Exploring the Foundations: An In-Depth Review of Numerical Techniques for Root Finding

Myasar Khalaf Obaid

mesrkalfobeed@gmail.com

Abstract

This paper presents a comparative analysis of the Bisection method and the Newton–Raphson method, focusing on their fundamental concepts, practical benefits, and inherent drawbacks. The discussion highlights aspects such as convergence behavior, computational performance, range of applicability to various function classes, and practical issues in implementation. The objective is to offer a thorough perspective that equips readers with clearer insights into the relative merits of both approaches, aiding in the selection of the most suitable technique for solving root-finding problems. Real-world illustrations are also included to demonstrate the effectiveness of these methods in diverse fields of science and engineering. Overall, the review seeks to strengthen the reader's understanding of numerical root-finding strategies and to encourage more effective use of these tools in computational analysis.

Keywords— Bisection algorithm; Newton–Raphson technique; Square root calculations; Polynomial and transcendental functions; Zero computation; RMS error; Precision; Convergence speed; Computational efficiency

1 Introduction

Root-finding represents a central challenge in numerical analysis, with extensive applications across science and engineering. The pursuit of determining function roots—such as square roots—has inspired the creation of a wide range of numerical algorithms, each designed with different principles and strategies. These computational methods have become essential in addressing sophisticated mathematical models and in handling real-world data analysis. Their utility spans diverse areas, including solving algebraic and transcendental equations, optimizing functions, and modeling physical systems, where accuracy and efficiency are crucial.

In scientific and engineering contexts, many mathematical problems are either extremely difficult or impossible to solve exactly. Numerical analysis bridges this gap by integrating mathematics and computer science to develop algorithms that approximate solutions effectively [1]. Among the numerous approaches, the **Bisection Method** and the **Newton–Raphson Method** are widely recognized. Both techniques approach root finding differently, offering unique advantages and limitations that make them valuable in specific contexts.

The Bisection Method is straightforward yet highly dependable. Its main strength lies in guaranteed convergence, provided that the function is continuous and the initial interval contains a sign change. This makes it a reliable option when other techniques may fail. It not only approximates the solution but also provides a clear interval within which the root exists. Despite its slower speed and inability to handle multiple or complex roots, the method's robustness, predictability, and role as a foundation for developing advanced algorithms make it a cornerstone of numerical analysis [2,3].

On the other hand, the Newton–Raphson Method is an iterative technique known for its rapid convergence when conditions are favorable. Although it may not always converge, in many cases it produces results in significantly fewer steps compared to other algorithms. Its iterative nature allows it to play a vital role in solving nonlinear equations, and its applications extend to optimization, control systems, signal processing, and even quantum mechanics. Furthermore, improvements on this method have led to higher-order iterative schemes that can be tailored for specific levels of accuracy [4].

A comparative study reveals that while the Bisection Method ensures reliability, the Newton–Raphson Method excels in speed and efficiency. The latter can determine roots to a high degree of precision and is particularly useful in evaluating nonlinear equations with complex or multiple roots. In practice, these two algorithms complement one another: one provides safety, the other speed. As modern research introduces hybrid techniques—such as combining Bisection with False Position or integrating genetic algorithms—root-finding continues to evolve, blending classical methods with innovative computational strategies [5].

This review aims to present a detailed overview of the Bisection and Newton–Raphson methods, highlighting their fundamental principles, applications, comparative performance, and potential as building blocks for advanced root-finding algorithms.

2 Concept of Methods

2.1 Bisection Method

The Bisection Method is a root-finding algorithm based on the repeated subdivision of an interval that contains a solution. If a continuous function changes sign within a given interval, then by the Intermediate Value Theorem, there must be at least one root inside that range. The method works by progressively halving the interval and selecting the subinterval where the sign change occurs, continuing the process until the approximation meets the required accuracy.

