J_{1/2}^r$(W) : W).

Corollary 3.38. If W is a Γ-nearring then J_0^r(W) is the largest ideal of W included in $J_{1/2}^r$(W).

Proof. If $J_{1/2}^r$(W) = W it follows that J_0^r(W) = W and that J_0^r(W) is the largest ideal of W included in $J_{1/2}^r$(W). Assuming that $J_{1/2}^r$(W) ≠ W. By definition, J_0^r(W) is intersection of all right 0-primitive ideals in W. J_0^r(W) = ∩Q = ∩$(\underline{K : W})$, Q is any right 0-primitive ideal of W and that K is any right 0-modular right ideal of W. By Propositions **3.32** and **3.35**, for any right 0-primitive ideal Q of W there corresponds a right 0-modular right ideal K of W provided that Q = $(\underline{K : W}) \subseteq K$. Since $J_{1/2}^r$(W) is intersection of all 0-modular right ideals of W and $J_{1/2}^r$(W) = ∩K, as seen above J_0^r(W) = ∩Q = ∩$(\underline{K : W})$ ⊆ ∩K = $J_{1/2}^r$(W). Consider C to be ideal of W provided that C ⊆ $J_{1/2}^r$(W). Presently WΓC ⊆ C ⊆ $J_{1/2}^r$(W). So C ⊆ ($J_{1/2}^r$(W) : W) = J_0^r(W). So, if any ideal C of W included in $J_{1/2}^r$(W), then C ⊆ J_0^r(W). Therefore J_0^r(W) is the largest ideal of W included in $J_{1/2}^r$(W).

Corollary 3.39. If W is a Γ-nearring then J_0^r(W) is the largest right quasiregular ideal of W included in $J_{1/2}^r$(W).

Proof. The proof is followed by Corollary **3.38** and Theorem **3.25**.

Corollary 3.40. J_0^r(W) includes all nilpotent(nil) right ideals of a Γ-nearring W.

Proof. Consider W to be Γ-nearring. From above Corollary **3.39,** J_0^r(W) is the largest right quasiregular ideal of W included in $J_{1/2}^r$(W). Consider C to be nil right ideal of W. Since C is nil, by Corollary **3.7**, C is right quasiregular in W. Therefore, C is right quasiregular right ideals of W. Hence C $\subseteq J_0^r(W)$. Consider C to be nilpotent right ideal of W. So C is a nil right ideal of W. As seen above, C $\subseteq J_0^r(W)$. Therefore $J_0^r(W)$ includes all nilpotent (nil) right ideals of a Γ-nearring W.

Theorem 3.41. A maximal right modular ideal C of W is a right 0-primitive ideal of W.

Proof. Suppose C is maximal right modular ideal in W. By Zron's lemma, C is included in a maximal right modular right ideal K of W. So W/K is a right ΓW-group of type-0 and that E $:= (\underline{0 : W/K}) = (\underline{K : W})$ is a right 0-primitive ideal of W. Also E is the largest ideal of W included in K. Since C is a maximal ideal of W included in K, $C \subseteq E \subseteq W$. As C is a maximal ideal of W and $E \neq W$, $C = E$. Therefore C is right 0-primitive ideal of W.

Theorem 3.42. Consider Q to be ideal of W. Q is a right 0-primitive ideal of W if and only if W/Q is a right 0-primitive Γ-nearring.

Proof. Suppose Q is a right 0-primitive ideal in W. By Theorem **3.41**, we have a maximal right modular right ideal K of W provided that Q is the largest ideal of W included in K. Presently K/Q is a maximal right modular right ideal of W/Q. As Q is the largest ideal of W included in K, the zero ideal of W/Q is the largest ideal of W/Q included in K/Q. Therefore W/Q is a right 0-primitive Γ-nearring. Conversely, Assuming that W/Q is a right 0-primitive Γ-nearring. So we have a maximal right modular right ideal K/Q of W/Q provided that the zero ideal of W/Q is the largest ideal of W/Q included in K/Q. Clearly K is a maximal right modular right ideal of W and that W/K is a right ΓW-group of type-0. As the zero ideal of W/Q is the largest ideal of W/Q included in K/Q, Q is the largest ideal of W included in K. So $Q = (\underline{K : W}) = (\underline{0 : W/K})$. Therefore, Q is a right 0-primitive ideal of W.

Theorem 3.43. A right 0-primitive ideal of a Γ-nearring W is a prime ideal of W.

Proof. Consider Q be a right 0-primitive ideal of W. We have a right ΓW-group D of type-0 which is monogenic by d for which there corresponds a $\gamma_0 \in \Gamma$ satisfying $d\gamma_0(u + v) = d\gamma_0 u + d\gamma_0 v$ for each $u, v \in W$ with $d\gamma_0 W = D$ provided that Q is the largest ideal of W included in $K = \{u \in W \mid d\gamma_0 u = 0\}$, K is a maximal right modular right ideal of W. by Proposition **3.20** and by Theorem **3.23**. So $Q = (\underline{K : W})$. Consider S and T be ideals of W and $S\Gamma T \subseteq Q$. Assuming that $S \nsubseteq Q$ and $T \nsubseteq Q$. Since $S \nsubseteq Q$, $d\gamma_0 S \neq \{0\}$. Clearly $d\gamma_0 S$ is a subgroup of ΓW-group D. We have $d\gamma_0 W = D$. Consider $h \in D$ and $s \in S$. Then $h = d\gamma_0 u$ for few $u \in W$. Presently $h + d\gamma_0 s - h = d\gamma_0 u + d\gamma_0 s - d\gamma_0 u = d\gamma_0(u + s - u) \in d\gamma_0 S$, as $u + s - u \in S$. So $d\gamma_0 S$ is a normal subgroup of ΓW-group D. Also, $(d\gamma_0 S)\gamma_0 W = d\gamma_0(S\gamma_0 W) \subseteq d\gamma_0 S$. This shows that $d\gamma_0 S$ is an ideal of D. Since $d\gamma_0 S \neq \{0\}$ and D is a right ΓW-group D of type-0, $d\gamma_0 S = D$. In the same way $d\gamma_0 T = D$. Presently $D \supseteq d\gamma_0 S\Gamma T = (d\gamma_0 S)\Gamma T = D\Gamma T \supseteq d\gamma_0 T = D$. Therefore $d\gamma_0 S\Gamma T = D$, on a contrary to $d\gamma_0 S\Gamma T = \{0\}$, so either $S \subseteq Q$ or $T \subseteq Q$. Hence, Q is a prime ideal of W.

Theorem 3.44. A commutative right 0-primitive Γ-nearring is a field.

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Proof. Consider W to be commutative right 0-primitive Γ-nearring. We have a maximal right modular ideal C of W provided that {0} is the largest ideal of W included in C. Suppose C is a right modular ideal of W with modular e_0, there exist $\gamma_0 \in \Gamma$ provided that $u - e_0\gamma_0 u \in C$ for each $u \in W$. Since W is commutative, C is an ideal of W. Therefore, C = {0}. Since $u - e_0\gamma_0 u \in C = \{0\}$, $u = e_0\gamma_0 u = u\gamma_0 e_0$. Hence, e_0 is the multiplicative identity in W. As W is a commutative Γ-nearring equipped with an identity. (W, +) is abelian and hence W is a ring. In a commutative ring with identity there is a maximal ideal {0}, then each non-zero element must be invertible, W is field. Therefore W is a field as C = {0} is a maximal ideal of the commutative ring W with identity. W is a field because C = {0} is a maximal ideal.

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