The steps of the algorithm can be outlined as follows:

Input: Function *f(x)*, initial interval $[x_1, x_2]$, and a tolerance value *ε*.
Formula: $x = \frac{x_1+x_2}{2}$

Algorithm – Bisection Method

1. Begin.
2. Read the initial estimates x_1, x_2, and the tolerance *ε*.
    - $x_1$and $x_2$are the starting guesses.
    - *ε* represents the absolute error (desired precision).
3. Evaluate $f(x_1)$and $f(x_2)$.
4. If $f(x_1) \cdot f(x_2) > 0$, report that the initial interval is invalid and terminate.
5. Compute midpoint: $x = \frac{x_1+x_2}{2}$.
6. If $| \frac{x_1-x_2}{x} | < \varepsilon$, output *x* as the root and stop.
7. Otherwise, calculate $f(x)$.
8. If $f(x) \cdot f(x_1) > 0$, set $x_1 = x$. Else, set $x_2 = x$.
9. Return to step 5 and repeat until the tolerance condition is satisfied.
10. End.

This iterative process guarantees convergence as long as the initial interval is valid, making the Bisection Method one of the most reliable techniques for locating roots of continuous functions.

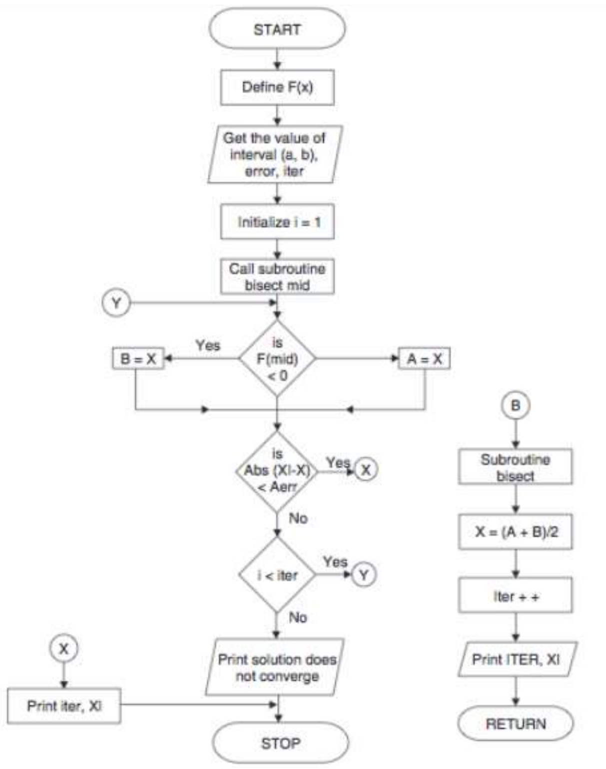

Figure 1 : flow chart of Bisection method

2.2 Newton–Raphson Method

The Newton–Raphson Method is a widely adopted iterative procedure for determining the roots of nonlinear equations. It is known for its speed and efficiency, relying on the idea of tangent-line approximations to progressively refine guesses until the solution is reached. Because of its rapid convergence when applied under suitable conditions, it has become an essential tool in both engineering and scientific computations.

Formula:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Algorithm – Newton–Raphson Method

1. Begin.
2. Read the initial approximation x_0, tolerance ε, maximum number of iterations *n*, and a threshold *d* for slope validation.
   - $x_0$is the starting guess.
   - $\varepsilon$is the absolute error (accuracy requirement).
   - *n* controls the loop iterations.
   - *d* ensures the derivative is not too small.
3. For each iteration $i = 1$to *n*:

a. Compute $f(x_0)$.

b. Compute $f'(x_0)$.

c. If $| f'(x_0) | < d$, report that the slope is too small and terminate.

d. Update estimate: $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$.

e. If $| \frac{x_1 - x_0}{x_1} | < \varepsilon$, accept $x_1$as the root and stop.

f. Set $x_0 = x_1$and continue.

4. If convergence is not achieved within *n* iterations, report possible divergence or oscillation.
5. End.

The Newton–Raphson Method is exceptionally fast when the initial guess is close to the actual solution, but it may fail if the slope is near zero or if the initial guess is poorly chosen. Despite these limitations, it remains one of the most powerful and commonly used techniques for root-finding in numerical analysis.

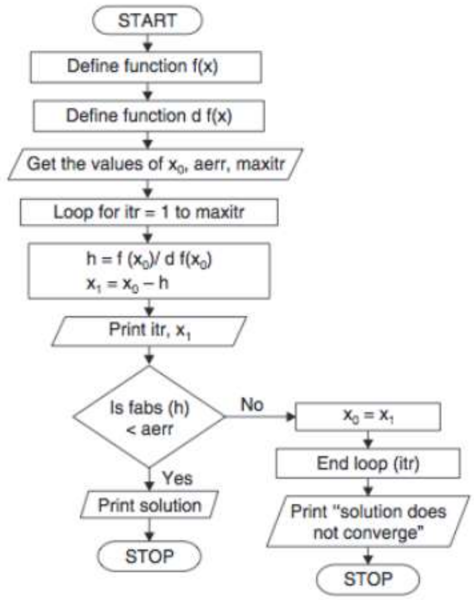

Figure 2 : flow chart of Newton-Raphson method

3 Rate of Convergence

3.1 Convergence Rate of the Bisection Method

The Bisection Method exhibits linear convergence. This indicates that at every step, the accuracy of the root estimate improves steadily, though at a relatively slow pace. Formally, the error bound can be expressed as:

$$| x_{n+1} - x^* | \leq (\frac{1}{2})^{n+1} | b - a |,$$

where $x_{n+1}$is the approximation after the $(n + 1)$-th iteration, $x^*$is the actual root, and $[a, b]$represents the starting interval.

This relationship shows that the error decreases by a factor of one-half with each iteration, which explains the method's classification as linearly convergent. Although the Bisection Method ensures convergence for continuous functions with a sign change, achieving high precision may demand a large number of iterations, particularly in cases involving complicated functions or intervals containing more than one root.

3.2 Convergence Rate of the Newton–Raphson Method

The Newton–Raphson Method typically demonstrates quadratic convergence. This means that the error shrinks much faster than in linear methods, with the number of accurate digits roughly doubling at each step. Its efficiency and rapid improvement per iteration make it significantly more powerful than the Bisection Method when the initial approximation is suitably chosen and the function behaves well.

$$| x_{n+1} - x^* | \leq C | x_n - x^* |^2,$$

where x_{n+1}denotes the approximation after the $(n+1)$-th step, $x^*$is the exact solution, and $C$is a constant factor.

This inequality illustrates that the Newton–Raphson Method achieves **quadratic convergence**, meaning the error decreases proportionally to the square of the previous error. As a result, the number of accurate digits roughly doubles with each successive iteration.

Nevertheless, this rapid rate of convergence is valid only if the initial estimate is sufficiently close to the actual root and the function behaves smoothly in that region. When the starting point is far from the root, or if the function has complications such as singularities or multiple nearby roots, convergence can slow down or even fail. In such situations, supplementary strategies—such as interval bracketing or hybrid algorithms—are often employed to secure convergence.

To validate the method, an algorithm suitable for computer implementation was designed and later applied to compute approximations of square roots of positive real numbers. The outcomes were then analyzed to evaluate both the accuracy and the computational efficiency of the Newton–Raphson approach. The following section presents the details of the MATLAB implementation process.

4 Comparative Analysis

A wide range of research has evaluated the performance of the Bisection and Newton–Raphson methods under different conditions. These comparisons typically consider convergence speed, computational workload, precision, and overall reliability.

In practice, the Newton–Raphson method converges more quickly than the Bisection method when the starting estimate is close to the actual solution. In contrast, the Bisection method provides guaranteed convergence within a chosen interval, making it the preferred option for non-differentiable functions or when only a broad interval is known. Based on the rate of convergence, the order can be expressed as:

$$\text{Newton–Raphson} > \text{Bisection}.$$

Example: Root of $f(x) = x^3 - 2x - 5$

Bisection Method:
Starting with the interval [2,3], the interval is repeatedly halved. After several steps, the approximate solution is obtained as $x \approx 2.710938$.

Newton–Raphson Method:
Using the initial guess $x_0 = 3$, iterations yield the result $x \approx 2.710268$after only three steps, showing significantly faster convergence.

Both approaches produce acceptable approximations, but their efficiency depends on the starting conditions and the function's nature.

4.1 Comparative Analysis in Tabular Form

Bisection Method:
With the interval [2,3], the procedure converges in 13 iterations to $x \approx 2.6305$.

Newton–Raphson Method:
With an initial guess $x_0 = 2$, the solution converges in only 4 iterations to $x \approx 2.6304$.

The tables illustrate the contrast: while Bisection guarantees steady progress, it requires more steps, whereas Newton–Raphson quickly achieves high accuracy.

5 Discussion of Results

Both methods are effective in approximating roots, but they differ significantly in behavior. The Bisection method always converges provided the interval is valid, yet its progress is relatively slow. Newton–Raphson, on the other hand, is highly efficient when the initial guess is suitable but may diverge or converge to an unintended root if the guess is poor.

The selection of the method depends on the problem setting. When an enclosing interval is known, Bisection is a safe choice. When a good starting value is available and the function is differentiable, Newton–Raphson offers superior speed. Hybrid approaches, such as Brent's method, combine the reliability of Bisection with the efficiency of Newton–Raphson to achieve both robustness and rapid convergence.

6 Conclusion

The Bisection and Newton–Raphson algorithms remain cornerstone techniques for root-finding in numerical computation. The Bisection method guarantees convergence under minimal assumptions, while the Newton–Raphson method provides faster results with suitable initial conditions. A solid understanding of their strengths and weaknesses is essential for choosing the appropriate tool in scientific and engineering applications. Modern research continues to enhance these methods by developing hybrid schemes that improve reliability and efficiency.

7 References

[1] A. S. Chouhan and E. Ijmtst, "Current Role of Numerical Analysis in Mathematics," June 2022. doi: 10.46501/IJMTST0806071.
[2] S. Namekar, "Role of Bisection Method," *Int. J. Res. Appl. Sci. Eng. Technol.*, vol. 7, no. 11, pp. 426–429, 2019. doi: 10.22214/ijraset.2019.11068.
[3] C. Solanki, P. Thapliya, and K. Tomar, "Role of Bisection Method," *Int. J. Comput. Appl. Technol. Res.*, vol. 3, no. 8, pp. 533–535, 2014. doi: 10.7753/ijcatr0308.1009.
[4] S. Akram and Q. Ann, "Newton–Raphson Method," *Financ. Simul. Model. Excel*, vol. 6, no. 7, pp. 183–185, 2015. doi: 10.1002/9781119200925.app2.
[5] C. L. Sabharwal, "Blended Root-Finding Algorithm Outperforms Bisection and Regula Falsi Algorithms," *Mathematics*, vol. 7, no. 11, 2019. doi: 10.3390/math7111118.
[6] U. L. Approximations and S. Equa-, "The Newton–Raphson Method," *Pure Appl. Math.*, vol. 9, pp. 53–55, 1973. doi: 10.1016/S0079-8169(08)62692-6.
[7] H. Rahman, K. C. Roy, S. K. Das, and S. A. Hossain, "A Study on the Numerical Accuracy and Efficiency of the Bisection Method in Finding Square Roots of Positive Real Numbers," *Int. J. Sci. Res. Comput. Sci. Eng.*, vol. 10, no. 3, pp. 7–12, 2022.
[8] Y. Jun and J. Jeon, "Modified Bisection Method for Solving Nonlinear Equations," *Int. J. Sci. Innov. Math. Res.*, vol. 7, no. 9, pp. 8–11, 2019. doi: 10.20431/2347-3142.0709003.
[9] V. V. Mehtre and S. Singh, "Obtaining Roots of Non-Linear Equations Using Newton–Raphson Method," vol. 3, no. 6, pp. 2–4, 2019.
[10] S. O. Ehiwario and J. C. Aghamie, "Comparative Study of Bisection, Newton–Raphson, and Secant Methods of Root-Finding Problems," *IOSR J. Eng.*, vol. 4, no. 4, pp. 1–7, 2014.

paper summary؟